<table>
<thead>
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<th>Title</th>
<th>INTRODUCTION TO A THEORY OF $b$-FUNCTIONS (Arc Spaces and Multiplier Ideals)</th>
</tr>
</thead>
<tbody>
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INTRODUCTION TO A THEORY OF \( b \)-FUNCTIONS

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We give an introduction to a theory of \( b \)-functions, i.e. Bernstein-Sato polynomials. After reviewing some facts from \( D \)-modules, we introduce \( b \)-functions including the one for arbitrary ideals of the structure sheaf. We explain the relation with singularities, multiplier ideals, etc., and calculate the \( b \)-functions of monomial ideals and also of hyperplane arrangements in certain cases.

1. D-modules.

1.1. Let \( X \) be a complex manifold or a smooth algebraic variety over \( \mathbb{C} \). Let \( \mathcal{D}_X \) be the ring of partial differential operators. A local section of \( \mathcal{D}_X \) is written as
\[
\sum_{\nu \in \mathbb{N}^n} a_\nu \partial_1^{\nu_1} \cdots \partial_n^{\nu_n} \in \mathcal{D}_X \quad \text{with} \quad a_\nu \in \mathcal{O}_X,
\]
where \( \partial_i = \partial / \partial x_i \) with \((x_1, \ldots, x_n)\) a local coordinate system.

Let \( F \) be the filtration by the order of operators i.e.
\[
F_p \mathcal{D}_X = \{ \sum_{|\nu| \leq p} a_\nu \partial_1^{\nu_1} \cdots \partial_n^{\nu_n} \}.
\]
where \( |\nu| = \sum \nu_i \). Let \( \xi_i = \text{Gr}_1^F \partial_i \in \text{Gr}_1^F \mathcal{D}_X \). Then
\[
\text{Gr}^F \mathcal{D}_X := \bigoplus_p \text{Gr}_p^F \mathcal{D}_X = \bigoplus_p \text{Sym}^p \Theta_X = \mathcal{O}_X[\xi_1, \ldots, \xi_n] \text{ locally},
\]
\[
\text{Spec}_X \text{Gr}^F \mathcal{D}_X = T^*X.
\]

1.2 Definition. We say that a left \( \mathcal{D}_X \)-module \( M \) is coherent if it has locally a finite presentation
\[
\bigoplus \mathcal{D}_X \to \bigoplus \mathcal{D}_X \to M \to 0.
\]

1.3. Remark. A left \( \mathcal{D}_X \)-module \( M \) is coherent if and only if it is quasi-coherent over \( \mathcal{O}_X \) and locally finitely generated over \( \mathcal{D}_X \). (It is known that \( \text{Gr}^F \mathcal{D}_X \) is a noetherian ring, i.e. an increasing sequence of locally finitely generated \( \text{Gr}^F \mathcal{D}_X \)-submodules of a coherent \( \text{Gr}^F \mathcal{D}_X \)-module is locally stationary.)

1.4. Definition. A filtration \( F \) on a left \( \mathcal{D}_X \)-module \( M \) is good if \( (M, F) \) is a coherent filtered \( \mathcal{D}_X \)-module, i.e. if \( F_p \mathcal{D}_X F_q M \subset M_{p+q} \) and \( \text{Gr}^F M := \bigoplus_p \text{Gr}_p^F M \) is coherent over \( \text{Gr}^F \mathcal{D}_X \).

1.5. Remark. A left \( \mathcal{D}_X \)-module \( M \) is coherent if and only if it has a good filtration locally.

\[\text{Date: Oct. 25, 2006, v.1.}\]
1.6. Characteristic varieties. For a coherent left $\mathcal{D}_X$-module $M$, we define the characteristic variety $\text{CV}(M)$ by
\[
\text{CV}(M) = \text{Supp} \text{Gr}^F M \subset T^* M,
\]
taking locally a good filtration $F$ of $M$.

1.7. Remark. The above definition is independent of the choice of $F$. If $M = \mathcal{D}_X/I$ for a coherent left ideal $I$ of $\mathcal{D}_X$, take $p_i \in F_{k_i}I$ such that the $\rho_i := \text{Gr}^F_{k_i}p_i$ generate $\text{Gr}^F I$ over $\text{Gr}^F D_X$. Then $\text{CV}(M)$ is defined by the $\rho_i \in \mathcal{O}_X[\xi_1, \ldots, \xi_n]$.

1.8. Theorem (Sato, Kawai, Kashiwara [39], Bernstein [2]). We have the inequality $\dim \text{CV}(M) \geq \dim X$. (More precisely, $\text{CV}(M)$ is involutive, see [39].)

1.9. Definition. We say that a left $\mathcal{D}_X$-module $M$ is holonomic if it is coherent and $\dim \text{CV}(M) = \dim X$.

2. De Rham functor.

2.1. Definition. For a left $\mathcal{D}_X$-module $M$, we define the de Rham functor $\text{DR}(M)$ by
\[
M \rightarrow \Omega^1_X \otimes_{\mathcal{O}_X} M \rightarrow \cdots \rightarrow \Omega^{\dim X}_X \otimes_{\mathcal{O}_X} M,
\]
where the last term is put at the degree 0. In the algebraic case, we use analytic sheaves or replace $M$ with the associated analytic sheaf $M^{an} := M \otimes_{\mathcal{O}_X} \mathcal{O}_X^{an}$ in case $M$ is algebraic (i.e. $M$ is an $\mathcal{O}_X$-module with $\mathcal{O}_X$ algebraic).

2.2. Perverse sheaves. Let $\mathcal{D}^b_c(X, \mathcal{C})$ be the derived category of bounded complexes of $\mathcal{C}$-modules $K$ with $\mathcal{H}^j K$ constructible. (In the algebraic case we use analytic topology for the sheaves although we use Zariski topology for constructibility.) Then the category of perverse sheaves $\text{Perv}(X, \mathcal{C})$ is a full subcategory of $\mathcal{D}^b_c(X, \mathcal{C})$ consisting of $K$ such that
\[
\dim \text{Supp} \mathcal{H}^{-j} K \leq j, \quad \dim \text{Supp} \mathcal{H}^{-j} \mathcal{D} K \leq j,
\]
where $\mathcal{D} K := \mathcal{R}\text{Hom}(K, \mathcal{C}[2 \dim X])$ is the dual of $K$, and $\mathcal{H}^j K$ is the $j$-th cohomology sheaf of $K$.

