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<th>On asymptotics of a second order linear O.D.E with a turning-regular singular point (Functional Equations Based upon Phenomena)</th>
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Kyoto University
§1. Introduction.

1.1. The differential equation studied is

\begin{equation}
\varepsilon^2 \frac{d^2 y}{dx^2} - \left( x^m - \frac{\varepsilon}{x} \right) y = 0,
\end{equation}

where \( x_0 \) and \( \varepsilon_0 \) are constants. This differential equation has a turning point and a regular singular point, both of which are situated at the origin. We do not have a one-step-method to obtain an asymptotic approximation to the solution as \( \varepsilon \to 0 \) in the whole domain \( D = \{ x : 0 < |x| \leq x_0 \} \), so we split (1.1) into two different types of the differential equation whose solutions are obtained separately (§2) and then we connect them by a so-called matching matrix in a common domain as shown in §4.

1.2. The differential equation (1.1) is represented in the matrix form:

\begin{equation}
\varepsilon \frac{dY}{dx} = \begin{bmatrix} 0 & 1 \\ x^m - \varepsilon/x & 0 \end{bmatrix} Y,
\end{equation}

where \( Y \) is a 2-by-2 matrix. (1.2) has the first two terms of

\begin{equation}
\varepsilon \frac{dY}{dx} = \left\{ \begin{bmatrix} 0 & 1 \\ x^m & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 \\ -1/x & 0 \end{bmatrix} + O(\varepsilon^2) \right\} Y.
\end{equation}

If \( O(\varepsilon^2) \) is small for \( x \in D \) and \( \varepsilon \), then a solution of (1.3) is a regular perturbation of one of (1.2) with respect to a small \( \varepsilon \). In this sense (1.2) is dominant to (1.3)

Our aim is to get two types of the formal solution of (1.1) and match them as \( \varepsilon \to 0 \). In order to do it, analyzing Stokes curve configuration is important (§3). The case of \( m = 1 \)
has been studied in Nakano [5].

**Remark:** We do not show any proofs or illustrations as they would take many pages.

§2. The reduced equations.

2.1. The differential equation (1.2) is written in the form

\[ x^{(m+1)/2}(x^{-m-1}\varepsilon)\frac{dZ}{dx} = \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (x^{-m-1}\varepsilon) \begin{bmatrix} 0 & 0 \\ -1 & -mx^{m/2}/2 \end{bmatrix} \right) Z, \]

where \( Y := \text{diag}[1, x^{m/2}] Z \). This differential equation is called an outer equation of (1.2) and it should be analyzed when \( x^{-m-1}\varepsilon \to 0 \), that is, for \( x \) in a sub-domain \( S := \{ x : K\varepsilon^{1/(m+1)} \leq |x| \leq x_0 \} \) \((K = \text{large constant})\) of the whole domain \( D \). A solution of (2.1) is called an outer solution of (1.2).

**Theorem 2.1.** The formal outer solution \( \tilde{Y}_{out} \) of (1.2) is given by

\[ \tilde{Y}_{out} := x^{m/4} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} e^{\frac{1}{\varepsilon} \alpha}, \]

\[ \alpha := \frac{2}{m+2\varepsilon} x^{(m+2)/2} + \frac{1}{m} \frac{1}{x^{m/2}}, \]

or

\[ \tilde{Y}_{out} := \begin{bmatrix} x^{-m/4} & 0 \\ 0 & x^{m/4} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{bmatrix}, \]

which is the leading term of an asymptotic expansion of a true outer solution of (1.2), namely, there exists a true outer solution \( Y_{out} \) such that

\[ Y_{out} \sim \tilde{Y}_{out} \quad (x^{-m-1}\varepsilon \to 0) \]

in an outer domain, i.e., in a sector

\[ S_m := \{ x : K\varepsilon^{1/(m+1)} \leq |x| \leq x_0, \quad -\frac{\pi}{m+2} < \arg x < \frac{3\pi}{m+2} \}. \]

Notice that the arguments of \( x \) in the above sector \( S_m \) correspond to the arguments of the boundaries of a canonical domain \( C_m^\infty \) (cf. (2.10)). \( \tilde{Y}_{out} \) is an outer WKB approximation.
to the solution of (1.2) of a matrix form.

