数理解析研究所講究録
第 1545 巻 2007 年 38-46

The Allen–Cahn type equation with multiple-well potentials and mean curvature flow equation

東京大学大学院数理科学研究科 大塚 岳 (Takeshi Ohtsuka)
Department of Mathematical Sciences,
The University of Tokyo

1. Introduction

In this paper we discuss about the convergence of the internal transition layers of the Allen–Cahn type equation with multiple-well potential of the form

$$u_t^\varepsilon - \Delta u^\varepsilon + \frac{1}{\varepsilon^2} f_\varepsilon(u^\varepsilon) = 0 \quad \text{in } \mathbb{R}^N \times (0,T) \tag{1.1}$$

with initial condition

$$u^\varepsilon(\cdot,0) = u_0 \in BUC(\mathbb{R}^N), \tag{1.2}$$

where f_ε is of the form

$$f_\varepsilon(r) = -\sin r - \varepsilon a(1+\cos r), \tag{1.3}$$

and a is a constant.

The equation (1.1) is called the Allen–Cahn equation if $f_\varepsilon(u) = 2u(u^2-1)$, which is introduced by [AC] as the equation which describes the motion of grain boundaries in a material. The function $u \mapsto 2u(u^2-1)$ is the derivative of the bistable potential of the form $u \mapsto (u^2-1)^2/2$. Here "bistable" means that the potential has exactly two local minima at $u = \pm 1$. By tending $\varepsilon \to 0$ we have a sharp interface, which is called internal transition layers, from the solution of the Allen–Cahn equation. The asymptotic analysis as in, for example, [RSK] yields that the internal transition layers approximates the motion of interfaces Γ_t which moves by

$$V = -H \quad \text{on } \Gamma_t,$$

where V is the normal velocity of Γ_t, and H is the mean curvature in the direction of the minus of the outer unit normal vector field of Γ_t. The rigorous proof of the convergence is given by [ESS]. This result is extended to the case that the interface moves by the mean curvature flow with some driving force by, for example, [BSS], or the Neumann boundary value broblems by

[KKR]. That is also extended to the anisotropic case by, for example, [ElS1], [ElPS], [ElS2], [GOS]. The set theoretic approach is provided by [BS]. It is extended to the Neumann type boundary value problems by [BD].

The function f_ε is the derivative of the multiple-well potential F_ε of the form

$$F_\varepsilon(r) = \cos u - \varepsilon a(r + \sin r). \tag{1.4}$$

This potential has local minima at $u = (2k+1)\pi$ for $k \in \mathbb{Z}$. Thereby the solution u^ε has a lot of internal transition layers in a neighborhood of the sets $\{x;\ u^\varepsilon(x,t) = 2\pi k\}$ for $k \in \mathbb{Z}$. The aim of this paper is to give an brief idea to prove the convergence of internal transition layers to the interface which moves by the mean curvature flow equation with driving force of the form

$$V = -H + A \quad \text{on } \Gamma_t,$$

where A is a constant. We remark that our problem is essentially same as that of the Allen–Cahn equation if we assume that the initial data u_0 satisfies $\sup_{\mathbb{R}^N} |u_0| \le \pi$ because of the comparison principle. Therefore we assume that u_0 satisfies

$$-\pi \le u_0 \le 3\pi \text{ in } \mathbb{R}^N,\ \inf_{\mathbb{R}^N} u_0 < 0, \text{ and } \sup_{\mathbb{R}^N} u_0 > 2\pi. \tag{1.5}$$

In this case the internal transition layers appear in a neighborhood of the sets $\{x;\ u^\varepsilon(x,t) = 2\pi k\}$ for $k = 0$ and $k = 1$, respectively.

For the proof, we adjust the method of the generation of interface by X. Chen in [C], and the construction of supersolutions for estimating the internal transition layers by L. C. Evans, H. M. Soner and P. E. Souganidis in [ESS]. The crucial difference between our problem and the Allen–Cahn equation is the way to construct a supersolution. The usual way to construct a supersolution as in [ESS] provides only the estimate of the motion of the internal transition layers in a neighborhood of $\{x;\ u^\varepsilon(x,t) = 2\pi\}$ from above. This is because of the height of the usual traveling wave. To overcome this difficulty, we construct a supersolution with twice heights of layers by using the property of a closedness of a viscosity supersolutions under infimum.