2.3. Theorem (Beilinson, Bernstein, Deligne [1]). $\text{Perv}(X, \mathcal{C})$ is an abelian category.

2.4. Theorem (Kashiwara). If $M$ is holonomic, then $\text{DR}(M)$ is a perverse sheaf.

Outline of proof. By Kashiwara [19], we have $\text{DR}(M) \in \mathcal{D}^b_c(X, \mathcal{C})$, and the first condition of (2.2.1) is verified. Then the assertion follows from the commutativity of the dual $\mathcal{D}$ and the de Rham functor $\text{DR}$.

2.5. Example. $\text{DR}(\mathcal{O}_X) = \mathcal{C}_X[\dim X]$.

2.6. Direct images. For a closed immersion $i : X \rightarrow Y$ such that $X$ is defined by $x_i = 0$ in $Y$ for $1 \leq i \leq r$, define the direct image of left $\mathcal{D}_X$-modules $M$ by
\[
i_* M := M[\partial_1, \ldots, \partial_r].\]
INTRODUCTION TO A THEORY OF $b$-FUNCTIONS

(Globally there is a twist by a line bundle.) For a projection $p : X \times Y \to Y$, define

$$p_+ M = Rp_* DR_X(M).$$

In general, $f_+ = p_+ i_+$ using $f = pi$ with $i$ graph embedding. See [4] for details.

2.7. Regular holonomic D-modules. Let $M$ be a holonomic $\mathcal{D}_X$-module with support $Z$, and $U$ be a Zariski-open of $Z$ such that $DR(M)|_U$ is a local system up to a shift. Then $M$ is regular if and only if there exists locally a divisor $D$ on $X$ containing $Z \setminus U$ and such that $M(*D)$ is the direct image of a regular holonomic $\mathcal{D}$-module 'of Deligne-type' (see [11]) on a desingularization of $(Z, Z \cap D)$, and $\text{Ker}(M \to M(*D))$ is regular holonomic (by induction on $\dim \text{Supp} M$).

Note that the category $M_{rh}(\mathcal{D}_X)$ of regular holonomic $\mathcal{D}_X$-modules is stable by subquotients and extensions in the category $M_{h}(\mathcal{D}_X)$ of holonomic $\mathcal{D}_X$-modules.

2.8. Theorem (Kashiwara-Kawai [24], [22], Mebkhout [28]).

(i) The structure sheaf $\mathcal{O}_X$ is regular holonomic.

(ii) The functor $DR$ induces an equivalence of categories

$$(2.8.1) \quad DR : M_{rh}(\mathcal{D}_X) \sim \text{Perv}(X, \mathbb{C}).$$

(See [4] for the algebraic case.)

3. $b$-Functions.

3.1. Definition. Let $f$ be a holomorphic function on $X$, or $f \in \Gamma(X, \mathcal{O}_X)$ in the algebraic case. Then we have

$$\mathcal{D}_X[s]f^s \subset \mathcal{O}_X[\frac{1}{f}]f^s \quad \text{where } \partial_t f^s = s(\partial_t f)f^{s-1},$$

and $b_f(s)$ is the monic polynomial of the least degree satisfying

$$b_f(s)f^s = P(x, \partial, s)f^{s+1} \quad \text{in } \mathcal{O}_X[\frac{1}{f}]f^s,$$

with $P(x, \partial, s) \in \mathcal{D}_X[s]$. Locally, it is the minimal polynomial of the action of $s$ on

$$\mathcal{D}_X[s]f^s/\mathcal{D}_X[s]f^{s+1}.$$ 

We define $b_{f,x}(s)$ replacing $\mathcal{D}_X$ with $\mathcal{D}_{X,x}$.

3.2. Theorem (Sato [38], Bernstein [2], Bjork [3]). The $b$-function exists at least locally, and exists globally in the case $X$ affine variety with $f$ algebraic.

3.3. Observation. Let $i_f : X \to \tilde{X} := X \times \mathbb{C}$ be the graph embedding. Then there are canonical isomorphisms

$$(3.3.1) \quad \tilde{M} := i_{f+}\mathcal{O}_X = \mathcal{O}_X[\partial_t]\delta(f - t) = \mathcal{O}_{X \times \mathbb{C}}[\frac{1}{f-t}]\mathcal{O}_{X \times \mathbb{C}},$$

where the action of $\partial_t$ on $\delta(f - t)$ ($= \frac{1}{f-t}$) is given by

$$(3.3.2) \quad \partial_t \delta(f - t) = -(\partial_t f)\partial_t \delta(f - t).$$

Moreover, $f^s$ is canonically identified with $\delta(f - t)$ setting $s = -\partial_t t$, and we have a canonical isomorphism as $\mathcal{D}_X[s]$-modules

$$(3.3.3) \quad \mathcal{D}_X[s]f^s = \mathcal{D}_X[s]\delta(f - t).$$
3.4. V-filtration. We say that V is a filtration of Kashiwara-Malgrange if V is exhaustive, separated, and satisfies for any $\alpha \in \mathbb{Q}$:

(i) $V^\alpha \widehat{M}$ is a coherent $\mathcal{D}_X[s]$-submodule of $\widehat{M}$.
(ii) $tV^\alpha \widehat{M} \subset V^{\alpha+1} \widehat{M}$ and $\delta$ holds for $\alpha \gg 0$.
(iii) $\partial_t V^\alpha \widehat{M} \subset V^{\alpha-1} \widehat{M}$.
(iv) $\partial_t - \alpha$ is nilpotent on $\text{Gr}^\alpha \widehat{M}$.

If it exists, it is unique.

3.5. Relation with the $b$-function. If $X$ is affine or Stein and relatively compact, then the multiplicity of a root $\alpha$ of $b_f(s)$ is given by the minimal polynomial of $s - \alpha$ on $\mathcal{D}_X[s]f^s$, using $\mathcal{D}_X[s]f^s = \mathcal{D}_X[s] \delta(f - t)$ with $s = -\partial_t$.

