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Topics from Competitive Game Theory

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Some of the recent works by the present author are given in the two parts A and B. A lot of interesting open problems are mentioned.

[A] 2- AND 3- PLAYER GAMES OF SCORE SHOWDOWN

1. 3-Player Games of Score Showdown. Let $X_{ij}(i=1,2,3; j=1,2)$ be the j-th r.v. observed by Player i. Assume that $\{X_{ij}\}$ are i.i.d. with $U[0,1]$ distribution. Each player i first observes $x_{i1}=x_{i2}$ and chooses $A/R$. If $A(R)$ is chosen, the $x_{i1}$ is accepted (rejected and the second r.v $X_{i,2}$ is observed). Player i’s score is

$S_i(x_{i1}, x_{i2}) = \{
\begin{cases}
x_{i1}, & \text{if } x_{i1} = x_{i2} \\
0, & \text{if } x_{i1} = x_{i2} \text{, rejected}
\end{cases}
\}$

We consider the cases

$\{x_{i1}, x_{i2}\} = \{x_{i2}, x_{i1}\}$

(Keep-or-Exchange, Risky-Exchange, Showcase-Showdown, Competing Average, resp)

Player who gets the highest score is the winner. Each player wants $\Pr(\text{he wins}) \rightarrow \max$

Also we consider the two versions of the games;

Simultaneous-move version —— Players’ are made indep. and are not known to his rivals.

Sequential-move version —— Players move sequentially, i.e., I at first, II at second, and III at third. After I’s move, II and III are informed of $X_{i1} = x_i$ and I’s choice of $A/R$. After II’s move, III is informed of $X_{i2} = x_2$ and II’s choice of $A/R$.

All players are intelligent, and each player should prepare for that any subsequent player must employ their optimal strategies.

In the present article, we use $1^\circ = \text{game}, C=\text{common}, EQ=\text{equilibrium}, S=\text{strategy},

V=\text{value}, D=\text{draw}$. Players 1, 2, 3 are sometimes written by I, II, III, resp.

2. Solutions to the Games. It is shown that games of $1^\circ SS$ and $1^\circ RE$ have the same solution $\{REF(3)\}$. Some other results are given below (Ref[1,2,3])
solution to \( \Gamma^{(3)} \) KE (simult.-move): CEQS is “Choose A(R) if his first observ. is \( \{<\} \)
\[ a^* = 0.618 \] i.e., a unique root in \( \langle 0,1 \rangle \) of the equation \( a^2 + a = 1 \).”
CEQV = \( \frac{1}{2} \). In \( \Gamma^{(3)} \) KE, the \( a^* \) the equation and CEQV change to \( a^* = 0.691, \quad 2a^2 = -a - a^3 \) and \( \frac{1}{3} \) resp.

Solution to \( \Gamma^{(3)} \) RE (simult.-move): CEQS is “Choose A(R), if his first observ. is \( \{<\} \)
\[ a^* = 0.544 \] i.e., a unique root in \( \langle 0,1 \rangle \) of the equation \( a^2 + a^4a = 1 \).” \( P(D) = \frac{1}{4} a^* \) \( \approx \) 0.022.
\[ P(W_1) = P(W_2) = \frac{1}{2}(1-P(D)) = 0.489. \]
In \( \Gamma^{(3)} \) RE, \( a^* \) the equation and the other result change to \( a^* = 0.656, \) \( 2a^2 + a^4a = -a^7 - a^3 \), \( P(D) = \frac{1}{8} a^* \) \( \approx \) 0.010, \( P(W_1) = P(W_2) = \frac{1}{2}(1-P(D)) \) \( \approx \) 0.330.

