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Ideal-adic tight closures and its applications

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INTRODUCTION

In 1980's, Hochster and Huneke introduced the notion of tight closure in positive characteristic. They gave a very short proof of Briançon–Skoda theorem (in positive characteristic), and proved that any pure subring of regular domain is Cohen–Macaulay using tight closures. Recently, tight closures give us a powerful technique in the theory of commutative algebra. However, we note that in the theory of tight closure, there are many open problems (Localization, completion, etc.); see e.g. [13], [17].

Fedder and Watanabe [6] defined the notion of F-rational rings using tight closures of parameter ideals, and have studied the relationship between them and rational singularities. Smith [29] proved that any excellent F-rational local ring is (pseudo–)rational, and Hara [7] and Mehta and Srinivas [25] proved the converse under some mild conditions.

In 2003, Hara and the author [11] introduced the notion of ideal-adic tight closure and defined the generalized test ideal, which is an analogue of the multiplier ideals in Algebraic Geometry (in equi-characteristic zero). Roughly speaking, the modulo p reduction of a multiplier ideal for large enough p coincides the generalized test ideal; see [11, Theorem 6.8]. On the other hand, there exist several differences between them. For instance, one needs the existence of resolution of singularities and some vanishing theorems to define the multiplier ideal. So one cannot define it in positive characteristic in the same manner. However, the generalized test ideal can be defined in any rings of positive characteristic.

The main purpose of this talk is to give a summary of the theory of ideal-adic tight closures and the generalized test ideals. Let us explain the organization of this report.

In Section 1, we summalize basic properties of tight closures associated descending filtration of ideals (e.g., ideal-adic tight closures). In particular, we prove an existence of $\mathfrak{a}_\bullet$-test element for all filtration $\mathfrak{a}_\bullet$ of ideals in an excellent reduced local ring.

In Section 2, we introduce the notion of the generalized test ideal $\tau(\mathfrak{a}_\bullet)$ with respect to filtration $\mathfrak{a}_\bullet$ of ideals, and give basic properties of them; see [11] and [9].

In Section 3, we give a proof of Skoda's theorem, which is an analogue of Skoda's theorem with respect to multiplier ideals; see e.g., [18, 19]. The simplest version of Skoda's theorem says that $\tau(\mathfrak{a}^n) = \tau(\mathfrak{a}^{n-1})\mathfrak{a}$ holds for all $n \geq d$ for any ideal $\mathfrak{a}$ of a d-dimensional complete local ring R.

In Section 4, we give a Howard-type theorem with respect to the generalized test ideals for monomial ideals; see [11, 3]. Let $\mathfrak{a}$ be a monomial ideal in a affine toric ring. Then one can define not only the generalized test ideal for $\mathfrak{a}$ but also the multiplier ideal for $\mathfrak{a}$. Our result implies that both ideals coincide. This is a generalization of Howald's theorem on the multiplier ideals of monomial ideals in a polynomial ring over a field k.

In Section 5, we prove the rationality of jumping exponents for any ideal in a polynomial ring over a perfect field k (after Blickle, Mustaţă and Smith [4]). As its application, we give an example of the multiplier ideal $\tau(f^{\mathrm{fpt}(f)})$ which is not integrally closed (not also radical!) in $k[x, y, z]$.

In Section 6, we give a brief explanation of *Restriction theorem* and *Subadditivity Theorem.*

In Section 7, we discuss about the question when $\tau(\mathfrak{a}) = \mathcal{J}(\mathfrak{a})$ holds in a 2-dimensional rational Gorenstein local domain. As a result, we prove that $\tau(\mathfrak{a})$ is integrally closed for any $\mathfrak{m}$-primary (integrally closed) ideal in such a ring. It seems to be open whether $\tau(\mathfrak{a}^t)$ is always integrally closed for a rational number t.

I was not able to discuss about some important properties of $\tau(\mathfrak{a}^t)$. I recommend that you refer Watanabe's, Hara's and Takagi's report for more details.

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1. Ideal-adic tight closures

1.1. **Basic properties.** Throughout this talk, let R be a Noetherian ring of characteristic $p > 0$, that is, R contains a prime field $F_p = \mathbb{Z}/p\mathbb{Z}$. The Frobenius map (denoted by F or F_R) is the homomorphism sending a to a^p. The ring R viewed as an R-module via the e-times iterated Frobenius map $F^e \colon R \to R$ $(a \mapsto a^{p^e})$ is denoted by ${}^eR$. Then the map $F^e \colon R \to {}^eR$ is an R-algebra homomorphism, and also it is identified with the natural inclusion map $R \hookrightarrow R^{1/p^e}$ provided that R is reduced. For an R-module M and $e \in \mathbb{N}$, we put $\mathbb{F}_R^e(M) = {}^eR \otimes_R M$ and regard it as an R-module by the action of $R = {}^eR$ from the left. Then we have the induced e-times iterated Frobenius map

$$F^e \colon M \to \mathbb{F}_R^e(M) \quad \left(m \mapsto m^{p^e} := F^e(m) := 1 \otimes m\right).$$

For an R-submodule N of M and $q = p^e$, we put

$$N_M^{[q]} = \mathrm{Im}(\mathbb{F}_R^e(N) \to \mathbb{F}_R^e(M)) = \mathrm{Ker}(\mathbb{F}_R^e(M) \to \mathbb{F}_R^e(M/N)).$$

Note that $\mathbb{F}_R^e(N) \to \mathbb{F}_R^e(M)$ is *not* injective in general. For an ideal I of R, $I^{[q]}$ denotes the ideal generated by all elements a^q for $a \in I$. Then $I^{[q]} = I_R^{[q]}$ in the above sense.

