Positive solutions for semilinear elliptic equations involving Dirac measures

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We are concerned with the problem of finding positive solutions with prescribed isolated singularities to semilinear elliptic equations. Choosing a finite set of points \( \{a_i\}_{i=1}^{m} \) in \( \mathbb{R}^{N} \) and a set of positive numbers \( \{c_i\}_{i=1}^{m} \), we consider the existence of positive solutions of the problem

\[
-\Delta u + u = u^p + \kappa \sum_{i=1}^{m} c_i \delta_{a_i} \quad \text{in } \mathcal{D}'(\mathbb{R}^{N}),
\]

with the condition at infinity

\[
u(x) \to 0 \quad \text{as } |x| \to \infty,
\]

where \( N \geq 3, 1 < p < N/(N-2) \), \( \kappa \geq 0 \) is a parameter, and \( \delta_a \) is the Dirac delta function supported at \( a \in \mathbb{R}^{N} \). We denote the Laplacian on \( \mathbb{R}^{N} \) by \( \Delta \) and the class of distributions on \( \mathbb{R}^{N} \) by \( \mathcal{D}'(\mathbb{R}^{N}) \).

We recall some known results concerning the singularities of possible solutions of the equation. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^{N} \) containing 0. By the works due to Lions [14] and Brezis and Lions [6], we obtain the following result.

**Theorem A** [14, 6]. Assume that \( u \in C^2(\Omega \setminus \{0\}) \) satisfies

\[
-\Delta u + u = u^q \quad \text{in } \Omega \setminus \{0\}
\]

with \( q > 1 \) and \( u \geq 0 \) a.e. in \( \Omega \). Then \( u \in L^q_{\text{loc}}(\Omega) \) and

\[
-\Delta u + u = u^q + \kappa \delta_0 \quad \text{in } \mathcal{D}'(\Omega)
\]

for some \( \kappa \geq 0 \). Furthermore, the following (i) and (ii) hold.

(i) In the case \( 1 < q < N/(N-2) \), if \( \kappa = 0 \) in (1.4) then \( u \in C^2(\Omega) \), and if \( \kappa > 0 \) then \( u \) behaves like a multiple of the fundamental solution \( E_0 \) for \( -\Delta \) in \( \mathbb{R}^{N} \), i.e.,

\[
-\Delta E_0 = \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^{N}).
\]
(ii) In the case \( q \geq N/(N - 2) \), there holds \( \kappa = 0 \) in (1.4).

For the proof, see Theorem 1 in [6] and Corollary 1, Theorem 2, and Remark 2 in [14].

It should be mentioned that Johnson, Pan, and Yi [13] showed the existence and asymptotic behaviour of singular positive radial solution \( u \) of (1.3) with \( 1 < q < (N + 2)/(N - 2) \). In particular, they showed that, if \( N/(N - 2) < q < (N + 2)/(N - 2) \), there exists a positive solution \( u \) of (1.3) satisfying \( u(x) \sim c|x|^{-2/(p-1)} \) as \( |x| \to 0 \) for some constant \( c > 0 \).

Then, in this case, the singularity of \( u \) at \( x = 0 \) exists, but is not visible in the sense of distribution.

In this paper, we investigate the existence of positive solutions with prescribed isolated singularities to the equation in \( \mathbb{R}^N \). By (ii) of Theorem A, if \( p \geq N/(N - 2) \) then (1.1) with \( \kappa > 0 \) has no positive solution \( u \in C^2(\mathbb{R}^N \setminus \{a_i\}_{i=1}^m) \). Hence, the condition \( 1 < p < N/(N - 2) \) is necessary for the existence of positive solutions \( u \in C^2(\mathbb{R}^N \setminus \{a_i\}_{i=1}^m) \) of (1.1) with \( \kappa > 0 \).

We review some known results concerning related problems. Lions [14] studied the existence of positive solutions of the problem

\[
\begin{cases}
-\Delta u = u^p + \kappa \delta_0 & \text{in } \mathcal{D}'(\Omega), \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]  

(1.5)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) containing 0 with smooth boundary \( \partial \Omega \). It was shown in [14] that there exists \( \kappa^* > 0 \) such that (1.5) has at least two positive solutions for each \( \kappa \in (0, \kappa^*) \) and no such solution for \( \kappa > \kappa^* \). Later, Baras and Pierre [4] studied the existence of positive solutions for the problem

\[
\begin{cases}
-\Delta u = u^p + \kappa \mu & \text{in } \mathcal{D}'(\Omega), \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]  