Solution to \( \Gamma^{(3)} \) RE (seq.-move): I’s opt. str. is “Choose A (R) \( X_{11} = x_1 \) if \( x_1 > \{<\} x_0 \)
where \( x_0 (\approx 0.570) \) is a unique root in \( \langle \frac{1}{2},1 \rangle \) of \( \tilde{x}^2 = 2x(1-3\tilde{x}) \)” II’s opt. str. is

“Choose A(R) \( X_{21} = x_2 \) if \( x_2 > \{<\} x_0 \) \( \tilde{x}^0 \) state \( \{ (x_2 | x_1, A) \}
\[ y_0(x_0) = (\frac{1}{2} - \frac{1}{2}) \tilde{x} \leq \{ \frac{3}{2} \} + (\frac{1}{2} \tilde{x} \geq \{ \frac{3}{2} \}) \tilde{x} \geq -1 \}. \] We obtain \( P(D) \) \( P(W_1) \)
\[ P(W_2) = 0.011, 0.477, 0.512, \text{resp.} \]

Solution to \( \Gamma^{(3)} \) CA (seq.-move): I’s opt. str. is “Choose A(R) \( X_{11} = x_{1i} \) if \( x_i > \{<\} x_{ij} \) where \( x_{ij} (\approx 0.549) \) is a unique root in \( \langle \frac{1}{2},1 \rangle \) of
\[ y_0(x_1) = \left\{ \begin{array}{ll}
\tilde{x}^0 \leq \{ \frac{3}{2} \} & \text{if } x_1 \geq \{ \frac{3}{2} \}
\tilde{x}^0 \geq \{ \frac{3}{2} \} & \text{if } x_1 < \{ \frac{3}{2} \}
\end{array} \right. \] We obtain \( P(W_1) = 1 - P(W_2) \) \( \approx \) 0.490.

Solution to \( \Gamma^{(3)} \) KE (seq.-move): It is too complicated to write here. We found
\[ P(W_1), P(W_2), P(W_1) = 0.3329, 0.3329, 0.33421, \text{resp.} \]

3. Information Types in 2-Player Games. We consider the two information types,
under which players decide their choices of A/R.

- \( \Gamma^{(1)} \) means that I chooses \( X_{11} = x_1 \), II chooses \( X_{21} = x_2 \) and each player informs
of his observed value to his opponent.

- \( \Gamma^{(2)} \) means that I observes \( X_{11} = x_1 \), II observes \( X_{21} = x_2 \) and I informs his \( x_1 \)
to II, and II doesn’t inform his \( x_2 \) to I.

It is clear that \( \Gamma^{(2)} \) is the information type discussed in Section 3.

Some results are given as follows (Ref.[4]).

Solution to \( \Gamma^{(2)} \) RE under \( \Gamma^{(1)} \); In state \( x_1 \neq x_2 \), there exists a unique
saddle point at R-A, R-R, A-R, if \( x_1 \vee x_2 \vee (x_3 - 1) = x_2, x_2 - 1, x_1 \), resp.
We obtain \( P(D) = \frac{1}{4} (\frac{1}{2} - 1) = 0.007 \), \( P(W_1) = P(W_2) = \frac{1}{4} (1 - P(D)) = 0.496 \).

- Solution to \( \Gamma^q \)KE under \( I^{(\eta-1)} \)
  - I's opt. str. is "Choose A (R) \( X_{i1} = x_i \),
  - if \( x_i > (\gamma) \quad a^* = \sqrt{3/2} \approx 0.6124".
  - II's opt. str. in state \( x_i \) is given by

<table>
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<th>II's opt. Choice</th>
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<td>( x_i &lt; a^* )</td>
<td>( \Lambda(R), ) if ( x_2 &gt; (\gamma) b_2 )</td>
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<tr>
<td>( a^* &lt; x_i &lt; x_2 )</td>
<td>( A )</td>
</tr>
<tr>
<td>( x_i &gt; a^* \lor x_2 )</td>
<td>( R )</td>
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- We obtain \( P(W_1) = \frac{1}{4} \), \( P(W_2) = \frac{1}{4} a^* \approx 0.4964 \).

4. 2-Player game of Continue-or-Stop. Player i observes \( (X_{i1}, X_{i2}, \ldots, X_{in}) \), sequentially one-by-one, and facing each \( X_{ij} \) chooses Cont/Stop. If Stop is chosen \( X_{ij} \) is accepted by player i and his play ends. If Cont. is chosen, \( X_{ij} \) is rejected and the next \( X_{ij+1} \) is observed and the game continues. Assume that \( \{X_{ij}\} \) are i.i.d. with \( U_{[0,1]} \) distribution.