Note that $V^\alpha \widehat{M}$ and $\mathcal{D}_X[s]f^{s+i}$ are 'lattices' of $\widehat{M}$, i.e.

$$V^\alpha \widehat{M} \subset \mathcal{D}_X[s]f^{s+i} \subset V^\beta \widehat{M} \quad \text{for} \ \alpha \gg i \gg \beta,$$

and $V^\alpha \widehat{M}$ is an analogue of the Deligne extension with eigenvalues in $[\alpha, \alpha + 1)$. The existence of $V$ is equivalent to the existence of $b_f(s)$ locally.

3.6. Theorem (Kashiwara [21], [23], Malgrange [27]). The filtration $V$ exists on $\widehat{M} := i_{f+}^* M$ for any holonomic $\mathcal{D}_X$-module $M$.

3.7. Remarks. (i) There are many ways to prove this theorem, since it is essentially equivalent to the existence of the $b$-function (in a generalized sense). One way is to use a resolution of singularities and reduce to the case where $CV(M)$ has normal crossings, if $M$ is regular.

(ii) The filtration $V$ is indexed by $\mathbb{Q}$ if $M$ is quasi-unipotent.

3.8. Relation with vanishing cycle functors. Let $\rho : X_t \to X_0$ be a 'good' retraction (using a resolution of singularities of $(X, X_0)$), where $X_t = f^{-1}(t)$ with $t \neq 0$ sufficiently near 0. Then we have canonical isomorphisms

$$\psi_f \mathcal{C}_X = R\rho_* \mathcal{C}_{X_t}, \quad \varphi_f \mathcal{C}_X = \psi_f \mathcal{C}_X / \mathcal{C}_{X_0},$$

where $\psi_f \mathcal{C}_X, \varphi_f \mathcal{C}_X$ are nearby and vanishing cycle sheaves, see [13].

Let $F_x$ denote the Milnor fiber around $x \in X_0$. Then

$$\mathcal{H}^j(\psi_f \mathcal{C}_X)_x = H^j(F_x, \mathcal{C}), \quad (\mathcal{H}^j(\varphi_f \mathcal{C}_X)_x = \mathcal{H}^j(F_x, \mathcal{C}).$$

For a $\mathcal{D}_X$-module $M$ admitting the $V$-filtration on $\widehat{M} = i_{f+}^* M$, we define $\mathcal{D}_X$-modules

$$\psi_f M = \bigoplus_{0 \leq \alpha \leq 1} \text{Gr}^\alpha \widehat{M}, \quad \varphi_f M = \bigoplus_{0 \leq \alpha \leq 1} \text{Gr}^\alpha \widehat{M}.$$

3.9. Theorem (Kashiwara [23], Malgrange [27]). For a regular holonomic $\mathcal{D}_X$-module $M$, we have canonical isomorphisms

$$\text{DR}_X \psi_f(M) = \psi_f \text{DR}_X(M)[-1],$$
$$\text{DR}_X \varphi_f(M) = \varphi_f \text{DR}_X(M)[-1],$$
INTRODUCTION TO A THEORY OF b-FUNCTIONS

and \(\exp(-2\pi i \partial_{t}t)\) on the left-hand side corresponds to the monodromy \(T\) on the right-hand side.

3.10. Definition. Let
\[
R_f = \{ \text{roots of } b_f(-s) \},
\]
\[
\alpha_f = \min R_f,
\]
\[
m_\alpha : \text{the multiplicity of } \alpha \in R_f.
\]
(Similarly for \(R_{f,x}\), etc. for \(b_{f,x}(s)\).)

3.11. Theorem (Kashiwara [20]). \(R_f \subset \mathbb{Q}_{>0}\).
(This is proved by using a resolution of singularities.)

3.12. Theorem (Kashiwara [23], Malgrange [27]).
(i) \(e^{-2\pi i R_f} = \{ \text{the eigenvalues of } T \text{ on } H^j(F_x, \mathbb{C}) \text{ for } x \in X_0, j \in \mathbb{Z} \},\)
(ii) \(m_\alpha \leq \min \{ i | N^i \psi_{f,\lambda}C_X = 0 \} \text{ with } \lambda = e^{-2\pi i \alpha},\)
where \(\psi_{f,\lambda} = \text{Ker}(T_s - \lambda) \subset \psi_f, N = \log T_u\text{ with } T = T_sT_u.\)
(This is a corollary of the above Theorem (3.9) of Kashiwara and Malgrange.)

4. Relation with other invariants.

4.1. Microlocal b-function. We define \(\tilde{R}_f, \tilde{m}_\alpha, \tilde{\alpha}_f\) with \(b_f(s)\) replaced by the microlocal (or reduced) b-function
\[
\tilde{b}_f(s) := b_f(s)/(s + 1).
\]
This \(\tilde{b}_f(s)\) coincides with the monic polynomial of the least degree satisfying
\[
\tilde{b}_f(s)\delta(f - t) = \tilde{P}\partial_{t}^{-1}\delta(f - t) \text{ with } \tilde{P} \in \mathcal{D}_X[s, \partial_{t}^{-1}].
\]
Put \(n = \dim X.\) Then

4.2. Theorem. \(\tilde{R}_f \subset [\tilde{\alpha}_f, n - \tilde{\alpha}_f], \quad \tilde{m}_\alpha \leq n - \tilde{\alpha}_f - \alpha + 1.\)
(The proof uses the filtered duality for \(\varphi_f,\) see [35].)

4.3. Spectrum. We define the spectrum by \(\text{Sp}(f, x) = \sum n_\alpha t^\alpha\) with
\[
n_\alpha := \sum_{j}(-1)^{j-n+1} \dim \text{Gr}_p^F \tilde{H}^j(F_x, \mathbb{C})_\lambda,
\]
where \(p = [n - \alpha], \lambda = e^{-2\pi i \alpha},\) and \(F\) is the Hodge filtration (see [12]) of the mixed Hodge structure on the Milnor cohomology, see [44]. We define
\[
E_f = \{ \alpha | n_\alpha \neq 0 \} \text{ (called the exponents).}
\]

4.4. Remarks. (i) If \(f\) has an isolated singularity at the origin, then \(\tilde{\alpha}_{f,x}\) coincides with the minimal exponent as a corollary of results of Malgrange [26], Varchenko [45], Scherk-Steenbrink [41].
(ii) If \(f\) is weighted-homogeneous with an isolated singularity at the origin, then by Kashiwara (unpublished)
\[
\tilde{R}_f = E_f, \quad \max \tilde{R}_f = n - \tilde{\alpha}_f, \quad \tilde{m}_\alpha = 1 (\alpha \in \tilde{R}_f).
\]
If $f = \sum_i x_i^2$, then $\tilde{\alpha}_f = n/2$ and this follows from the above Theorem (4.2).