2.2. We reduce (1.2) to another form in the complement $C := \{ x : 0 < |x| < K \varepsilon^{1/(m+1)} \}$ of the sub-domain $S$, i.e., $D = C \cup S$. Let $x := \varepsilon^{1/(m+1)} t$ (a stretching transform) and $Y := \text{diag}[1, \varepsilon^{m/2(m+1)}] U$, then (1.2) becomes a form such as

\[
\varepsilon^{m/2(m+1)} \frac{dU}{dt} = \begin{bmatrix} 0 & 1 \\ p(t) & 0 \end{bmatrix} U \quad \left( p(t) := t^m - \frac{1}{t} \right),
\]

which has a very similar form to (1.2) but lacks a term of $\varepsilon$ and is called an inner equation of (1.2). The origin $t = 0$ is a regular singular point and zeros of $p(t)$ are turning points of (2.6), which are called secondary turning points of (1.2). A solution of (2.6) is called an inner solution of (1.2).

**Theorem 2.2.** The formal inner solution $\tilde{Y}_{in}$ of (1.2) is given by

\[
\tilde{Y}_{in} := \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{m/2(m+1)} \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & p^{1/4} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} e^\beta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\]

\[
\beta := \frac{1}{\varepsilon^{m/2(m+1)}} \int^t \sqrt{p} \, dt,
\]

or

\[
\tilde{Y}_{in} := \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{m/2(m+1)} \end{bmatrix} \begin{bmatrix} p^{-1/4} & 0 \\ 0 & p^{1/4} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^\beta & 0 \\ 0 & e^{-\beta} \end{bmatrix},
\]

which is the leading term of the asymptotic expansion of a true inner solution of (1.2), namely, there exists a true inner solution $Y_{in}$ of (1.2) such that

\[
Y_{in} \sim \tilde{Y}_{in} \quad \text{as} \quad \begin{cases} \varepsilon \to 0 \\ t \to \infty \end{cases}
\]

in a canonical domain

\[
\mathbb{C}_m^\infty := \left\{ t : 0 < |t| < \infty, -\frac{\pi}{m+2} < \arg t < \frac{3\pi}{m+2} \text{ near } t = \infty \right\}.
\]

$\tilde{Y}_{in}$ is an inner WKB approximation to the solution of (1.2) of a matrix form. The property (2.9) is called the double asymptotic property (Fedoryuk [2]).
§3. Stokes curves and the canonical domains.

3.1. A Stokes curve for (2.6) is, by definition, a set of points $t$'s given by

\[(3.1) \quad \{t : \Re \xi(a, t) = 0\},\]

where

\[(3.2) \quad \xi(a, t) := \int_a^t \sqrt{p} \, dt \quad (p(a) = 0).\]

An anti-Stokes curve of (2.6) is defined by an equation

\[(3.3) \quad \Im \xi(a, t) = 0 \quad (p(a) = 0).\]

These curves are particular level curves defined by $\Re \xi(a, t) =$ const. and $\Im \xi(a, t) =$ const., namely, they are the curves of level zero.

The global property of Stokes curve configuration for a general rational function $p(t)$ is well known in Evgrafov-Fedoryuk [1], Fedoryuk [2] and Nakano [6]-[7], and Fukuhara [3], Hukuhara [4] and Paris-Wood [8] for a local property of Stokes curves. The outline of the Stokes curve configuration for (2.6) is as follows:

**Theorem 3.1.** The Stokes and anti-Stokes curves for (2.6) possess the following properties:

(i) The origin $t = 0$ is a regular singular point from which one Stokes curve and one anti-Stokes curve emerge.

When $m =$ odd, two lines $t < -1, 0 < t < 1$ on the real axis are Stokes curves, and two lines $-1 < t < 0, 1 < t$ are anti-Stokes curves.

When $m =$ even, a line $0 < t < 1$ on the real axis is a Stokes curve and two lines $t < 0, 1 < t$ on the real axis are anti-Stokes curves.

(ii) The point at infinity $t = \infty$ is an irregular singular point and $m+3$ Stokes curves emerge from (or tend to) $t = \infty$ at angles $\pm \frac{\pi}{m+2}, \pm \frac{3\pi}{m+2}, \pm \frac{5\pi}{m+2}, \ldots$.

