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Convergence rates of asymptotic solutions to Hamilton-Jacobi equations in Euclidean n space

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This is a survey of the paper [F]. Let us consider the Cauchy problem for the Hamilton-Jacobi equation

(0.1) $$u_t(x,t)+H(x,Du(x,t)) = 0 \quad \text{in } \mathbb{R}^n\times(0,\infty),$$

(0.2) $$u(\cdot,0) = u_0 \quad \text{in } \mathbb{R}^n.$$

By [FIL, I], under suitable assumptions on H, u_0, it is shown that the Cauchy problem (0.1)-(0.2) admits a unique solution $u\in C(\mathbb{R}^n\times[0,\infty))$ and there is a pair $(c,v)\in\mathbb{R}\times C(\mathbb{R}^n)$ such that

(0.3) $$\lim_{t\to\infty}(u(x,t)+ct)=v(x) \quad \text{locally uniformly in } \mathbb{R}^n.$$

Furthermore, v is a solution of

(0.4) $$H(x,Dv(x))=c \quad \text{in } \mathbb{R}^n.$$

In this talk, we are intersted in rates of convergence of (0.3). We assume the following:

(A1) $H\in C(\mathbb{R}^n\times\mathbb{R}^n)$.

(A2) For each $x\in\mathbb{R}^n$, $H(x,\cdot)$ is convex in $\mathbb{R}^n$.

(A3) $\lim_{r\to\infty}\inf\{H(x,p)\mid x\in B(0,R),\ p\in\mathbb{R}^n\setminus B(0,r)\}=\infty$ for $R>0$.

(A4) There exists a pair $(\theta_0,c,w^+,w^-)\in(0,\infty)\times\mathbb{R}\times C(\mathbb{R}^n)\times C^1(\mathbb{R}^n)$ such that w^+ and w^- are, respectively, a subsolution and a supersolution of (0.4) and

(0.5) $$0\le w^+(x)-w^-(x)\le\frac{1}{\theta_0}(c-H(x,Dw^-(x))) \quad \text{in } \mathbb{R}^n.$$

If (A1)-(A4) is fuffilled, then we define the function $\hat{w}$ by

(0.6) $$\hat{w}(x)=\sup\{w(x)\mid w\in W\} \quad \text{in } \mathbb{R}^n,$$

where W is the set of all $w\in C(\mathbb{R}^n)$ such that w^-+w is a viscosity subsolution of (0.4)

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for the constant c of (A4), and w satisfies the inequality

$$0 \le w(x) \le w^+(x) - w^-(x) \quad \text{in } \mathbb{R}^n. \tag{0.7}$$

Remark 1. $0 \le \hat{w}(\cdot) \in \mathrm{Lip}_{loc}(\mathbb{R}^n)$, and $w^- + \hat{w}$ is a solution of (0.4). □

Besides (A1)-(A4), we assume:

(A5) There exists a function $\varphi \in C(\mathbb{R}^n)$ such that $\inf\{w^-(x) + \hat{w}(x) + \varphi(x) | x \in \mathbb{R}^n\} > -\infty$ and the following comparison principle holds: Let

$$\Phi := \Big\{u \in C(\mathbb{R}^n \times [0,\infty)) \,\Big|\, \inf\{u(x,t) + \varphi(x) \,|\, x \in \mathbb{R}^n,\ t \in [0,T)\} > -\infty \text{ for } T > 0\Big\}.$$

If $u_1 \in C(\mathbb{R}^n \times [0,\infty))$ and $u_2 \in \Phi$ are, respectively, a subsolution and a supersolution of (0.1) and satisfy $u_1(\cdot,0) \le u_2(\cdot,0)$ in $\mathbb{R}^n$, then $u_1 \le u_2$ in $\mathbb{R}^n \times [0,\infty)$.