Let R^o denote the complement of the union of all minimal prime ideals of R. Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_n\}_{n \in \mathbb{N}}$ be a collection of ideals of R. Then $\mathfrak{a}_\bullet$ is called a (*descending*) *filtration of ideals in R* if the following conditions are satisfied:

$$\begin{aligned}
&(a) \quad \mathfrak{a}_m\mathfrak{a}_n \subseteq \mathfrak{a}_{m+n} \text{ for all } m,\, n \in \mathbb{N};\\
&(b) \quad \mathfrak{a}_1 \cap R^o \neq \emptyset;\\
&(c) \quad \mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \mathfrak{a}_3 \supseteq \cdots .
\end{aligned}$$

Example 1.1. Let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \cap R^o \neq \emptyset$. Then one can get several examples of descending filtration of ideals $\mathfrak{a}_\bullet$ as follows:

(1) $\mathfrak{a}_n = R$ for all $n \in \mathbb{N}$.
(2) $\mathfrak{a}_n = \mathfrak{a}^n$ for all $n \in \mathbb{N}$.
(3) $\mathfrak{a}_n = \overline{\mathfrak{a}^n}$ (the *integral closure* of $\mathfrak{a}^n$) for all $n \in \mathbb{N}$.
(4) $\mathfrak{a}_n = \mathfrak{a}^{\lceil tn \rceil}$ for all $n \in \mathbb{N}$, where t is a given positive real number.
(5) $\mathfrak{a}_n = \mathfrak{a}^{(n)} = \mathfrak{a}^n R_W \cap R$ (the *symbolic power* of $\mathfrak{a}^n$), where W is the complement of the union of all associated prime divisor of I over R.

Proof. We check the condition (a) in the case (4) only. Put $tm = M - \epsilon$ and $tn = N - \delta$, where M, N are integers and $0 \leq \epsilon, \delta < 1$. Then $\lceil tm \rceil = M$ and $\lceil tn \rceil = N$. Moreover, since $\lceil t(m+n) \rceil = \lceil M+N-(\epsilon+\delta) \rceil = M+N-\lfloor \epsilon+\delta \rfloor$, we have

$$\mathfrak{a}_m\mathfrak{a}_n = \mathfrak{a}^M\mathfrak{N} = \mathfrak{a}^{M+N} \subseteq \mathfrak{a}^{M+N-\lfloor \epsilon+\delta \rfloor} = \mathfrak{a}_{m+n},$$

as required. □

Remark 1.2. Let $\mathfrak{a}_\bullet$ be a descending filtration of ideals in R. Then for any real number $t > 0$, we can define another filtration of ideals $\mathfrak{a}^t_\bullet$ as follows: $\mathfrak{a}^t_n = \mathfrak{a}_{\lceil tn \rceil}$ for every $n \in \mathbb{N}$. Indeed, since $\lceil tm \rceil + \lceil tn \rceil \geq \lceil t(m+n) \rceil$ we have

$$\mathfrak{a}^t_m\mathfrak{a}^t_n = \mathfrak{a}_{\lceil tm \rceil}\mathfrak{a}_{\lceil tn \rceil} \subseteq \mathfrak{a}_{\lceil tm \rceil + \lceil tn \rceil} \subseteq \mathfrak{a}_{\lceil t(m+n) \rceil}.$$

For any (decending) filtration of ideals in R, we define the notion of $\mathfrak{a}_\bullet$-tight closure, which gives a slight generalization of ($\mathfrak{a}$-)tight closure.

Definition 1.3 ([11, 9]). Let $\mathfrak{a}_\bullet$ be a (decending) filtration of ideals in R, and let $N \subseteq M$ be R-modules. For $z \in M$,

$$z \in N_M^{*\mathfrak{a}_\bullet} \quad \overset{def}{\Longleftrightarrow} \quad \exists c \in R^o \text{such that } cz^q\mathfrak{a}_q \subseteq N_M^{[q]} \text{ for all } q = p^e,\ e \gg 0.$$

The R-module $N_M^{*\mathfrak{a}_\bullet}$ is called the *$\mathfrak{a}_\bullet$-tight closure* of N in M.

For an ideal I of R, we define $I^{*\mathfrak{a}_\bullet} = I_R^{*\mathfrak{a}_\bullet}$.