R. Jerrard proved the another type of the convergence result in [J]. He consider the equation of the form

$$u_t^\varepsilon - \Delta u^\varepsilon + \frac{1}{\varepsilon^{1+\gamma}} f_\varepsilon\left(\frac{u^\varepsilon}{\varepsilon^{1-\gamma}}\right) = 0 \quad \text{in } \mathbb{R}^N \times (0,T)$$

for $\gamma \in [0,1]$ instead of (1.1). He proved a locally uniform convergence of $u = \lim_{\varepsilon \to 0} u^\varepsilon$ provided that $\gamma \in (0, \gamma_0)$ for some γ_0, and u solve the mean curvature flow equation if $\gamma > 0$.

2. Equations

2.1. Allen–Cahn equation with multi-well potential

Consider the Cauchy problem (1.1) with initial condition (1.2). The usual theory of viscosity solutions are valid for (1.1)–(1.2). Especially, we have the comparison principle, the existence and uniqueness of viscosity solutions. See [CIL] for the proof of them.

For sufficiently small $\varepsilon > 0$, the function $f_\varepsilon \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ has exactly three zeros in $[-\pi, \pi]$ at $r = \pm\pi$ and $r = \alpha_\varepsilon$. By straightforward calculation we have

$$f_\varepsilon'(\pm\pi) = 1, \;\; f_\varepsilon'(\alpha_\varepsilon) = -1,$$

and f_ε satisfies

$$f_\varepsilon > 0 \text{ in } (-\pi, \alpha_\varepsilon), \;\; f_\varepsilon < 0 \text{ in } (\alpha_\varepsilon, \pi).$$

Therefore the f_ε satisfies the assumptions for the nonlinear term of the Allen–Cahn equation in $[-\pi, \pi]$. Since f_ε is periodic with the period 2π, several internal transition layers appear. By the assumption (1.5), the internal transition layers appear around the sets $\{x;\ u^\varepsilon(x,t) = 2\pi k\}$ for $k = 0, 1$.

Remark 2.1. *In this paper we give an explicit form of f_ε. Fortunately, we can extend the results of this paper to the case that $f_\varepsilon = f_0 + \varepsilon f_1$, and satisfy the condition*

(i) $f_0, f_1 \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$,

(ii) $f_0(r)$ *has exactly three zeros in* $[-\pi, \pi]$ *at* $r = \pm\pi$ *and* $r = 0$, $f_1(r)$ *has exactly two zeros in* $[-\pi, \pi]$ *at* $r = \pm\pi$,

(iii) $f_0'(\pm\pi) > 0$ *and* $f'(0) < 0$,

(iv) $\int_{-\pi}^{\pi} f_0(r)dr = 0$.

The important property is that the periods of f_0 and f_1 are same.

2.2. Asymptotic expansion

To find an interface evolution equation for the internal transition layers, we consider the formal asymptotic expansion of solutions of (1.1) as in [RSK]. Set

$$u^\varepsilon(x,t) = Q(x, t, \varepsilon^{-1}\varphi(x,t) - \varepsilon^{-2}ct) + \varepsilon P(x, t, \varepsilon^{-1}\varphi(x,t) - \varepsilon^{-2}ct) + O(\varepsilon^2)$$

for a neighborhood of $\Gamma_k^\varepsilon(t) := \{x;\ u^\varepsilon(x,t) = 2\pi k\}$ for $k = 0, 1$. Then we obtain

$$u_t^\varepsilon - \Delta u^\varepsilon + \frac{1}{\varepsilon^2} f_\varepsilon(u^\varepsilon) = \varepsilon^{-2} I_0 + \varepsilon^{-1} I_1 + O(1),$$

where the order $O(1)$ is as $\varepsilon \to 0$,

$$\begin{aligned} I_0 &= -\ |\nabla\varphi|^2 Q'' - cQ' + f_0(Q), \\ I_1 &= -\ |\nabla\varphi|^2 P'' - cP' + f_0(Q)P + Q'(\varphi_t - \Delta\varphi) - 2\langle \nabla Q', \nabla\varphi\rangle + f_1(Q), \end{aligned}$$