By Steenbrink [42], we have moreover

$$\text{Sp}(f, x) = \prod_i (t - t^{w_i})/(t^{w_i} - 1),$$

where $(w_1, \ldots, w_n)$ is the weights of $f$, i.e. $f$ is a linear combination of monomials $x_1^{m_1} \cdots x_n^{m_n}$ with $\sum_i w_i m_i = 1$.

4.5. Malgrange’s formula (isolated singularities case). We have the Brieskorn lattice [5] and its saturation defined by

$$H_f'' = \Omega_{X,x}^{n-1}/df \wedge d\Omega_{X,x}^{n-2}, \quad \tilde{H}_f'' = \sum_{i \geq 0} (t \partial_t)^i H_f'' \subset H_f''[t^{-1}].$$

These are finite $\mathbb{C}\{t\}$-modules with a regular singular connection.

4.6. Theorem (Malgrange [26]). The reduced b-function $\tilde{b}_f(s)$ coincides with the minimal polynomial of $-\partial_t$ on $\tilde{H}_f''/t\tilde{H}_f''$.

(The above formula of Kashiwara on $b$-function (4.4.1) can be proved by using this together with Brieskorn’s calculation.)

4.7. Asymptotic Hodge structure (Varchenko [45], Scherk-Steenbrink [41]). In the isolated singularity case we have

$$F^p H^{n-1}(F_x, \mathbb{C})_\lambda = \text{Gr}_V^\alpha H_f''$$

using the canonical isomorphism

$$H^{n-1}(F_x, \mathbb{C})_\lambda = \text{Gr}_V^\alpha H_f''[t^{-1}],$$

where $p = [n - \alpha], \lambda = e^{-2\pi i \alpha}$, and $V$ on $H_f''[t^{-1}]$ is the filtration of Kashiwara and Malgrange.

(This can be generalized to the non-isolated singularity case using mixed Hodge modules.)

4.8. Reformulation of Malgrange’s formula. We define

$$\tilde{F}^p H^{n-1}(F_x, \mathbb{C})_\lambda = \text{Gr}_V^\alpha \tilde{H}_f''$$

using the canonical isomorphism (4.7.2), where $p = [n - \alpha], \lambda = e^{-2\pi i \alpha}$. Then

$$\tilde{m}_\alpha = \text{the minimal polynomial of } N \text{ on } \text{Gr}^p \tilde{F} H^{n-1}(F_x, \mathbb{C})_\lambda.$$

4.9. Remark. If $f$ is weighted homogeneous with an isolated singularity, then

$$\tilde{F} = F, \quad \tilde{R}_f = E_f \text{ (by Kashiwara).}$$

If $f$ is not weighted homogeneous (but with isolated singularities), then

$$\tilde{R}_f \subset \bigcup_{k \in \mathbb{N}} (E_f - k), \quad \tilde{\alpha}_f = \min \tilde{R}_f = \min E_f.$$

4.10. Example. If $f = x^5 + y^4 + x^3 y^2$, then

$$E_f = \left\{ \frac{i}{5} + \frac{j}{4} : 1 \leq i \leq 4, 1 \leq j \leq 3 \right\}, \quad \tilde{R}_f = E_f \cup \left\{ \frac{11}{20} \right\} \setminus \left\{ \frac{31}{20} \right\}.$$
4.11. Relation with rational singularities [34]. Assume $D := f^{-1}(0)$ is reduced. Then $D$ has rational singularities if and only if $\alpha_f > 1$. Moreover, $\omega_D/\rho_* \omega_{\tilde{X}} \simeq F_{1-n} \varphi_f O_X$, where $\rho : \tilde{X} \to X$ is a resolution of singularities.

In the isolated singularities case, this was proved in 1981 (see [31]) using the coincidence of $\alpha_f$ and the minimal exponent.

4.12. Relation with the pole order filtration [34]. Let $P$ be the pole order filtration on $O_X(*D)$, i.e. $P_i = O_X((i+1)D)$ if $i \geq 0$, and $P_i = 0$ if $i < 0$. Let $F$ be the Hodge filtration on $O_X(*D)$. Then $F_i \subset P_i$ in general, and $F_i = P_i$ on a neighborhood of $x$ for $i \leq \alpha_{f,x} - 1$.

(For the proof we need the theory of microlocal $b$-functions [35].)

4.13. Remark. In case $X = \mathbb{P}^n$, replacing $\alpha_{f,x}$ with $[(n-r)/d]$ where $r = \dim \text{Sing } D$ and $d = \deg D$, the assertion was obtained by Deligne (unpublished).

5. Relation with multiplier ideals.

5.1. Multiplier ideals. Let $D = f^{-1}(0)$, and $\mathcal{J}(X, \alpha D)$ be the multiplier ideals for $\alpha \in \mathbb{Q}$, i.e.

\[
\mathcal{J}(X, \alpha D) = \rho_* \omega_{\tilde{X}/X}(-\sum_i [\alpha m_i]\tilde{D}_i),
\]

where $\rho : (\tilde{X}, \tilde{D}) \to (X, D)$ is an embedded resolution and $\tilde{D} = \sum_i m_i \tilde{D}_i := \rho^* D$. There exist jumping numbers $0 < \alpha_0 < \alpha_1 < \cdots$ such that

\[
\mathcal{J}(X, \alpha_j D) = \mathcal{J}(X, \alpha D) \neq \mathcal{J}(X, \alpha_{j+1} D) \quad \text{for } \alpha_j \leq \alpha < \alpha_{j+1}.
\]

Let $V$ denote also the induced filtration on

$O_X \subset O_X[\partial_t] \delta(f-t)$.

5.2. Theorem (Budur, S. [10]). If $\alpha$ is not a jumping number,

\[
\mathcal{J}(X, \alpha D) = V^\alpha O_X.
\]

For $\alpha$ general we have for $0 \ll \epsilon < 1$

\[
\mathcal{J}(X, \alpha D) = V^{\alpha+\epsilon} O_X, \quad V^\alpha O_X = \mathcal{J}(X, (\alpha - \epsilon)D).
\]

Note that $V$ is left-continuous and $\mathcal{J}(X, \alpha D)$ is right-continuous, i.e.

\[
V^\alpha O_X = V^{\alpha-\epsilon} O_X, \quad \mathcal{J}(X, \alpha D) = \mathcal{J}(X, (\alpha + \epsilon)D).
\]

The proof of (5.2) uses the theory of bifiltered direct images [32], [33] to reduce the assertion to the normal crossing case.