Score is

\[
S_i(X_{i1}, \ldots, X_{in}) = \begin{cases} \{X_{i1} \} & \text{stop at } X_{i1} \\ \{X_{i1}, X_{i2}, \ldots, X_{in} \} & \text{rejected and stop at } X_{i1} \\
\end{cases}
\]

Consider the situation where player i has \( n-1 \) decision thresholds \( (l >) a_{i1} > a_{i2} > \ldots > a_{in-1} \) \( (\geq 0) \), so that \( \gamma \) chooses Stop (Cont.) if \( X_{ij} > (\gamma) a_{ij} \). Player who gets the higher score than his opponent is the winner. Each player aims \( \text{Pr} \) (he wins) \( \rightarrow \) max.

- Solution when \( n = 3 \) (Ref[5]): The common optimal thresholds are \( (b_{i1}^0, b_{i2}^0) = (0.74, 3, 0.657) \), where \( (b_{i1}^0, b_{i2}^0) \) is a unique root in \( (a_1, 1) \)

\[
\begin{align*}
&b_{i2}^2 + b_2 - (b_{i1}^{-1} + b_1) = 0 \\
&b_{i2}^2 + (1 + b_2)^{-1} b_1 - 1 = 0
\end{align*}
\]

- \( P(W_1) = P(W_2) = \frac{1}{2} \).

5. Remark. There are many open problems of interest around the topic—games of score showdown. Three among them are given below.

1. Let \( (X_{ij}, Y_{ij}), i = 1, 2, j = 1, 2 \), are i.i.d. with joint p.d.f.

\[
\{ (X_{ij}, Y_{ij}) = 1 + \gamma \} (1 - 2x)(1 - 2y) \quad \forall (x, y) \in [0, 1]^2,
\]

with \( |Y| \leq 1 \). In the game \( \Gamma^{(2)} \)KE (simult.-move), the score is

\[
S(x_{ij}, y_{i1}, x_{i2}, y_{i2}) = \begin{cases} (X_{i1}, Y_{i1}) & \text{accepted} \\
(X_{i2}, Y_{i2}) & \text{rejected}
\end{cases}
\]

Define that \((x, y)\) is higher than \([\text{lower than, intermediate to}]\ (x', y')\), if \((x, y)\) 
\(\geq [\ <, \ otherwise] \ (x', y')\). Player with score \((x, y)\) gets \(1 [-1, 0] \). Solve this zero-sum game.

(2). Sequential-move games \(\Gamma^{(3)}\)RE and \(\Gamma^{(3)}\)CA remain unsolved.

(3). 3-player games under various information types are of interest, \(\Gamma^{(3)}\)KE, \(\Gamma^{(3)}\)RE and \(\Gamma^{(3)}\)CA, under \(I^{111-111-111}\), \(I^{110-110-110}\), etc. remain unsolved. The last one \(I^{110-011-101}\), for example, means that each player observes his own \(X_{i1} = x_{i1}\) and, in addition, I knows \(x_{21}\), II knows \(x_{31}\) and III knows \(x_{41}\).

REFERENCES


\[\boxed{B}\] MULTISTAGE OPTIMAL STOPPING GAMES

1. Each Player has his Priority. Player I, II, and III observe \(X_1, X_2, \ldots, X_n\) with i.i.d. \(U[0, 1]\) distribution sequentially one-by-one. They have their previously given priorities \(p_1, p_2, p_3\).

Facing \(X_j\), each player chooses \(R / A\), independently of his rivals.

If only one player chooses \(A\), he gets \(X_j\) with his priority, dropping out from the game thereafter, and the remaining two players continue their two-player game with the "revised" priorities. If two players choose \(A\), one player selected according to the "revised" priorities gets \(X_j\), drops out from the game thereafter, and the remaining two players continue their two-player game with the "revised" priorities. If the choices are \(A\)-\(A\)-\(A\) player \(i\) gets \(X_j\) with prob. \(p_i\) dropping out from the game, and the remaining two players continue their two-player game with the revised" priorities. If the choices are \(R\)-\(R\)-\(R\) then \(X_j\) is rejected, the next \(X_{j+1}\) is observed and the subsequent 3-player game continues. Each player aims to maximize the ENV he can get (\(N\) in ENV means net, i.e., no-observation-cost and no-discounting).