(A6) $u_0 \in C(\mathbb{R}^n)$, and there exists a pair $(K,F) \in \mathbb{R} \times C([0,\infty))$ such that

$$F \ge 0 \quad \text{in } [0,\infty), \qquad \limsup_{s \searrow 0} \frac{F(s)}{s} < \infty, \tag{0.8}$$

$$K + w^-(x) \le u_0(x) \le K + w^-(x) + F(\hat{w}(x)) \text{ in } \mathbb{R}^n. \tag{0.9}$$

Remark 2. We give a sufficient condition for (A5). Assume that (A1), (A3) and (A4) hold and that $H(x,\cdot)$ is strictly convex in $\mathbb{R}^n$ for each $x \in \mathbb{R}^n$ instead of (A2). Furthermore, assume that there exist functions $\psi_i \in \mathrm{Lip}_{loc}(\mathbb{R}^n)$ and $\sigma_i \in C(\mathbb{R}^n)$, with $i = 0,1$, such that for $i = 0,1$,

$$\begin{cases} \lim_{|x|\to\infty} \sigma_i(x) = \infty, \quad \inf\{w^-(x) + \hat{w}(x) + \psi_0(x) | x \in \mathbb{R}^n\} > -\infty, \\ H(x, -D\psi_i(x)) \le -\sigma_i(x) \text{ almost every } x \in \mathbb{R}^n. \\ \lim_{|x|\to\infty} (\psi_1(x) - \psi_0(x)) = \infty. \end{cases} \tag{0.10}$$

Then (A5) holds for $\varphi = \psi_0$ by [I, Theorem 4.1]. As for examples satisfying these conditions, we give in our talk. □

Lemma 1. Let F be the function of (A6). Then, there exists a function $G \in C([0,\infty)) \bigcap C^1((0,\infty))$ such that

$$G(0) = 0, \ \ s + F(s) \le G(s) \le sG'(s) \text{ in } (0,\infty). \tag{0.11}$$ □

In the following, we assume (A1)-(A6). We define the constant $\theta \in (0,\infty]$ by

$$\theta = \sup\{\theta_0 \,|\, \theta_0 \text{ fulfills (0.5)}\}. \tag{0.12}$$

Theorem 1. Assume (A1)-(A6).

(i) $\theta = \infty$ if and only if w^- is a solution of (0.4).

(ii) Let $u \in \Phi$ be a solution of the Cauchy problem (0.1)-(0.2).

(a) If $\theta = \infty$, then $\hat{w} = 0$ in $\mathbb{R}^n$ and $u(x,t) + ct = K + w^-(x)$ in $\mathbb{R}^n \times [0,\infty)$.

(b) If $\theta < \infty$, then

$$(0.13) \quad -\hat{w}(x)e^{-\theta t} \le u(x,t) + ct - (K + w^-(x) + \hat{w}(x)) \le [G(\hat{w}(x)) - \hat{w}(x)]\, e^{-\theta t} \quad \text{in } \mathbb{R}^n \times [0,\infty),$$

where G is the function of Lemma 1. □

Next, we give an example such that even if (A1)-(A5) hold, the rate of convergence in (0.3) is just equal to t^{-1} as $t \to \infty$ provided (A6) is violated. For $a, b \in \mathbb{R}$, let $a^+ = \max\{a, 0\}$, $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

Example 1. For $\alpha > 0$, let

$$\begin{aligned} H(x,p) &= \alpha x \cdot p + \frac{1}{2}|p|^2 - \frac{\alpha^2}{2}(1 - |x|^2)^+ \quad \text{in } \mathbb{R}^n \times \mathbb{R}^n, \\ u_0(x) &= \frac{\alpha}{2} \quad \text{in } \mathbb{R}^n. \end{aligned}$$

Then, we have:

(i) The assumptions (A1)-(A5) hold for the constants $c = -\alpha^2/2$, $\theta_0 = \alpha$ and the functions

$$w^+(x) = \zeta_1(x), \qquad w^-(x) = \zeta_k(x) \ (k \in (0,1)), \qquad \varphi(x) = \frac{\alpha}{2}|x|^2 \quad \text{in } \mathbb{R}^n,$$

where

$$\zeta_\ell(x) = \frac{\alpha}{2}(1 - |x|^2) + \alpha\ell \int_1^{|x| \vee 1} \sqrt{r^2 - 1}\, dr \quad \text{for } x \in \mathbb{R}^n,\ \ell \in (0,1].$$

However, there is no pair $(K, F) \in \mathbb{R} \times C([0,\infty))$ for which (A6) holds.