(1.6)

where \( \mu \) is a positive bounded Radon measure in \( \Omega \). In [4] they showed that (1.6) has at least one positive solution for each sufficiently small \( \kappa > 0 \) by investigating the corresponding integral equations. See also Roppongi [16]. Amann and Quittner [3] exhibited the existence of \( \kappa^* > 0 \) such that (1.6) has at least two positive solutions for \( 0 < \kappa < \kappa^* \) and no solution for \( \kappa > \kappa^* \). Bidaut-Veron and Yarur [5] gave the existence results and a priori estimates for (1.6) including the case where \( \mu \) is unbounded. In [3], [5], they also consider the problems involving measures as boundary data. We also refer a survey by Veron [19], [20], and the references therein. In [17] the second author studied the existence of positive solutions for the problem

\[ -\Delta u + f(u) = \sum_{i=1}^m c_i \delta_{a_i} \text{ in } \mathcal{D}'(\mathbb{R}^N) \]
in the cases where $f$ is nonnegative. In [17] he also showed the nonexistence of positive solutions for some $f$ with sign changing.

Concerning nonhomogeneous semilinear elliptic problems of the form

$$-\Delta u + u = u^q + \kappa f(x) \quad \text{in} \quad \mathbb{R}^N$$

with $q > 1$ and $f \in H^{-1}(\mathbb{R}^N)$, we refer to Zhu [21], Deng and Li [10], [11], Cao and Zhou [7], and Hirano [12]. They successfully showed the existence of at least two positive solutions of the problems under suitable conditions. See also [18, 8] for closely related problems.

In order to state our results, we introduce some notations. Let $E_1$ denote the fundamental solution for $-\Delta + I$ in $\mathbb{R}^N$, that is,

$$E_1(x) = E_1(|x|) = \frac{1}{(2\pi)^{N/2}|x|^{(N-2)/2}}K_{(N-2)/2}(|x|) \quad \text{for} \quad x \in \mathbb{R}^N \setminus \{0\},$$

where $K_{\nu}$ is the modified Bessel function of order $\nu$. We see that $E_1$ has the following properties:

$$E_1(x) \sim \frac{1}{(N-2)N\omega_N|x|^{N-2}} \quad \text{as} \quad |x| \to 0, \quad \text{and}$$

$$E_1(x) \sim c_1|x|^{-(N-1)/2}e^{-|x|} \quad \text{as} \quad |x| \to \infty,$$

where $\omega_N$ denotes the volume of the unit ball in $\mathbb{R}^N$ and $c_1 > 0$ is a constant depends on $N$. In particular, $E_1 \in C^\infty(\mathbb{R}^N \setminus \{0\})$ and $E_1 \in L^r(\mathbb{R}^N)$ for all $1 \leq r < N/(N-2)$. Define $f_0$ by

$$f_0(x) = \sum_{i=1}^m c_i E_1(x-a_i).$$

Then $f_0 \in C^\infty(\mathbb{R}^N \setminus \{a_i\}_{i=1}^m)$ and $f_0 \in L^r(\mathbb{R}^N)$ for all $1 \leq r < N/(N-2)$, and $f_0$ satisfies

$$-\Delta f_0 + f_0 = \sum_{i=1}^m c_i \delta_{a_i} \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N).$$

In this paper we refer to $u$ as a positive solution of (1.1) if $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ satisfies (1.1) in the sense of distribution and $u > 0$ a.e. in $\mathbb{R}^N$.

**Proposition 1.1.** Let $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ be a positive solution of (1.1) with $\kappa > 0$. Then $u \in C^2(\mathbb{R}^N \setminus \{a_i\}_{i=1}^m)$ and $u(x) > 0$ for $x \in \mathbb{R}^N \setminus \{a_i\}_{i=1}^m$. Assume, in addition, that (1.2) holds. Then $u \in L^q(\mathbb{R}^N)$ for all $q \in [1, N/(N-2))$ and $u$ satisfies

$$u = E_1 * [u^p] + \kappa f_0 \quad \text{a.e. in} \quad \mathbb{R}^N \quad (1.7)$$

and $u(x) = O(E_1(x))$ as $|x| \to \infty$, where the symbol $*$ denotes the convolution.