As a corollary we get another proof of the results of Ein, Lazarsfeld, Smith and Varolin [16], and of Lichtin, Yano and Kollár [25]:
5.3. Corollary.
(i) \{Jumping numbers \leq 1\} \subset R_f, see [16].
(ii) \(\alpha_f\) = minimal jumping number, see [25].

Define \(\alpha'_{f,x} = \min_{y \neq x}\{\alpha_{f,y}\}\). Then

5.4. Theorem. If \(\xi f = f\) for a vector field \(\xi\), then

\[(5.4.1)\quad R_f \cap (0, \alpha'_{f,x}) = \{\text{Jumping numbers}\} \cap (0, \alpha'_{f,x}).\]

(This does not hold without the assumption on \(\xi\) nor for \([\alpha'_{f,x}, 1)\).)

For the constantness of the jumping numbers under a topologically trivial deformation of divisors, see [14].

6. \(b\)-Functions for any subvarieties.

6.1. Let \(Z\) be a closed subvariety of a smooth \(X\), and \(f = (f_1, \ldots, f_r)\) be generators of the ideal of \(Z\) (which is not necessarily reduced nor irreducible). Define the action of \(t_j\) on

\[O_X[\frac{1}{f_1 \cdots f_r}][s_1, \ldots, s_r]\prod_i f_i^{s_i},\]

by \(t_j(s_i) = s_i + 1\) if \(i = j\), and \(t_j(s_i) = s_i\) otherwise. Put \(s_{i,j} := s_i t_i^{-1} t_j\), \(s = \sum_i s_i\).

Then \(b_f(s)\) is the monic polynomial of the least degree satisfying

\[(6.1.1)\quad b_f(s) \prod_i f_i^{s_i} = \sum_{k=1}^{r} P_k t_k \prod_i f_i^{s_i},\]

where \(P_k\) belong to the ring generated by \(\mathcal{D}_X\) and \(s_{i,j}\).

Here we can replace \(\prod_i f_i^{s_i}\) with \(\prod_i \delta(t_i - f_i)\), using the direct image by the graph of \(f : X \to \mathbb{C}^r\). Then the existence of \(b_f(s)\) follows from the theory of the \(V\)-filtration of Kashiwara and Malgrange. This \(b\)-function has appeared in work of Sabbah [30] and Gyoja [18] for the study of \(b\)-functions of several variables.

6.2. Theorem (Budur, Mustaţă, S. [8]). Let \(c = \text{codim}_X Z\). Then \(b_Z(s) := b_f(s - c)\) depends only on \(Z\) and is independent of the choice of \(f = (f_1, \ldots, f_r)\) and also of \(r\).

6.3. Equivalent definition. The \(b\)-function \(b_f(s)\) coincides with the monic polynomial of the least degree satisfying

\[(6.3.1)\quad b_f(s) \prod_i f_i^{s_i} \in \sum_{|c| = 1} D_X[s] \prod_{c_i < 0} (-c_i) \prod_i f_i^{s_i + c_i},\]

where \(c = (c_1, \ldots, c_r) \in \mathbb{Z}^r\) with \(|c| := \sum_i c_i = 1\). Here \(D_X[s] = D_X[s_1, \ldots, s_r]\).

This is due to Mustaţă, and is used in the monomial ideal case. Note that the well-definedness does not hold without the term \(\prod_{c_i < 0} (-c_i)\).

We have the induced filtration \(V\) by

\[O_X \subset i_f+ O_X = O_X[\delta_1, \ldots, \delta_r] \prod_i \delta(t_i - f_i).\]

6.4. Theorem (Budur, Mustaţă, S. [8]). If \(\alpha\) is not a jumping number,

\[(6.4.1)\quad J(X, \alpha Z) = V^\alpha O_X.\]
INTRODUCTION TO A THEORY OF b-FUNCTIONS

For $\alpha$ general we have for $0 < \varepsilon \ll 1$

(6.4.2) $\mathcal{J}(X, \alpha Z) = V^{\alpha+\varepsilon}\mathcal{O}_X$, $V^\alpha\mathcal{O}_X = \mathcal{J}(X, (\alpha - \varepsilon)Z)$.

6.5. Corollary (Budur, Mustaţă, S. [8]). We have the inclusion

(6.5.1) $\{\text{Jumping numbers}\} \cap [\alpha_f, \alpha_f + 1) \subset R_f$.

6.6. Theorem (Budur, Mustată, S. [8]). If $Z$ is reduced and is a local complete intersection, then $Z$ has only rational singularities if and only if $\alpha_f = r$ with multiplicity 1.

7. Monomial ideal case.

7.1. Definition. Let $a \subset C[x] := C[x_1, \ldots, x_n]$ a monomial ideal. We have the associated semigroup defined by

$$\Gamma_a = \{ u \in \mathbb{N}^n \mid x^u \in a \}.$$  

Let $P_a$ be the convex hull of $\Gamma_a$ in $\mathbb{R}_{\geq 0}^n$. For a face $Q$ of $P_a$, define

$$M_Q : \text{the subsemigroup of } \mathbb{Z}^n \text{ generated by } u - v \text{ with } u \in \Gamma_a, v \in \Gamma_a \cap Q.$$  

$$M'_Q = v_0 + M_Q \text{ for } v_0 \in \Gamma_a \cap Q \text{ (this is independent of } v_0).$$

For a face $Q$ of $P_a$ not contained in any coordinate hyperplane, take a linear function with rational coefficients $L_Q : \mathbb{R}^n \to \mathbb{R}$ whose restriction to $Q$ is 1. Let

$$V_Q : \text{the linear subspace generated by } Q.$$  

$$e = (1, \ldots, 1).$$

$$R_Q = \{ L_Q(u) \mid u \in (e + (M_Q \setminus M'_Q)) \cap V_Q \},$$

$$R_a = \{ \text{roots of } b_a(-s) \}.$$  

7.2. Theorem (Budur, Mustaţă, S. [9]). We have $R_a = \bigcup_Q R_Q$ with $Q$ faces of $P_a$ not contained in any coordinate hyperplanes.