Let \(W_n, V_n\) be CEQV, for 3-player (2-player) equal-priority game. Then the Opt. Eq. is

\((W_n, W_n, W_n) = E \left[ \text{eq. val. } M_4(X) \right] \quad (n \gg 1, \ \overline{W}_0 = \overline{V}_0 = \emptyset)\)
\[ M_n(x) = \begin{array}{c}
R \text{ by II} \\
A \text{ by II}
\end{array}
\]

\[
M_n(x) =
\begin{array}{c|c|c}
& R \text{ by II} & A \text{ by II} \\
\hline
W, W, W & V, x, V \\
\hline
x, V, V & \frac{1}{2}(x + V), \frac{1}{2}(x + V), V \\
\hline
\frac{1}{2}(x + V), V & \frac{1}{2}(x + V), \frac{1}{2}(x + V), \\
\hline
\frac{1}{3}(x + 2V), \frac{1}{3}(x + 2V) & \frac{1}{3}(x + 2V), \frac{1}{3}(x + 2V), \\
\end{array}
\]

\[
[W(\forall) \mu \, \mathcal{W}_{n-1}(\mathcal{V}_{n-1})] = E\{\text{val.} \begin{array}{c|c|c}
R & V_{n-1}, V_{n-1} & U_{n-1}, x \\
\hline
A & \frac{1}{2}(x + U_n) & \frac{1}{2}(x + U_{n-1})
\end{array} \}
\]

where \( U_n = \frac{1}{2}(1 + U_{n-1}) \), \( n \geq 1, U_0 = 0 \).

Common opt. str. for each player is derived, and CEQ \( V \) is computed as

\[
U_n = \begin{cases}
\frac{1}{2}, & n \leq 4, \\
0.7417, & 0.8364, & 0.8791, & \text{for } n = 1, 4, 8, 12, \text{ resp.}
\end{cases}
\]

\[
V_n = \begin{cases}
\frac{1}{4}, & 0.6466, & 0.7182, & 0.8361, & \text{ibid.}
\end{cases}
\]

\[
W_n = \begin{cases}
\frac{1}{2}, & 0.5596, & 0.7186, & 0.7936, & \text{ibid.}
\end{cases}
\]

The cases \( \langle p_1, p_2 \rangle = (\frac{1}{2}, \frac{1}{2}) \) and \( \langle p_1, p_2, p_3 \rangle = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) are also solved (Ref. [194]).

Player with low priority stands at disadvantage, since he remains late in the game, and faces less offers.

The cases where player’s aim is Pr. (he gets a r.v. better than opponent) \( \rightarrow \text{max. are solved in Ref. [192].} \)

In the 2-player No-Information case, for example, let us define state \( (i, y) \) to mean that (1) both players remain in the game, and (2) players currently face the r.v. \( Y_i = y \). Let \( V(i, y) \) be the value of the game in state \( (i, y) \), for the \( n \)-problem. Note that \( n \) is fixed throughout, players should choose A-A in state \( (n, y) \) and hence draw of the game cannot occur.

Then the Opt. Eq. in state \( (i, y) \) is

\[
V((i, y)) = \text{val.} \begin{array}{c}
R \left[ \begin{array}{c|c|c}
\mu_{i+1} & 1 - g(i, y) \\
g(i, y) & (p - \bar{p})g(i, y) + \bar{p}
\end{array} \right] \end{array} \begin{array}{c}
A \end{array} \begin{array}{c} 1 \leq i \leq n - 1, V(n, y) = p \in [\frac{1}{2}, 1], \forall y \in [\frac{1}{2}, 1] \end{array}
\]

where \( \mu_{i+1} = \left( \sum_{y=1}^{y=n-1} V(i, y') \right) \), and

\[
g(i, y) = \prod_{j=1}^{n} \left( 1 - \frac{y_j}{i+y} \right) = (\frac{i}{i+y})^{y} = (\frac{i}{y})^{\frac{i}{y}}
\]
The solution is: Opt. str. pair in state \( (s, y) \) is
\[ R-R, \ R-A, \ A-A, \ \text{if } 0 < g(s, y) < \bar{p}_n, \ \text{Fin} < g(s, y) < \frac{1}{2}, \ \frac{1}{2} < g(s, y) < 1, \ \text{resp.} \]