Remark 1.4. In case of $\mathfrak{a}_\bullet = \{R\}$, $N_M^{*\mathfrak{a}_\bullet} = N_M^*$ is the *tight closure* introduced by Hochster and Huneke [14]. Moreover, in case of $\mathfrak{a}_\bullet = \{\mathfrak{a}^{\lceil tn \rceil}\}$, $N_M^{*\mathfrak{a}_\bullet} = N^{*\mathfrak{a}^t}$ is the *$\mathfrak{a}^t$-tight closure* introduced by Hara and the author [11].

Remark 1.5 ([13, 14, 6]). A Noetherian ring R is called *weakly F-regular* if any ideal I is *tightly closed* (i.e., $I^* = I$). The ring R is *F-regular* if the localization R_P is weakly F-regular for any prime ideal P in R. In fact, R is weakly F-regular if and only if $R_\mathfrak{m}$ is weakly F-regular for all maximal ideals $\mathfrak{m}$. But it is *not* known whether any weakly F-regular ring is F-regular or not; see also [17].

Assume that R is reduced and *F-finite*, that is, ${}^1R$ is finitely generated as an R-module. Then R is *strongly F-regular* if for every element $c \in R^o$ there exists $q = p^e$ such that $c^{1/q}R \hookrightarrow R^{1/q}$ splits as an R-linear map. In fact, strongly F-regular ring are F-regular.

Suppose that $(R, \mathfrak{m})$ is local. Then R is *F-rational* if any parameter ideal is tightly closed. Clearly, weakly F-regular rings are F-rational. Moreover, F-rational rings are normal and Cohen–Macaulay if it is a homomorphic image of a Cohen–Macaulay local ring.

Lemma 1.6. *Let $\mathfrak{a}_\bullet$ be a (descending) filtration of ideals in R. Let $N \subseteq M$ be R-modules. Then*

1. $N \subseteq N_M^* \subseteq N_M^{*\mathfrak{a}_\bullet} \subseteq M$.
2. *If $N \subseteq N' \subseteq M$, then $N^{*\mathfrak{a}_\bullet} \subseteq (N')^{*\mathfrak{a}_\bullet}$. In particular, $N^{*\mathfrak{a}_\bullet} \subseteq (N^{*\mathfrak{a}_\bullet})^{*\mathfrak{a}_\bullet}$.*
3. *If $\mathfrak{b}_\bullet \subseteq \mathfrak{a}_\bullet$, then $N^{*\mathfrak{b}_\bullet} \supseteq N^{*\mathfrak{a}_\bullet}$. If, in addition, $\mathfrak{b}_\bullet$ is a reduction of $\mathfrak{a}_\bullet$, that is, there exists $k \in \mathbb{N}$ such that $\mathfrak{b}_n \subseteq \mathfrak{a}_n \subseteq \mathfrak{b}_{n-k}$ for every integer $n > k$, then equality holds.*
4. $(0)_{M/N}^{*\mathfrak{a}_\bullet} = N_M^{*\mathfrak{a}_\bullet}/N$.
5. *Put $(0)_M^{*\mathfrak{a}_\bullet fg} := \bigcup (0)_{M'}^{*\mathfrak{a}_\bullet}$, where M' runs through all finitely generated R-submodules of M which contains N. Then $(0)_M^{*\mathfrak{a}_\bullet fg} \subseteq (0)_M^{*\mathfrak{a}_\bullet}$.*

Proof. We prove the latter statement of (3) only. Suppose that there exists an integer $k \in \mathbb{N}$ such that $\mathfrak{b}_n \subseteq \mathfrak{a}_n \subseteq \mathfrak{b}_{n-k}$ for every integer $n > k$. Then we must show that $N^{*\mathfrak{b}_\bullet} \subseteq N^{*\mathfrak{a}_\bullet}$. Let $z \in N^{*\mathfrak{b}_\bullet}$. Then there exists an element $c \in R^o$ such that $cz^q\mathfrak{b}_q \in N_M^{[q]}$ for all $q = p^e \gg 0$. Take an element $d \in \mathfrak{a}_k \cap R^o$ and fix it. Then since $d\mathfrak{a}_q \subseteq \mathfrak{a}_{q+k} \subseteq \mathfrak{b}_q$, we have $(cd)z^q\mathfrak{a}_q \subseteq N_M^{[q]}$, as required. □

The following property is known as "*Contraction property*" in the theory of tight closure.

Proposition 1.7 (Contraction). *Let $R \hookrightarrow S$ be a module-finite extension of Noetherian domains. Let $\mathfrak{a}_\bullet$ be a filtration of ideals in R. For an ideal I of R, we have*

$$I^{*\mathfrak{a}_\bullet} = (IS)^{*\mathfrak{a}_\bullet} \cap R.$$

Proof. We omit it here. □

For example, we can apply the above proposition to the case of excellent local domains.