$Q' = Q_\sigma$ and $P' = P_\sigma$ for $Q = Q(x,t,\sigma)$ and $P = P(x,t,\sigma)$, respectively. The equation (1.1) yields that $I_0 = I_1 = 0$. We now assume that $Q(x,t,\pm\infty) = \pm\pi + 2\pi k$ for $k = 0, 1$. Then the methods in [RSK, Section 3] yields that

$$\varphi_t - \Delta\varphi + \frac{\langle \nabla^2\varphi\nabla\varphi, \nabla\varphi\rangle}{|\nabla\varphi|^2} - A_k|\nabla\varphi| = 0,$$

where

$$A_k = -\frac{\int_{-\pi+2\pi k}^{\pi+2\pi k} f_1(u)du}{\int_{\mathbb{R}} (q_k'(\sigma))^2 d\sigma}, \tag{2.1}$$

and q_k is the solution of the ordinary differential equation of the form

$$\begin{aligned} q_k'' &= f_0(q) \quad \text{in } \mathbb{R}, \\ q_k(\pm\infty) &= \pm\,\pi + 2\pi k, \\ q_k(0) &= 2\pi k. \end{aligned}$$

We remark that $q_k = q_0 + 2\pi k$ and $\int_{-\pi+2\pi k}^{\pi+2\pi k} f_1(u)du = \int_{-\pi}^{\pi} f_1(u)du$, which yields $A_k = A_0 =: A$.

Here and hereafter we consider the level set equation for $V = -H + A$ of the form

$$u_t - \Delta u + \frac{\langle \nabla^2 u \nabla u, \nabla u\rangle}{|\nabla u|^2} - A|\nabla u| = 0 \quad \text{in } \mathbb{R}^N \times (0,T) \tag{2.2}$$

with initial condition

$$u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^N. \tag{2.3}$$

The usual methods for viscosity solution are valid for (2.2). See [CGG], [ES], or [G] for more precise properties.

3. Convergence result

We prepare some notations to state our main result.

Let u be a solution of (2.2)–(2.3). For $k = 0, 1$, we define

$$\begin{aligned} I_t^k &= \{x;\ u(x,t) > 2\pi k\}, \\ O_t^k &= \{x;\ u(x,t) < 2\pi k\}, \\ \Gamma_t^k &= \{x;\ u(x,t) = 2\pi k\}. \end{aligned}$$

We also define for $k = 0, 1$,

$$\begin{aligned} I^k &= \{(x,t) \in \mathbb{R}^N \times (0,T);\ u(x,t) > 2\pi k\}, \\ O^k &= \{(x,t) \in \mathbb{R}^N \times (0,T);\ u(x,t) < 2\pi k\}. \end{aligned}$$

Theorem 3.1. *Let u^ε be a viscosity solution of (1.1) with $u^\varepsilon(\cdot,0) = u_0$. Assume that the initial data u_0 satisfies (1.5). Let u be a viscosity solution of (2.2) with $u(\cdot,0) = u_0$. Then we have the followings.*

(i) For $k = 0, 1$ and any compact subset $K \in I^k$, we have

$$\varliminf_{\varepsilon \to 0} \sup_{(x,t)\in K} u^\varepsilon(x,t) \geq (2k+1)\pi.$$

(ii) For $k = 0, 1$ and any compact subset $k \in O^k$, we have

$$\varlimsup_{\varepsilon \to 0} \inf_{(x,t)\in K} u^\varepsilon(x,t) \leq (2k-1)\pi.$$

By Theorem 3.1 and the comparison principle it is easy to obtain

Corollary 3.2. *Under the same hypothesis of Theorem 3.1 we have*

$$u^\varepsilon \to (2k+1)\pi \quad \textit{in } I^k \cap O^{k+1}$$

for $k = -1, 0, 1$ locally uniformly as $\varepsilon \to 0$.

The proof of Theorem 3.1 is devided into two steps, which are described in the following two lemmas.

Lemma 3.3. *Let u^ε be a viscosity solution of* (1.1) *with $u^\varepsilon(\cdot,0) = u_0$. Assume that u_0 satisfies* (1.5). *Then, for any $b > 0$ and $m > 0$, there exist positive constants $\bar{\varepsilon} = \bar{\varepsilon}(b,m)$ and $\tau_0 = \tau_0(b)$ such that*

$$\begin{aligned} &u^\varepsilon(x, \tau_0\varepsilon^2|\log\varepsilon|) \geq (2k+1)\pi - b\varepsilon \\ &\qquad \text{if } x \in \{y \in \mathbb{R}^N;\ u_0(y) \geq 2\pi k + m\}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &u^\varepsilon(x, \tau_0\varepsilon^2|\log\varepsilon|) \leq (2k-1)\pi + b\varepsilon \\ &\qquad \text{if } x \in \{y \in \mathbb{R}^N;\ u_0(y) \leq 2\pi k - m\} \end{aligned} \tag{3.2}$$

for $k = 0$, 1 provided that $\varepsilon \in (0.\bar{\varepsilon})$.