Also, $m+3$ anti-Stokes curves emerge from (or tend to) $t = \infty$ at middle angles between neighboring two Stokes curves.

(iii) All the zero $t = e^{2k\pi i/(m+1)}$ ($k = 0, 1, 2, 3, \ldots$) of $p(t)$ are situated on the unit circle $|t| = 1$ symmetrically with respect to the real axis and they are simple secondary
turning points. From a turning point \( t = e^{2k\pi i/(m+1)} \) three Stokes curves emerge at angles
\[
\pm \frac{\pi}{3} + \frac{4k\pi}{3(m+1)}, \quad \pi + \frac{4k\pi}{3(m+1)}.
\]
Three anti-Stokes curves emerge from every zero at middle angles between neighboring two Stokes curves.

(iv) There is a Stokes curve connecting \( \alpha := e^{2k\pi i/(m+1)} \) and \( \alpha^* := e^{2\pi i - 2k\pi t/(m+1)} \). This Stokes curve crosses the anti-Stokes curve \(-1 < t < 0\) and can not cross lines \( t < -1 \) or \( 0 < t < 1 \).

(v) There is an anti-Stokes curve connecting \( \alpha := e^{2k\pi i/(m+1)} \) and \( \bar{\alpha} := e^{-2k\pi i/(m+1)} \). This anti-Stokes curve crosses only the Stokes curve \( 0 < t < 1 \).

(vi) Any Stokes curve (resp., any anti-Stokes curve) can not cross other Stokes curves (resp., anti-Stokes curves) except for at turning points or at \( t = \infty \).

(vii) A Stokes curve and an anti-Stokes curve emerging from a turning point tend to another turning point or to \( t = \infty \).

(viii) Any Stokes curve or any anti-Stokes curve can not cross itself.

(ix) When a point \( t = \alpha \) is a turning point or a simple pole, there are no (sums of) Stokes or anti-Stokes curves homotopic to a circle surrounding \( \alpha \). Therefor there are no circle-like Stokes or anti-Stokes curves for (6.1).

3.2. A canonical domain on the \( t \)-plane (or the Riemann surface) is, by definition, a simply connected domain bounded by Stokes curves which is mapped by \( \xi = \xi(a, t) \) onto the whole \( \xi \)-plane except several slits. Refering Theorem 3.1 we can get several canonical domains whose illustration is omitted here.


Existence domains \( S_m \) and \( C_m^\infty \) of the outer and the inner solutions have a common part where two solutions relate linearly. This linear relation is represented by a so-called matching matrix. The matrcing matrix \( M := [m_{ij}] \) between \( Y_{out} \) and \( Y_{in} \) is defined by the equality \( Y_{out}M = Y_{in} \); i.e.,

\[
(4.1) \quad \tilde{Y}_{out}M \sim \tilde{Y}_{in} \quad (\varepsilon \to 0).
\]
Theorem 4.1. The matching matrix defined by (4.1) is given by

\[
M \sim \varepsilon^{m/4(m+1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (\varepsilon \to 0).
\]

§5. The main theorem.

Summing up the results so far, we can get

The main theorem. The differential equation (1.1) (or (1.2)) possesses a formal outer solution (an outer WKB approximation) (2.2) (or (2.3)) which is an asymptotic expansion of the true outer solution in a sector (i.e., an outer domain) (2.5) as \( x^{-m-1} \varepsilon \to 0 \). The differential equation (1.1) possesses a formal inner solution (an inner WKB approximation) (2.7) (or (2.8)) which is an asymptotic expansion of the true inner solution in a canonical domain (i.e., an inner domain) as \( \varepsilon \to 0 \) or \( t \to \infty \). The arguments of the outer domain's boundaries are \( -\pi/(m+2) \) and \( 3\pi/(m+2) \), and those of the inner domain's boundaries are identical for a large \( t \), and two domains have a common part in which the outer and the inner solutions are related by the matching matrix (4.2).

References

[1] Evgrafov, M. A. and M. V. Fedoryuk, Asymptotic behavior as \( \lambda \to \infty \) of solutions of the equation \( w''(z) - p(z, \lambda)w(z) = 0 \) in the complex \( z \)-plane. Uspehi Mat. Nauk 21, or Russian Math. Surveys 21 (1966), 1-48.