Example 1.8. Let R be an excellent local domain. Let $\overline{R}$ denote the integral closure of R in the quotient field of R. Let $\mathfrak{a}_\bullet$ be a filtration of ideals in R. For an ideal I of R, we have

$$I^{*\mathfrak{a}_\bullet} = (I\overline{R})^{*\mathfrak{a}_\bullet} \cap R.$$

1.2. **Test element.** In this subsection, we prove an existence of $\mathfrak{a}_\bullet$-test element for any filtration $\mathfrak{a}_\bullet$ under some mild conditions.

Now let us recall the definition of $\mathfrak{a}_\bullet$-test element.

Definition 1.9. Let $\mathfrak{a}_\bullet$ be a filtration of ideals in R. An element $c \in R^o$ is called an *$\mathfrak{a}_\bullet$-test element* of R if $cz^q\mathfrak{a}_q \subseteq I^{[q]}$ holds whenever I is an ideal of R and $z \in I^{*\mathfrak{a}_\bullet}$ and $q = p^e$.

Remark 1.10. Suppose that R is an excellent reduced local ring. Let $c \in R^o$ be an $\mathfrak{a}_\bullet$-test element of R. Then for any finitely generated R-module M, $cz^q\mathfrak{a}_q = 0$ holds in $\mathbb{F}_R^e(M)$ whenever $z \in (0)_M^{*\mathfrak{a}_\bullet}$ and $q = p^e$,

Theorem 1.11 (Existence of $\mathfrak{a}_\bullet$-test element). *Let R be a Noetherian reduced ring, and $c \in R^o$. Assume that either one of the following conditions holds:*

1. *R is F-finite and R_c is strongly F-regular.*
2. *R is of finite type over an excellent local ring and R_c is Gorenstein F-regular (e.g., regular).*

Then there exists a power c^n ($n \in \mathbb{N}$) is an $\mathfrak{a}_\bullet$-test element for any filtration $\mathfrak{a}_\bullet$ of ideals in R. Note that n is independent on $\mathfrak{a}_\bullet$.

In particular, any excellent reduced local ring has an element $c \in R^o$ which is an $\mathfrak{a}_\bullet$-test element for any filtration $\mathfrak{a}_\bullet$ of ideals in R.

Proof. Let us give a sketch of the proof of (2) in the case R_c is regular.

Step 1.: F-finite case.

The following lemma due to Hochster and Huneke is a key lemma in the proof.

Lemma 1.12 ([13]). *Assume that R is F-finite and reduced. If the localization R_c of R at $c \in R^o$ is strongly F-regular, then there exists an integer $n \geq 0$, depending only on R and c, satisfying the following property: For any $d \in R^o$, there exists a power q' of p and an R-linear map $\varphi\colon R^{1/q'} \to R$ sending $d^{1/q'}$ to c^n.*

Assume that R is F-finite, $c \in R^o$ and R_c is strongly F-regular. In what follows, we fix an intger $n \geq 0$ for which the above lemma holds. We claim that c^n is an $\mathfrak{a}_\bullet$-test element. Let I be an ideal of R and $z \in I^{*\mathfrak{a}_\bullet}$. Fix $q = p^e$. By definition, there exists $d \in R^o$ such that $dz^Q\mathfrak{a}_Q \subseteq I^{[Q]}$ for all $Q = p^E$. By the above lemma, we can take a power $q' = p^{e'}$ and an R-linear map $\varphi\colon R^{1/q'} \to R$ sending $d^{1/q'}$ to c^n. For such a power $q' = p^{e'}$ we have

$$dz^{qq'}\mathfrak{a}_q^{[q']} \subseteq dz^{qq'}\mathfrak{a}_{qq'} \subseteq (I^{[q]})^{[q']}.$$

Taking a q'-root and applying φ to both sides, we obtain that $c^nz^q\mathfrak{a}_q \subseteq I^{[q]}$, as required. Hence we get (1).

Step 2.: excellent case.

Assume that R is of finite type an excellent local ring B, and that R_c is Gorenstein, F-regular.

First we assume that B is complete local. Using Γ-construction argument (see [15]), we can find an F-finite reduced Noetherian local ring R^Γ such that $R \to R^\Gamma$ is faithfully flat, $(R_\Gamma)_c$ is Gorenstein, F-regular (and thus strongly F-regular). By Step 1, there exists an integer $n \geq 0$ such that c^n is $\mathfrak{a}_\bullet R^\Gamma$-test element in R_Γ. This yields that c^n is an $\mathfrak{a}_\bullet$-test element in R. Indeed, let $z \in I^{*\mathfrak{a}_\bullet}$ and $q = p^e$. Then $c^n z^q \mathfrak{a}_q \subseteq I^{[q]}R^\Gamma \cap R = I^{[q]}$.