Outline of the proof. Let $f_j = \prod_i x_i^{a_{i,j}}, \ell_i(s) = \sum_j a_{i,j}s_j$. Define

$$g_c(s) = \prod_{c_i < 0}(-s_i^{c_i})\prod_{\ell_i(c) > 0}(\ell_i(s)^{c_i} + \ell_i(c)).$$

Let $I_a \subset C[s]$ be the ideal generated by $g_c(s)$ with $c \in \mathbb{Z}^r$, $\sum_i c_i = 1$. Then

7.3. Proposition (Mustaţă). The $b$-function $b_a[s]$ of the monomial ideal $a$ is the monic generator of $C[s] \cap I_a$, where $s = \sum_i s_i$.

Using this, Theorem (7.2) follows from elementary computations.

7.4. Case $n = 2$. Here it is enough to consider only 1-dimensional $Q$ by (7.2). Let $Q$ be a compact face of $P_a$ with $\{v^{(1)}, v^{(2)}\} = \partial Q$, where $v^{(i)} = (v^{(i)}_1, v^{(i)}_2)$ with $v^{(1)}_1 < v^{(2)}_1, v^{(1)}_2 > v^{(2)}_2$. Let

$$G : \text{the subgroup generated by } u - v \text{ with } u, v \in Q \cap \Gamma_a.$$  

$v^{(3)} \in Q \cap \mathbb{N}^2$ such that $v^{(3)} - v^{(1)}$ generates $G$.

$$S_Q = \{(i, j) \in \mathbb{N}^2 \mid i < v^{(3)}_1, j < v^{(1)}_2 \}.$$
$S_{Q}^{[1]} = S \cap M_{Q}^{1}$, $S_{Q}^{[0]} = S_{Q} \setminus S_{Q}^{[1]}$.

Then

$R_{Q} = \{L_{Q}(u + e) - k \mid u \in S_{Q}^{[k]} \ (k = 0, 1)\}$.

In the case $Q \subset \{x = m\}$, we have $R_{Q} = \{i/m \mid i = 1, \ldots, m\}$.

7.5. Examples. (i) If $a = (x^{a_{0}}, xy^{b})$, with $a, b \geq 2$, then

$$R_{a} = \left\{ \frac{(b - 1)i + (a - 1)j}{ab - 1} \mid 1 \leq i \leq a, \ 1 \leq j \leq b \right\}.$$

(ii) If $a = (xy, x^{3}y^{2}, x^{5}y)$, then $S_{Q}^{[1]} = \emptyset$ and

$$R_{a} = \left\{ \frac{5}{13}, \frac{i}{13} \ (7 \leq i \leq 17), \frac{19}{13}, \frac{j}{6} \ (3 \leq j \leq 9) \right\}.$$

(iii) If $a = (xy, x^{3}y^{2}, x^{4}y)$, then $S_{Q}^{[1]} = \{(2, 4)\}$ for $\partial Q = \{(1, 5), (3, 2)\}$ with $L_{Q}(v_{1}, v_{2}) = (3v_{1} + 2v_{2})/13$, and

$$R_{a} = \left\{ \frac{i}{13} \ (5 \leq i \leq 17), \frac{j}{5} \ (2 \leq j \leq 6) \right\}.$$}

Here $19/13$ is shifted to $6/13$.

7.6. Comparison with exponents. If $n = 2$ and $f$ has a nondegenerate Newton polygon with compact faces $Q$, then by Steenbrink [43]

$$E_{f} \cap (0, 1] = \bigcup_{Q} E_{Q}^{\leq 1} \text{ with } E_{Q}^{\leq 1} = \{L_{Q}(u) \mid u \in \{0\} \cup Q \cap \mathbb{Z}_{>0}^{2}\},$$

where $\{0\} \cup Q$ is the convex hull of $\{0\} \cup Q$. Here we have the symmetry of $E_{f}$ with center 1.

7.7. Another comparison. If $a = (x_{1}^{a_{1}}, \ldots, x_{n}^{a_{n}})$, then

$$R_{a} = \{\sum_{i}p_{i}/a_{i} \mid 1 \leq p_{i} \leq a_{i}\}.$$ 

On the other hand, if $f = \sum_{i} x_{i}^{a_{i}}$, then

$$\tilde{R}_{f} = E_{f} = \{\sum_{i}p_{i}/a_{i} \mid 1 \leq p_{i} \leq a_{i} - 1\}.$$ 

8. Hyperplane arrangements.

8.1. Let $D$ be a central hyperplane arrangement in $X = \mathbb{C}^{n}$. Here, central means an affine cone of $Z \subset \mathbb{P}^{n-1}$. Let $f$ be the reduced equation of $D$ and $d := \deg f > n$. Assume $D$ is not the pull-back of $D' \subset \mathbb{C}^{n'} (n' < n)$.

8.2. Theorem. (i) $\max R_{f} < 2 - \frac{1}{d}$. (ii) $m_{1} = n$.

Proof of (i) uses a partial generalization of a solution of Aomoto's conjecture due to Esnault, Schechtman, Viehweg, Terao, Varchenko ([17], [40]) together with a generalization of Malgrange's formula (4.8) as below:
8.3. Theorem (Generalization of Malgrange’s formula) [36]. There exists a pole order filtration $P$ on $H^{n-1}(F_0, C)_\lambda$ such that if $(\alpha + N) \cap R'_f = \emptyset$, then

$$(8.3.1) \quad \alpha \in R'_f \Leftrightarrow \text{Gr}^p_f H^{n-1}(F_0, C)_\lambda \neq 0,$$

with $p = [n - \alpha], \lambda = e^{-2\pi i \alpha}$, where $R'_f = \cup_{x \neq 0} R_{f,x}$.

This reduces the proof of (8.2)(i) to

$$(8.3.2) \quad P^i H^{n-1}(F_0, C)_\lambda = H^{n-1}(F_0, C)_\lambda,$$

for $i = n - 1$ if $\lambda = 1$ or $e^{2\pi i / d}$, and $i = n - 2$ otherwise.