Lemma 3.4. *Let u be a viscosity solution of* (2.2) *with $u(\cdot,0) = u_0 \in BUC(\mathbb{R}^N)$. Let $\Gamma_t = \{x;\ u(x,t) = C\}$ with $C \in \mathbb{R}$ and $d(x,t)$ be a function defined by*

$$d(x,t) = \begin{cases} dist(x,\Gamma_t) & \text{if } x \in \{y;\ u(y,t) \geq C\}, \\ -dist(x,\Gamma_t) & \text{if } x \in \{y;\ u(y,t) < C\}. \end{cases}$$

For any $\beta > 0$, there exist a constant $\varepsilon_0 = \varepsilon(\delta) > 0$ and a viscosity supersolution $v = v^{\varepsilon,\delta}$ of (1.1) *provided that $\varepsilon \in (0,\varepsilon_0)$ satisfying*

(i) $v(x,t) \geq 3\pi$ if (x,t) satisfies $d(x,t) > \beta$,

(ii) $v(x,t) \leq -\pi + \varepsilon\tilde{C}$ if (x,t) satisfies $d(x,t) < -\beta$,

where $\tilde{C}$ is a positive constant.

We remark that we can construct a viscosity subsolution satisfying (i) and (ii) of Lemma 3.4 by similar way, so that we only mentioned about the construction of a supersolution.

The crucial observation for our problem is Lemma 3.4. In the method of the construction as in [ESS] we consider the traveling wave $q\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$q'' + cq' = f_\varepsilon(q) \quad \text{in } \mathbb{R}, \tag{3.3}$$

where c is a constant determined only by f_ε. By applying the method as in [AW, Section 4], we obtain the existence and uniqueness of (q,c) with $q(\pm\infty) = 2\pi k \pm \pi$ for $k \in \mathbb{Z}$. When we try to construct a supersolution as in Lemma 3.4, we attempt to consider the traveling wave q satisfying (3.3) with the boundary condition

$$q(-\infty) = -\pi, \ q(\infty) = 3\pi$$

instead of $q(\pm\infty) = 2\pi k \pm \pi$. Unfortunately, however, there is no such a solution if $a = 0$, i.e., $f_\varepsilon = -\sin u$ (see [O]). To overcome this difficulty, we adjust the method in [ESS] for our problem.

Let q be a traveling wave satisfying (3.3) with $q(\pm\infty) = \pm\pi$. Let η be a truncating function as in [ESS] satisfying $\eta \in C^\infty(\mathbb{R})$,

$$\eta(\sigma) = \begin{cases} \sigma - \delta & \text{if } \sigma > \delta/2, \\ -\delta & \text{if } \sigma < \delta/4, \end{cases}$$
$$0 \leq \eta' \leq C_\eta \text{ in } \mathbb{R},$$
$$|\eta''| \leq C_\eta/\eta \text{ in } \mathbb{R}$$

for $\delta > 0$, where C_η is a numerical constant. Define $\psi_j^{\varepsilon,b}\colon \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ by

$$\psi_j^{\varepsilon,b}(x,t) = q\left(\frac{\eta(d(x,t)) + K_1 t + jb}{\varepsilon}\right) + 2\pi(1-j) + \varepsilon(K_2 + jb).$$

We remark that $q(s) + 2\pi$ is a solution of (3.3) with $q(\pm\infty) = 2\pi \pm \pi$. By give more precise estimates in the proof of [ESS, Theorem 3.2], we obtain the following lemma.

Lemma 3.5. *Under the hypothesis on above, for $\delta > 0$, there exist positive constants $b_0 = b_0(\delta)$, $K_1 = K_1(\delta)$ and $K_2 = K_2(\delta)$ such that, for any $b \in (0, b_0)$, there exists $\hat{\varepsilon} = \hat{\varepsilon}(\delta, b)$ such that $\psi_j^{\varepsilon,b}$ is a viscosity supersolution of (1.1) provided that $\varepsilon \in (0, \hat{\varepsilon})$ and $j = 0, 1$.*

We define $v(x,t)$ by

$$v(x,t) = \begin{cases} \min\{\psi_0^{\varepsilon,b}(x,t), \psi_1^{\varepsilon,b}(x,t)\} & \text{if } \eta(d(x,t)) + K_1 t \leq -b/2, \\ \psi_0^{\varepsilon,b}(x,t) & \text{if } \eta(d(x,t)) + K_1 t > -b/2. \end{cases}$$