Next, we consider the general case. Assume that R_c is regular [1]. Put $R' = R \otimes_B \widehat{B}$, where $\widehat{B}$ is the $\mathfrak{m}_B$-adic completion of B. Since B is excellent, if R_c is regular, so is R'_c. By Step 2, there exists an integer $n \geq 0$ such that c^n is an $\mathfrak{a}_\bullet R'$-test element. Since $R \to R'$ is faithfully flat, one can conclude that c^n is an $\mathfrak{a}_\bullet$-test element using a similar argument as above. □

1.3. **Completion.** In the theory of tight closure, it is important problem to show that any tight closure of an excellent local ring commute with completion. In this subsection, we discuss about a similar problem for $\mathfrak{a}_\bullet$-tight closure.

The following proposition gives a partial answer, which is known for original tight closures.

Proposition 1.13 (Commute with completion). *Let R be an excellent reduced local ring, and I an $\mathfrak{m}$-primary ideal. Then for any filtration $\mathfrak{a}_\bullet$ of ideals, we have $I^{*\mathfrak{a}_\bullet}\widehat{R} = (I\widehat{R})^{*\mathfrak{a}_\bullet}$.*

Proof. Since R is excellent and reduced, there exists an element $c \in R^o$ such that R_c and $\widehat{R}_c$ are regular rings. By Theorem 1.11, we can pick $d = c^n$ such that d is an $\mathfrak{a}_\bullet$-test element and $\mathfrak{a}_\bullet\widehat{R}$-test element. This yileds that $(I\widehat{R})^{*\mathfrak{a}_\bullet} \cap R = I^{*\mathfrak{a}_\bullet}$. In particular, if I is $\mathfrak{m}$-primary, then

$$(I\widehat{R})^{*\mathfrak{a}_\bullet} = ((I\widehat{R})^{*\mathfrak{a}_\bullet} \cap R)\widehat{R} = I^{*\mathfrak{a}_\bullet}\widehat{R},$$

as required. □

Remark 1.14. There exists a non-excellent local ring for which the above proposition does not hold; Loepp and Rotthaus [22]. It is difficult to drop the assumption that $\mathfrak{a}$ is $\mathfrak{m}$-primary. For non-reduced case, the problem remains open.

We do not have any proof without reducedness of R in Proposition 1.13. But in the case of ideal-adic filtration, we can drop this assumption by virtue of the following lemma.

Lemma 1.15 (Reduction to the domain case). *Let $t \geq 0$ be a real number and $\mathfrak{a}$ an ideal of R. Let $\mathfrak{a}_\bullet$ be a filtration of ideals defined by $\mathfrak{a}_n = \mathfrak{a}^{\lceil tn \rceil}$. Then for an ideal I of R, we have*

(1) $I^{*\mathfrak{a}_\bullet}/\sqrt{0} = I^{*\mathfrak{a}_\bullet}R_{\mathrm{red}} = (IR_{\mathrm{red}})^{*\mathfrak{a}_\bullet}$.

(2) *For any $z \in R$, $z \in I^{*\mathfrak{a}_\bullet}$ if and only if $z + \mathfrak{p} \in (I \cdot R/\mathfrak{p})^{*\mathfrak{a}_\bullet}$.*

[1]Note that you can relax this assumption to that R_c is Gorenstein F-regular by the argument as in [15, Section 7]

Proof. (1) It suffices to show that $(IR_{\mathrm{red}})^{*\mathfrak{a}_\bullet} \subseteq I^{*\mathfrak{a}_\bullet} R_{\mathrm{red}}$. Take a power $q' = p^{e'}$ such that $(\sqrt{0})^{[q']} = 0$. Since $\mathfrak{a} \cap R^o \neq \emptyset$, we can choose elements $a_1, \ldots, a_n \in \mathfrak{a} \cap R^o$ which generates $\mathfrak{a}$. Put $a = a_1^{2q'} \cdots a_n^{2q'} \in R^o$. Then for any $q = p^e$,

$$a\mathfrak{a}^{\lceil tqq' \rceil} \subseteq (\mathfrak{a}^{\lceil tq \rceil})^{[q']} = (\mathfrak{a}^{[q']})^{\lceil tq \rceil}.$$

Now suppose that $z + \sqrt{0} \in (IR_{\mathrm{red}})^{*\mathfrak{a}_\bullet}$. Then there exists $c \in R^o$ such that $cz^q \mathfrak{a}^{\lceil tq \rceil} \subseteq I^{[q]} + \sqrt{0}$. This yields that $c^{q'} z^{qq'} (\mathfrak{a}^{\lceil tq \rceil})^{[q']} \subseteq I^{[qq']}$. Since the left-hand side contains $ac^{q'} z^{qq'} \mathfrak{a}^{\lceil tqq' \rceil}$, we get the required result.

(2) We omit the proof. □

Question 1.16. *Does the lemma hold for general filtration $\mathfrak{a}_\bullet$ of ideals ?*

1.4. **Localization.** The localization problem in the theory of tight closure remains open in general.