8.4. Construction of the pole order filtration $P$. Let $U = \mathbb{P}^{n-1} \setminus Z$, and $F_0 = f^{-1}(0) \subset C^n$. Then $F_0 = \tilde{U}$ with $\pi : \tilde{U} \to U$ a $d$-fold covering ramified over $Z$. Let $L^{(k)}$ be the local systems of rank 1 on $U$ such that $\pi_*C = \bigoplus_{0 \leq i < d} L^{(k)}$ and $T$ acts on $L^{(k)}$ by $e^{-2\pi ik/d}$. Then

$$(8.4.1) \quad H^j(U, L^{(k)}) = H^j(F_0, C)_{e(-k/d)},$$

and $P$ is induced by the pole order filtration on the meromorphic extension $L^{(k)}$ of $L^{(k)} \otimes_C O_U$ over $\mathbb{P}^{n-1}$, see [15], [36], [37]. This is closely related to:

8.5. Solution of Aomoto’s conjecture ([17], [40]). Let $Z_i$ be the irreducible components of $Z$ $(1 \leq i \leq d)$, $g_i$ be the defining equation of $Z_i$ on $\mathbb{P}^{n-1} \setminus Z_d$ $(i < d)$, and $\omega := \sum_{i < d} \alpha_i \omega_i$ with $\omega_i = dg_i/g_i$, $\alpha_i \in \mathbb{C}$. Let $\nabla$ be the connection on $O_U$ such that $\nabla u = du + \omega \wedge u$. Set $\alpha_d = -\sum_{i < d} \alpha_i$. Then $H^*_\text{DR}(U, (O_U, \nabla))$ is calculated by

$$(\mathcal{A}^*_\alpha, \omega \wedge) \quad \text{with} \quad \mathcal{A}^*_\alpha = \sum C\omega_{i_1} \cdots \wedge \omega_{i_p},$$

if $\sum_{Z_i \supset L} \alpha_i \notin \mathbb{N} \setminus \{0\}$ for any dense edge $L \subset Z$ (see (8.7) below). Here an edge is an intersection of $Z_i$.

For the proof of (8.2)(ii) we have

8.6. Proposition. $N^{n-1} \psi_{f,\lambda} C \neq 0$ if $\text{Gr}^W_{2n-2} H^{n-1}(F_z, C)_\lambda \neq 0$.

(Indeed, $N^{n-1} : \text{Gr}^W_{2n-2} \psi_{f,\lambda} C \stackrel{\sim}{\longrightarrow} \text{Gr}^0 W \psi_{f,\lambda} C$ by the definition of $W$, and the assumption of (8.6) implies $\text{Gr}^W_{2n-2} \psi_{f,\lambda} C \neq 0$.)

Then we get (8.2)(ii), since $\omega_{i_1} \wedge \cdots \wedge \omega_{i_{n-1}} \neq 0$ in $\text{Gr}^W_{2n-2} H^{n-1}(\mathbb{P}^{n-1} \setminus Z, C) = \text{Gr}^W_{2n-2} H^{n-1}(F_z, C)_1$.

8.7. Dense edges. Let $D = \cup_i D_i$ be the irreducible decomposition. Then $L = \cap_{i \in I} D_i$ is called an edge of $D$ $(I \neq \emptyset)$.

We say that an edge $L$ is dense if $\{D_i/L \mid D_i \supset L\}$ is indecomposable. Here $C^n \supset D$ is called decomposable if $C^n = C^{n'} \times C^{n''}$ such that $D$ is the union of the pull-backs from $C^{n'}, C^{n''}$ with $n', n'' \neq 0$.

Set $m_L = \#\{D_i \mid D_i \supset L\}$. For $\lambda \in \mathbb{C}$, define

$${\mathcal{D}E}(D) = \{\text{dense edges of } D\}, \quad {\mathcal{D}E}(D, \lambda) = \{L \in {\mathcal{D}E}(D) \mid \lambda^{m_L} = 1\}.$$
We say that $L, L'$ are strongly adjacent if $L \subset L'$ or $L \supset L'$ or $L \cap L'$ is non-dense.

Let

$$m(\lambda) = \max\{ |S| \mid S \subset \mathcal{D}(D, \lambda) \text{ such that } \text{any } L, L' \in S \text{ are strongly adjacent} \}.$$ 

8.8. Theorem [37]. $m_\alpha \leq m(\lambda)$ with $\lambda = e^{-2\pi i \alpha}$.

8.9. Corollary. $R_f \subset \bigcup_{L \in \mathcal{D}(D)} Z m_L^{-1}$.

8.10. Corollary. If $\gcd(m_L, m_{L'}) = 1$ for any strongly adjacent $L, L' \in \mathcal{D}(D)_f$, then $m_\alpha = 1$ for any $\alpha \in R_f \setminus Z$.

Theorem 2 follows from the canonical resolution of singularities $\pi : (\tilde{X}, \tilde{D}) \to (\mathbb{P}^{n-1}, D)$ due to [40], which is obtained by blowing up along the proper transforms of the dense edges. Indeed, mult $\tilde{D}(\lambda)_{\text{red}} \leq m(\lambda)$, where $\tilde{D}(\lambda)$ is the union of $\tilde{D}_i$ such that $\lambda^{\tilde{m}_i} = 1$ and $\tilde{m}_i = \text{mult}_{\tilde{D}_i} \tilde{D}$.

8.11. Theorem (Mustaţă [29]). For a central arrangement,

$$(8.11.1) \mathcal{J}(X, \alpha D) = I_0^k \text{ with } k = [\alpha] - n + 1 \text{ if } \alpha < \alpha'_f,$$

where $I_0$ is the ideal of 0 and $\alpha'_f = \min_{x \neq 0} \{ \alpha_{f,x} \}$.

(This holds for the affine cone of any divisor on $\mathbb{P}^{n-1}$, see [36].)

8.12. Corollary. We have $\dim F^{n-1} H^{n-1}(F_0, C)_{e(-k/d)} = \binom{n-1}{k-1}$ for $0 < \frac{k}{d} < \alpha'_f$, and the same holds with $F$ replaced by $\mathbb{P}$.

8.13. Corollary. $\alpha_f = \min(\alpha'_f, \frac{\alpha}{d}) < 1$.

(Note that $\alpha_f$ coincides with the minimal jumping number.)

8.14. Generic case. If $D$ is a generic central hyperplane arrangement, then

$$(8.14.1) b_f(s) = (s + 1)^{n-1} \prod_{j=n}^{2d-2}(s + \frac{j}{d})$$

by U. Walther [46] (except for the multiplicity of $-1$). He uses a completely different method.