The winning prob. for \( n=10 \) are computed downward in \( i \) until reaching \( M_i = V(1, 1) \). We obtain
\[ P_i(I \ \text{wins}) = 0.578, \ 0.619, \ \text{for } p = \frac{3}{4}, \ 1, \ \text{resp.} \]

3. Committee's Selection

I and II observe \( (X_j, Y_j), j = 1, \cdots, n, \) i.i.d. with \( U_{q_j, x} \)
distribution sequentially one-by-one, and each player chooses \( R/A \). \( X_j, (Y_j) \) is I's
(II's) evaluation of \( j \)-th applicant's ability (Ref.[9]). The Opt. Eq. is

\[ (u_n, v_n) = E[u_{n+1} M_n(X, Y) ; M_n(x, y) = R \begin{pmatrix} u_{n-1}, v_{n-1} \\ u_{n-1} + \bar{p}x, v_{n-1} + \bar{p}y \end{pmatrix} A \begin{pmatrix} u_{n-1} + \bar{p}x, v_{n-1} + \bar{p}y \\ x, y \end{pmatrix} \]

The solution is: In state \( (X = x, Y = y) \),

I chooses \( A(R) \), if \( x > (\triangleleft) v_{n-1} \), indep. of \( y \),

II chooses \( A(R) \), if \( y > (\triangleleft) v_{n-1} \), indep. of \( x \),

where
\[ u_n = \frac{1}{2} \left\{ p u_{n-1} + \bar{p} (2 v_{n-1} - 1) v_{n-1} + 1 \right\}, \ \ \ \ \ \ \ \ \ \ \ v_n = \text{same with } p \to \bar{p} \text{ and } u \to v. \]

It is shown that \( u_n \uparrow u_\infty, \ v_n \uparrow v_\infty \), and \( (u_\infty, v_\infty) \) is a unique root in
\( (0, 1)^2 \) of
\[ u = \frac{\sqrt{1-pv}}{\sqrt{1-pv} + \sqrt{p}} \quad \quad v = \frac{\sqrt{1-pu}}{\sqrt{1-pu} + \sqrt{pu}} \]

Convergence is quick:

\[ u_0 = 0.6592, \ 0.6867, \ 0.7530, \ 0.8611, \ \text{for } p = 0.5, \ 0.6, \ 0.8, \ 1.0, \ \text{resp.} \]
\[ v_0 = 0.6592, \ 0.6331, \ 0.5788, \ \frac{1}{2}, \ \text{for } p = 0.5, \ 0.6, \ 0.8, \ 1.0, \ \text{resp.} \]
\[ u_\infty = 2/3, \ 0.6946, \ 0.7663, \ \frac{1}{2}, \ \text{for } p = 0.5, \ 0.6, \ 0.8, \ 1.0, \ \text{resp.} \]
\[ v_\infty = 2/3, \ 0.6408, \ 0.5899, \ \frac{1}{2}, \ \text{for } p = 0.5, \ 0.6, \ 0.8, \ 1.0, \ \text{resp.} \]

Various interesting open problems arise. 1. If \( X_j, (Y_j) \) is the ability of management
( foreign language ), then \( X_j \) and \( Y_j \) are not independent. 2. The case where players aim
ENV of \( (X_j | Y_j > A) \to \max, \) (Ref.[11]), 3. 3-player game where \( (X_j, Y_j, Z_j) \) is
observed.

We first consider a simple n-round poker. Each of two players I and II receives a hand \( x \) and \( y \), respectively, in \([0,1]\), according to a uniform distribution, and chooses one of two alternatives Reject or Accept. If choice-pair is R–R, the game proceeds to the next round and both players are dealt new hands \( x \) and \( y \). If the choice-pair is A–A showdown occurs and the game ends with I’s reward \( sgn(x-y) \). If players choose different choices, then arbitration comes in, and forces them to take the same choices as I’s (II’s) with probability \( p \) (\( \bar{p} \)). This zero-sum game is played in n-rounds, and player I(II) aims to maximize(minimize) the expected reward to I.