Question 1.17. *Let W be a multiplicatively closed subset in R. For any ideal I of R, does $I^* R_W = (IR_W)^*$ hold?*

Of course, we can generalize the above question to the $\mathfrak{a}_\bullet$-tight closure. As a partial answer, we can prove the following theorem:

Theorem 1.18. *Assume that R is regular. Then the above question is true.*

Moreover, it is not difficult to generalize the above theorem to the case where pure subrings of regular rings. On the other hand, the above theorem also follows from the following theorem (under some mild conditions).

Theorem 1.19. *Let $(R, \mathfrak{m})$ be an excellent regular local ring. Let $\mathfrak{a}_\bullet$ be a filtration of ideals in R. Then for any element $c \in R^o$ there exists $e_0 = e_0(c, \mathfrak{a}_\bullet)$ such that for any ideal I of R and $z \in R$, $z \in I^*$ holds whenever $cz^q \mathfrak{a}_q \subseteq I^{[q]}$ for some $q = p^e$, $e \geq e_0$.*

2. A GENERALIZATION OF TEST IDEALS

In the theory of tight closure, Hochster and Huneke [14] defined the test ideal of R:

$$\tau(R) = \bigcap_{M:\text{f.g.}A\text{-module}} \mathrm{Ann}_R(0)^*_M.$$

In fact, the test ideal $\tau(R)$ is generated by test elements, and all elements of $\tau(R) \cap R^o$ are test elements of R.

In this section, we introduce the notion of the generalized test ideals, which gives an analogue of multiplier ideals; see [11].

Definition 2.1. Put $E = \oplus_{\mathfrak{m} \in \mathrm{Max}(R)} E_R(R/\mathfrak{m})$. Let $\mathfrak{a}_\bullet$ be a filtration of ideals in R. Then we define

$$\tau(\mathfrak{a}_\bullet) = \bigcap_M \mathrm{Ann}_R(0)^{*\mathfrak{a}_\bullet}_M = \mathrm{Ann}_R(0)^{*\mathfrak{a}_\bullet^{\mathrm{fg}}}_E,$$

where M runs through all finitely generated A-modules (in E). We call this ideal *the (generalized) test ideal with respect to* $\mathfrak{a}_\bullet$.

On the other hand, we define $\widetilde{\tau}(\mathfrak{a}_\bullet) = \mathrm{Ann}_R(0)^{*\mathfrak{a}_\bullet}_E$.

For an ideal $\mathfrak{a}$ of R and a real number $t \geq 0$, we use $\tau(\mathfrak{a}^t)$ instead of $\tau(\{\mathfrak{a}^{\lceil tn \rceil}\})$.

Remark 2.2. There is a little bit difference between the generalized test ideal and the original one.

(1) In general, $\tau(\mathfrak{a}_\bullet) \cap R^o \neq \{c \in R^o \ : \ c \text{ is an } \mathfrak{a}_\bullet\text{-test element.}\}$. For example, if R is regular, the right-hand side is equal to R^o, but the left-hand side is not.

(2) Assume that R is a Gorenstein local ring. Let $a_1, \ldots, a_d$ be a system of parameters. Then one can compute $\tau(\mathfrak{a}_\bullet)$ as follows:

$$\tau(\mathfrak{a}_\bullet) = \bigcap_{t=1}^{\infty} (a_1^t, \ldots, a_d^t) : (a_1^t, \ldots, a_d^t)^{*\mathfrak{a}_\bullet}.$$

More generally, if R is an approximately Gorenstein local ring, then there exists a sequence of ideals I_t such that

(a) $I_1 \supseteq I_2 \supseteq \cdots$.

(b) R/I_t is a 0-dimensional Gorenstein local ring.

(c) For any integer $n \geq 1$, there exists an intger $t \geq 1$ such that $I_t \subseteq \mathfrak{m}^n$.

(d) $\tau(\mathfrak{a}_\bullet) = \bigcap_{t=1}^{\infty} I_t \colon I_t^{*\mathfrak{a}_\bullet}$.

(3) Assume that $(R, \mathfrak{m})$ is a complete local ring. Then

$$\mathrm{Ann}_E\, \tau(\mathfrak{a}_\bullet) = \bigcup_M (0)^{*\mathfrak{a}_\bullet}_M,$$

where M runs through all finitely generated submodules of $E = E_R(R/\mathfrak{m})$. This follows from the Matlis duality theorem. In fact, if R is a complete local ring, then $W = \mathrm{Ann}_E(\mathrm{Ann}_R\, W)$ holds for any submodule W of E.

In the next section, we give a way to compute $\tau(\mathfrak{a}^t)$ for monomial ideals.