Since $q(\sigma) \to \pm\pi$ exponentially fast as $\sigma \to \pm\infty$, we observe that

$$\psi_0^{\varepsilon,b} < \psi_1^{\varepsilon,b} \text{ on } \{(y,s);\ \eta(d(y,s)) + K_1 s \in [-3b/4, -b/4]\}$$

for sufficiently small b and ε. Then we observe that v is a viscosity supersolution of (1.1). Moreover we observe that

$$v(x,t) > 3\pi \quad \text{for} \quad (x,t) \in \{(y,s);\ \eta(d(y,s)) + K_1 s > b/4\},$$
$$v(x,t) < -\pi + \varepsilon\tilde{C} \quad \text{for} \quad (x,t) \in \{(y,s);\ \eta(d(y,s)) + K_1 s < -5b/4\},$$

where $\tilde{C}$ is a positive constant.

References

[AC] S. Allen and J. Cahn, A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsing, Acta Metall., 27(1979), 1084–1095

[AW] D. G. Aronson and H. F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, Adv. in Math., 30(1978), 33–76

[BSS] G. Barles, H. M. Soner and P. E. Souganidis, Front propagation and phase field theory, SIAM J. Cont. Opt., 31(1993), 439–469

[BS] G. Barles and P. E. Souganidis, A new approach to front propagation problems: Theory and applications, Arch. Rational Mech. Anal., 141(1998), 237–296

[BD] G. Barles and F. Da Lio, A geometrical approach to front propagation problems in bounded domains with Neumann-type boundary conditions, Interfaces and Free Boundaries, 5(2003), 239–274

[C] X. Chen, Generation and propagation of interface in reaction-diffusion equations, J. Differential Equations, 96(1992), 116–141

[CGG] Y.-G. Chen, Y. Giga and S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, J. Differential Geometry, 33(1991), 749–786

[CIL] M. G. Crandall, H. Ishii and P. L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc., 27(1992), 1–67

[ES] L. C. Evans and J. Spruck, Motion of level sets by mean curvature, I, J. Differential Geometry, 33(1991), 635–681

[ElPS] C. M. Elliott, M. Paolini and R. Schätzle, Interface estimates for the fully anisotropic Allen-Cahn equation and anisotropic mean-curvature flow, Math. Models Methods. Appl. Sci., 6(1996), 1103–1118

[ElS1] C. M. Elliott and R. Schätzle, The limit of the anisotropic double-obstacle Allen-Cahn equation, Proc. Royal Soc. Edinburgh, 126A(1996), 1217–1234

[ElS2] C. M. Elliott and R. Schätzle, The limit of the fully anisotropic double-obstacle Allen-Cahn equation in the nonsmooth case, SIAM J. Math. Anal., 28(1997), 274–303

[ESS] L. C. Evans, H. M. Soner and P. E. Souganidis, Phase transitions and generalized motion by mean curvature, Comm. Pure Appl. Math., 45(1992), 1097–1123

[G] Y. Giga, *Surface Evolution Equations – A Level Set Approach*, Birkhäuser, 2006.

[GGIS] Y. Giga, S. Goto, H. Ishii and M.-H. Sato, Comparison principle and convexity perserving properties for singular degenerate parabolic equations on unbounded domains, Indiana Univ. Math. J., 40(1991), 443–470

[GOS] Y. Giga, T. Ohtsuka and R. Schätzle, On a uniform approximation of motion by anisotropic curvature by the Allen–Cahn equations, Interfaces and Free Boundaries, 8(2006), 317–348

[J] R. L. Jerrard, Singular limits of scalar Ginzburg–Landau equations with multiple-well potentials, Advances in Differential Equations, 2(1997), 1–38

[KKR] M. A. Katsoulakis, G. Kossioris and Reitich, Generalized motion by mean curvature with Neumann conditions and the Allen–Cahn model for phase transitions, J. Geom. Anal., 5(1995), 255–279

[O] T. Ohtsuka, Motion of interfaces by the Allen–Cahn type equation with multiple-well potentials, Preprint Series, Graduate School of Mathematical Sciences, The University of Tokyo, 2006-8, 2006.

[RSK] J. Rubinstein, P. Sternberg and J. B. Keller, Fast reaction, slow diffusion, and curve shortening, SIAM J. Appl. Math., 49(1989), 116–133