Let \( \phi_n(x)(\psi_n(y)) \) be the probability that player I (II) chooses A on the hand \( x \) (\( y \)). Also let \( v_n \) be the value (for I) of the n-round game. Then we have

\[
v_n = \max_{\phi_n(\cdot)} \min_{\psi_n(\cdot)} E_{x,y}[(\tilde{\phi}_n(x), \phi_n(x))M_n(x,y)(\tilde{\psi}_n(y), \psi_n(y))^T]
\]

where

\[
M_n(x,y) = R \begin{pmatrix} v_{n-1} & \bar{p}sgn(x-y) + pv_{n-1} \\ \bar{p}sgn(x-y) + pv_{n-1} & sgn(x-y) \end{pmatrix}
\]

The solution is

\[
\phi_n^*(x) = \mathbb{I}(x > a_n), \quad \psi_n^*(y) = \mathbb{I}(y > a_n), \quad v_n = 2a_{n+1} - 1
\]

where \( \{a_n\} \) is determined by

\[
a_{n+1} = a_n + \frac{1}{2} \left( p\bar{a}_n^2 - \bar{p} a_n^2 \right) \quad (n > 1), \quad a_1 = \frac{1}{2}.
\]

We obtain \( a_n \uparrow a_\infty = \sqrt{\sqrt{p} \left( \sqrt{p} + \sqrt{\bar{p}} \right)} \), \( v_n \uparrow 2a_\infty - 1 = \sqrt{\sqrt{p} \left( \sqrt{p} + \sqrt{\bar{p}} \right)} \).

The bilateral-move version, when \( p = \frac{1}{2} \), has an interesting solution. Let \( w_n \) be the value of the n-stage game. \( w_n < 0 \), since I must move first and inevitably gives some information about his true hand. We find that \( w_n \downarrow w_\infty = -\left( -\frac{3}{2} \right) \left( \frac{1}{1 + \sqrt{3}} \right)^{\frac{1}{2}} \), where \( g = \frac{1}{2} \left( \sqrt{3} - 1 \right) \approx 0.618 \), the golden bisection number.

Also we find that disadvantage disappears when \( p = 0.6 \), i.e.

\[
w_n = 0, \quad \forall n \geq 1. \quad (Ref.[10]).
\]

3-player High-Hand-Wins poker, under simple-majority rule and with bet \( B > 0 \), is also an interesting open problem.

4. Odd-Man-Wins and Odd-Man-Out. In the three-player two-choice games there often appear Odd-Man and Even-Men. What is the reasonable partition among players of each \( X_j \)?

Let \( v_n \) (\( w_n \)) be CEQV of n-stage Odd-Man-Wins (Odd-Man-Out). Then the payoff matrices \( M_n(X) \) are
for Odd-Man-Wins, Odd-Man-Out, resp.

Each player must think about: (1) He wants to become the odd-man (an even-man), when the game is Odd-Man-Wins (Odd-Man-Out), especially when he faces a very large $X_j$, and (2) Since $X_j$ is a random variable, he can expect a larger one may come up in the future.

It is shown that (Ref.[67]), $v_{n} \uparrow v_{\infty} \doteq 0.205$ and $w_{n} \downarrow w_{\infty} \doteq 0.160$. So, multi-stage play yields each player a merit of size $v_{\infty} - \frac{1}{6} \doteq 0.39$ and a demerit of size $\frac{1}{6} - w_{\infty} \doteq 0.064$.

The extension to the many-player two-choice simple-majority games is an interesting open problem.

The case where the odd-man has priority $p$, and the even-men has priority $\frac{p}{2}$ is solved in Ref.[7]. When $p = \frac{1}{2}$, CEQV = $\frac{1}{3} \nu_{n}$, where $\{\nu_{n}\}$ is the Moser's sequence. When $p = 0 (1)$, the game reduces to Odd-Man-Out (Odd-Man-Wins).

REFERENCES


In some of these, names of co-author(s) are omitted.