Example 2.3. Let $R = k[[x,y]]$ be a formal power series ring over a field k. Let $\mathfrak{m} = (x,y)R$ and $\mathfrak{a} = (x^3, xy, y^3)R$. Then for each real number $t \geq 0$,

$$\tau(\mathfrak{a}^t) = \begin{cases} R, & & (0 \leq t < 1) \\ \mathfrak{m} & = (x,y), & (1 \leq t < \frac{4}{3}) \\ \mathfrak{m}^2 & = (x^2, xy, y^2), & (\frac{4}{3} \leq t < \frac{5}{3}) \\ \mathfrak{a} & = (x^3, xy, y^3), & (\frac{5}{3} \leq t < 2) \\ \mathfrak{a}\mathfrak{m} & = (x^4, x^2y, xy^2, y^4), & (2 \leq t < \frac{7}{3}) \\ \cdots & \cdots & \cdots \end{cases}$$

$$\tau(\mathfrak{m}^t) = \begin{cases} R, & (0 \le t < 2) \\ \mathfrak{m} \quad = (x, y), & (2 \le t < 3) \\ \mathfrak{m}^2 \quad = (x^2, xy, y^2), & (3 \le t < 4) \\ \mathfrak{m}^3 \quad = (x^3, x^2y, xy^2, y^3), & (4 \le t < 5) \\ \mathfrak{m}^4 \quad = (x^4, x^3y, x^2y^2, xy^3, y^4), & (5 \le t < 6) \\ \cdots \quad \cdots & \cdots \end{cases}$$

Now let us gather basic properties of generalized test ideals.

Proposition 2.4. *Let $\mathfrak{a}$, $\mathfrak{b}$ be ideals of R such that $\mathfrak{a} \cap R^o \neq \emptyset$, $\mathfrak{b} \cap R^o \neq \emptyset$. Let $\mathfrak{a}_\bullet$, $\mathfrak{b}_\bullet$ be filtrations of ideals in R. Then the following statements hold:*

1. *If $\mathfrak{b}_\bullet \subseteq \mathfrak{a}_\bullet$, then $\tau(\mathfrak{b}_\bullet) \subseteq \tau(\mathfrak{a}_\bullet)$. If $\mathfrak{b}_\bullet$ is a reduction of $\mathfrak{a}_\bullet$, then equality holds.*
2. *$\tau(\mathfrak{a}_\bullet)\mathfrak{b} \subseteq \tau(\mathfrak{a}_\bullet\mathfrak{b})$, where $\mathfrak{a}_\bullet\mathfrak{b} = \{\mathfrak{a}_n\mathfrak{b}^n\}$ is a filtration of ideals.*
3. *If R admits a test element, then $\tau(\mathfrak{a}_\bullet) \cap R^o \neq \emptyset$.*
4. *Assume that R is weakly F-regular. Then $\mathfrak{a} \subseteq \tau(\mathfrak{a})$. If, in addition, $\mathfrak{a}$ has pure height 1, then equality holds.*

Proof. (1) It follows from $(0)_M^{*\mathfrak{b}_\bullet} \supseteq (0)_M^{*\mathfrak{a}_\bullet}$.

(2) Let M be a finitely generated A-module. If $z \in (0)_M^{*\mathfrak{a}_\bullet\mathfrak{b}}$, then there exists $c \in R^o$ such that $cz^q\mathfrak{a}_q\mathfrak{b}^q = 0$ for all sufficiently large $q = p^e$. This implies that $c(z^q\mathfrak{b}^{[q]})\mathfrak{a}_q = 0$ and thus $z \in [(0)_M^{*\mathfrak{a}_\bullet} : \mathfrak{b}]_M$. Hence $(0)_M^{*\mathfrak{a}_\bullet\mathfrak{b}} \subseteq [(0)_M^{*\mathfrak{a}_\bullet} : \mathfrak{b}]_M$.
If $c \in \tau(\mathfrak{a}_\bullet)$, then $c(0)_M^{*\mathfrak{a}_\bullet} = 0$ for any finitely generated A-module M. The above argument implies that $(c\mathfrak{b})(0)_M^{*\mathfrak{a}_\bullet\mathfrak{b}} = 0$. Therefore

$$\tau(\mathfrak{a}_\bullet)\mathfrak{b} \subseteq \bigcap_{M:\mathrm{f.g}} \mathrm{Ann}_R(0)_M^{*\mathfrak{a}_\bullet\mathfrak{b}} = \tau(\mathfrak{a}_\bullet\mathfrak{b})$$

by definition.

(3) It follows from $\tau(R)\mathfrak{a} \subseteq \tau(\mathfrak{a})$.

(4) The first statement immediately follows from (3). For the second statement, see [11, Proposition 1.11]. □

The following theorem is essetially due to Hara [9], which indicates the importance of ideal-adic filtration. Note that we need to assume that the filtration satisfies descending condition: $\mathfrak{a}_n \supseteq \mathfrak{a}_{n+1}$ in the proof of the second statement.

Theorem 2.5. *Assume that $(R, \mathfrak{m})$ is an excellent reduced local ring. Let $\mathfrak{a}_\bullet$ be a filtration of ideals in R. Then*

$$\tau(\mathfrak{a}_\bullet) = \max\{\tau(\mathfrak{a}_{p^e}^{1/p^e}) \ : \ e \in \mathbb{N}\} = \max\{\tau(\mathfrak{a}_\ell^{1/\ell}) \ : \ \ell \in \mathbb{N}\}.$$

Proof. First we prove the following claim.