For each $\kappa > 0$, we define $U_j^\kappa$ for $j = 0, 1, 2, \ldots$, inductively, by

$$U_0^\kappa = \kappa f_0 \quad \text{and} \quad U_j^\kappa = E_1 * [(U_{j-1}^\kappa)^p] + \kappa f_0 \quad \text{for} \quad j = 1, 2, \ldots \quad (1.8).$$
Take \( q_0 \in (p, N/(N - 2)) \) arbitrarily, and define \( \{q_j\} \) by
\[
\frac{1}{q_j} = \frac{1}{q_0} - \left( \frac{2}{N} - \frac{p - 1}{q_0} \right) j = \frac{1}{q_{j-1}} - \left( \frac{2}{N} - \frac{p - 1}{q_0} \right) \quad \text{for } j = 1, 2, \ldots .
\]
(1.9)

From \( p < N/(N - 2) \) and \( q_0 > p \), it follows that \( 2/N - (p - 1)/q_0 > 0 \). Then, by choosing suitable \( q_0 \) if necessary, there exists a positive integer denoted by \( j_0 \) satisfying
\[
\frac{1}{q_{j_0 - 1}} > 0 > \frac{1}{q_{j_0}}.
\]
(1.10)

We use the notation \( C_0(R^N) = \{ u \in C(R^N) : u(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty \} \).

**Proposition 1.2.** For each \( \kappa \in (0, \infty) \), the following (i) - (iii) are equivalent to each other:

(i) \( u = w + U_{j_0}^\kappa \in L_{1oc}^p(R^N) \) is a positive solution of \( (1.1)_\kappa - (1.2) \);

(ii) \( w \in C_0(R^N) \) is positive in \( R^N \) and satisfies
\[
w = E_1 * [(w + U_{j_0}^\kappa)^p - (U_{j_0 - 1}^\kappa)^p] \quad \text{in } R^N; \tag{1.11}_\kappa
\]

(iii) \( w \in H^1(R^N) \) is a weak positive solution of
\[
-\Delta w + w = (w + U_{j_0}^\kappa)^p - (U_{j_0 - 1}^\kappa)^p \quad \text{in } R^N, \tag{1.12}_\kappa
\]

that is, \( w > 0 \text{ a.e. in } R^N \) and satisfies
\[
\int_{R^N} (\nabla w \cdot \nabla \psi + w \psi) \, dx = \int_{R^N} [(w + U_{j_0}^\kappa)^p - (U_{j_0 - 1}^\kappa)^p] \psi \, dx \quad \text{(1.13)}_\kappa
\]
for any \( \psi \in H^1(R^N) \).

By Proposition 1.2, the problem \( (1.1)_\kappa - (1.2) \) can be reduced to the problems \( (1.11)_\kappa \) in \( C_0(R^N) \) and \( (1.12)_\kappa \) in \( H^1(R^N) \). We will investigate the problems \( (1.11)_\kappa \) and \( (1.12)_\kappa \) by an approach based on adaptation of the methods by [1, 2, 9, 14].

Our main results are stated in the following theorems.

**Theorem 1.** There exists \( \kappa^* \in (0, \infty) \) such that

(i) if \( 0 < \kappa < \kappa^* \) then the problem \( (1.1)_\kappa - (1.2) \) has a positive minimal solution \( u_\kappa \), that is, \( u_\kappa \leq u \text{ a.e. in } R^N \) for any positive solution \( u \) of \( (1.1)_\kappa - (1.2) \). Furthermore, if \( 0 < \kappa < \kappa < \kappa^* \) then \( u_\kappa < u_\kappa \text{ a.e. in } R^N \);

(ii) if \( \kappa > \kappa^* \) then the problem \( (1.1)_\kappa - (1.2) \) has no positive solution.
**Theorem 2.** If $\kappa = \kappa^*$ then the problem \((1.1)_\kappa-(1.2)\) has a unique positive solution.

**Theorem 3.** If $0 < \kappa < \kappa^*$ then the problem \((1.1)_\kappa-(1.2)\) has a positive solution $\varpi_\kappa$ satisfying $\overline{u}_\kappa > \underline{u}_\kappa$.

Proofs of Theorems 1-3 can be found in [15]. In the proof of Theorem 1, we will employ the bifurcation results and the comparison argument for solutions of \((1.12)_\kappa\) and \((1.11)_\kappa\), respectively, to obtain the minimal solutions. We will prove Theorem 2 by establishing a priori bound for the solutions of \((1.12)_\kappa\). We will prove Theorem 3 by employing the variational method with the Mountain Pass Lemma. In the proofs of Theorems 2 and 3, the results concerning the eigenvalue problems to the linearized equations around the minimal solutions play a crucial role.

**REFERENCES**