(ii) The Cauchy problem (0.1)-(0.2) admits a unique solution $u \in \Phi$ given by

$$\frac{\alpha}{2(\alpha t + 1)}(|x|^2 \wedge 1) \le u(x,t) - \frac{\alpha^2}{2}t - \zeta_1(x) \le \frac{\alpha}{2(\alpha t + 1)}|x|^2 \quad \text{in } \mathbb{R}^n \times [0,\infty),$$

where Φ is the set of (A5) for φ of (i). □

Finally, we give an example such that the precise rate of convergence in (0.3) is obtained by our sufficient condition.

Example 2. For $\alpha, \beta > 0$, let

$$H(x,p) = \alpha x \cdot p + \frac{1}{2}|p|^2 - \frac{\beta}{2}|x|^2 \quad \text{in } \mathbb{R}^n \times \mathbb{R}^n.$$

Assume that $u_0 \in C(\mathbb{R}^n)$ and that there is a constant $\ell \in (1, \infty)$ such that

$$\frac{A}{2}|x|^2 \le u_0(x) \le \frac{A\ell}{2}|x|^2 \quad \text{in } \mathbb{R}^n, \tag{0.14}$$

where $A = \sqrt{\alpha^2 + \beta} - \alpha$. Then, we have:

(i) Let $k \in (0,1)$. The assumptions (A1)-(A6) hold for

$$\begin{aligned} &c = 0, \quad \theta_0 \in (0, A(1+k) + 2\alpha] \\ &w^+(x) = \frac{A}{2}|x|^2, \quad w^-(x) = \frac{Ak}{2}|x|^2, \quad \varphi(x) = \frac{\alpha}{2}|x|^2 \quad \text{in } \mathbb{R}^n, \\ &K = 0, \quad F(s) = \frac{\ell - k}{1-k}s \quad \text{in } [0, \infty). \end{aligned}$$

In this case, θ is equal to $A(1+k) + 2\alpha (=: \theta_k)$, and $\hat{w}(x) = A(1-k)|x|^2/2$ in $\mathbb{R}^n$.

(ii) Let Φ be the set of (A5) which is defined for φ of (i). Then, the Cauchy problem (0.1)-(0.2) admits a unique solution u in Φ. By letting $G(s) = F(s) + s$ in $[0, \infty)$, Theorem 1 leads to

$$-\frac{A(1-k)}{2}|x|^2 e^{-\theta_k t} \le u(x,t) - \frac{A}{2}|x|^2 \le \frac{A(\ell - k)}{2}|x|^2 e^{-\theta_k t} \quad \text{in } \mathbb{R}^n \times [0, \infty).$$

In particular, letting $k \nearrow 1$, we obtain

$$0 \le u(x,t) - \frac{A}{2}|x|^2 \le \frac{A(\ell - 1)}{2}|x|^2 e^{-\lambda t} \quad \text{in } \mathbb{R}^n \times [0, \infty),$$

where $\lambda = 2\sqrt{\alpha^2 + \beta}$.

(iii) When $u_0(x) = A\ell|x|^2/2$ in $\mathbb{R}^n$, a unique solution $u \in \Phi$ is given by

$$u(x,t) = \frac{A|x|^2}{2}\left(1 + \frac{\lambda(\ell - 1)}{A(\ell - 1)(e^{\lambda t} - 1) + \lambda e^{\lambda t}}\right) \quad \text{in } \mathbb{R}^n \times [0, \infty).$$

Hence, the rate $e^{-\lambda t}$ which is obtained in (ii) is optimal in this case. □

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