Note that Theorems (8.2) and (8.8) imply that the left-hand side divides the right-hand side of (8.14.1), and the equality follows using also (8.12).

8.15. Explicit calculation. Let $\alpha = k/d$, $\lambda = e^{-2\pi i \alpha}$ for $k \in \{1, \ldots, d\}$. If $\alpha \geq \alpha'_f$, we assume there is $I \subset \{1, \ldots, d-1\}$ such that $|I| = k - 1$, and the condition of [40]

$$(8.15.1) \sum_{Z_i \supset L} \alpha_i \notin N \setminus \{0\} \text{ for any dense edge } L \subset Z,$$

is satisfied for

$$(8.15.2) \alpha_i = 1 - \alpha \text{ if } i \in I \cup \{d\}, \text{ and } -\alpha \text{ otherwise.}$$

Let $V(I)$ be the subspace of $H^{n-1} \mathcal{A}_\alpha$ generated by

$$\omega_{i_1} \wedge \cdots \wedge \omega_{i_{n-1}} \text{ for } \{i_1, \ldots, i_{n-1}\} \subset I.$$

8.16. Theorem. Let $\alpha = k/d$, $\lambda = e^{-2\pi i \alpha}$ for $k \in \{1, \ldots, d\}$. Then
8.17. Theorem [37]. Assume \( n = 3 \), \( \text{mult}_{x}Z \leq 3 \) for any \( z \in Z \subset \mathbb{P}^{2} \), and \( d \leq 7 \). Let \( \nu_{3} \) be the number of triple points of \( Z \), and assume \( \nu_{3} \neq 0 \). Then

\[
(8.17.1) \quad b_{f}(s) = (s + 1) \prod_{i=2}^{4}(s + \frac{1}{2}) \prod_{j=3}^{7}(s + \frac{1}{3}),
\]

with \( r = 2d - 2 \) or \( 2d - 3 \). We have \( r = 2d - 2 \) if \( \nu_{3} < d - 3 \), and the converse holds for \( d < 7 \). In case \( d = 7 \), we have \( r = 2d - 3 \) for \( \nu_{3} > 4 \), however, for \( \nu_{3} = 4 \), \( r \) can be both \( 2d - 2 \) and \( 2d - 3 \).

8.18. Remarks. (i) We have \( \nu_{3} < d - 3 \) if and only if

\[
(8.18.1) \quad \chi(U) = \frac{(d-2)(d-3)}{2} - \nu_{3} > \frac{(d-3)(d-4)}{2} = \binom{d-3}{2}.
\]

(ii) By (8.4.1) we have \( \chi(U) = h^{2}(F_{0}, C)_{\lambda} - h^{1}(F_{0}, C)_{\lambda} \if \lambda^{d} = 1 \text{ and } \lambda \neq 0. \fi \)

(iii) Let \( \nu'_{i} \) be the number of \( i \)-ple points of \( Z' := Z \cap C^{2} \). Then by [6]

\[
(8.18.2) \quad b_{0}(U) = 1, \quad b_{1}(U) = d - 1, \quad b_{2}(U) = \nu'_{2} + 2\nu'_{3},
\]

8.19. Examples. (i) For \((x^{2} - 1)(y^{2} - 1) = 0 \) in \( \mathbb{C}^{2} \) with \( d = 5 \), (8.17.1) holds with \( r = 7 \), and \( 8/5 \notin R_{f} \). In this case we do not need to take \( I \), because \( (d - 2)/d = 5/7 \notin R_{f} \). We have \( b_{1}(U) = b_{2}(U) = 4 \) and \( h^{2}(F_{0}, C)_{\lambda} = \chi(U) = 1 \) if \( \lambda^{d} = 1 \) and \( \lambda \neq 0 \). So \( j/5 \in R_{f} \) for \( 3 \leq j \leq 7 \) by (a), (b), (c), and \( 8/5 \notin R_{f} \) by (d).

(ii) For \((x^{2} - 1)(y^{2} - 1)(x + y) = 0 \) in \( \mathbb{C}^{2} \) with \( d = 6 \), (8.17.1) holds with \( r = 9 \), and \( 10/6 \notin R_{f} \). In this case we have \( b_{1}(U) = 5, b_{2}(U) = 6, \chi(U) = 2, h^{1}(F_{0}, C)_{\lambda} = 1, h^{2}(F_{0}, C)_{\lambda} = 3 \) for \( \lambda = e^{\pm 2\pi i/3} \). Then \( 4/6 \in R_{f} \) by (e) and \( 10/6 \notin R_{f} \) by (f), where \( I^{c} \) corresponds to \((x + 1)(y + 1) = 0 \). For other \( j/6 \), the argument is the same as in (i).

(iii) For \((x^{2} - y^{2})(x^{2} - 1)(y + 2) = 0 \) in \( \mathbb{C}^{2} \) with \( d = 6 \), (8.17.1) holds with \( r = 10 \), and \( 10/6 \in R_{f} \). In this case we have \( b_{1}(U) = 5, b_{2}(U) = 9, \chi(U) = 5, h^{1}(F_{0}, C)_{\lambda} = 0, \lambda^{2}(F_{0}, C)_{\lambda} = 5 \) for \( \lambda = e^{\pm 2\pi i/3} \). Then \( 4/6 \in R_{f} \) by (e) and \( 10/6 \in R_{f} \) by (c), where \( I^{c} \) corresponds to \((x + 1)(y + 2) = 0 \).

(iv) For \((x^{2} - y^{2})(x^{2} - 1)(y^{2} - 1) = 0 \) in \( \mathbb{C}^{2} \) with \( d = 7 \), (8.17.1) holds with \( r = 11 \), and \( 12/7 \notin R_{f} \). In this case we have \( b_{1}(U) = 6, b_{2}(U) = 9, \chi(U) = 4, h^{2}(F_{0}, C)_{\lambda} = 4 \) if \( \lambda^{7} = 1 \) and \( \lambda 
eq 1 \). Then \( 5/7 \in R_{f} \) by (e) and \( 12/7 \notin R_{f} \) by (f), where \( I^{c} \) corresponds to \((x + 1)(y + 1) = 0 \). Note that \( 5/7 \) is not a jumping number.
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INTRODUCTION TO A THEORY OF $b$-FUNCTIONS


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