Claim 1: $(0)_M^{*\mathfrak{a}_{k\ell}^{1/k\ell}} \subseteq (0)_M^{*\mathfrak{a}_k^{1/k}}$ for any finitely generated A-module M and for all integers $k, \ell > 0$. In particular, $\tau(\mathfrak{a}_k^{1/k}) \subseteq \tau(\mathfrak{a}_{k\ell}^{1/k\ell})$.

If we write $q = nk\ell - \epsilon$, where $0 \le \epsilon < k\ell$, then $\lceil \frac{q}{k\ell} \rceil = n$ and $\lceil \frac{q}{k} \rceil = \lceil n\ell - \frac{\epsilon}{k} \rceil = n\ell - \lfloor \frac{\epsilon}{k} \rfloor (> n\ell - \ell)$. Pick $a \in \mathfrak{a}_1 \cap R^o$ and fix it. Then we have

$$a^{k\ell}\mathfrak{a}_k^{\lceil q/k \rceil} \subseteq \mathfrak{a}_1^{k\ell}\mathfrak{a}_k^{n\ell - \lfloor \epsilon/k \rfloor} \subseteq \mathfrak{a}_k^{n\ell} \subseteq \mathfrak{a}_{k\ell}^{n} = \mathfrak{a}_{k\ell}^{\lceil q/k\ell \rceil}.$$

The claim follows from this. //

Since R is Noetherian, there exists a maximal element in $\{\tau(\mathfrak{a}_k^{1/k}) : k \in \mathbb{N}\}$. By Claim 1, such an element is unique.

Next, we prove the following claim.

Claim 2: $\tau(\mathfrak{a}_\ell^{1/\ell}) \subseteq \tau(\mathfrak{a}_\bullet)$ for every $\ell \in \mathbb{N}$.

Let $\ell \in \mathbb{N}$ and $q = p^e > \ell$ and fix them. If we write $q = n\ell - \epsilon$, where $0 \le \epsilon < \ell$, then $\lceil \frac{q}{\ell} \rceil = n$. Thus we get

$$\mathfrak{a}_\ell^{\lceil q/\ell \rceil} = \mathfrak{a}_\ell^n \subseteq \mathfrak{a}_{n\ell} \subseteq \mathfrak{a}_{n\ell-\epsilon} (= \mathfrak{a}_q),$$

where we need the descending property of $\mathfrak{a}_\bullet$ in the last inequality. (But one does *not* need to assume this property in the case ℓ is a power of p.) This implies that $(0)_M^{*\mathfrak{a}_\bullet} \subseteq (0)_M^{*\mathfrak{a}_\ell^{1/\ell}}$ for any finitely generated A-module M. Hence we get the assertion of the claim. //

Fix $q = p^e$ such that $\tau(\mathfrak{a}_k^{1/q}) = \max\{\tau(\mathfrak{a}_Q^{1/Q}) : Q \text{ is a power of } p\}$. In order to prove the theorem, it is enough to show that $\tau(\mathfrak{a}_\bullet) \subseteq \tau(\mathfrak{a}_q^{1/q})$.

Take $c \in \tau(R) \cap R^o$ and fix it. Let $z \in (0)_E^{*\mathfrak{a}_q^{1/q}\mathrm{fg}}$. Since R is excellent reduced (and thus approximately Gorenstein), there exists a sequence of finitely generated (cyclic) submodules $\{M_t\}_{t=1,2,\ldots}$ of $E = E_R(R/\mathfrak{m})$ such that $(0)_E^{*\mathfrak{a}_\bullet^{\mathrm{fg}}} = \bigcup_{t=1}^\infty (0)_{M_t}^{*\mathfrak{a}_\bullet}$. Hence $z \in (0)_{M_t}^{*\mathfrak{a}_q^{1/q}}$ for some $t \in \mathbb{N}$. Since M_t is Artinian, there exists a minimal element $(0)_{M_t}^{*\mathfrak{a}_{q'}^{1/q'}}$ of the set $\{(0)_{M_t}^{*\mathfrak{a}_Q^{1/Q}}\}$. In fact, this ideal is the minimum element in this set by Claim 1. If necessary, replacing q with qq', we may assume that $q' = q$. Hence $z \in (0)_{M_t}^{*\mathfrak{a}_{q''}^{1/q''}}$ for all powers $q'' = p^{e''}$ of p. In particular, $cz^{q''}\mathfrak{a}_{q''} = 0$ in $\mathbb{F}^{e''}(M_t)$. This implies that $z \in (0)_{M_t}^{*\mathfrak{a}_\bullet} \subseteq (0)_E^{*\mathfrak{a}_\bullet^{\mathrm{fg}}}$. Therefore $\tau(\mathfrak{a}_\bullet) \subseteq \tau(\mathfrak{a}_q^{1/q})$, as required. □

Remark 2.6. We have no counterexamples of filtrations for which the above theorem fails without its descending condition.

Many good properties of $\tau(\mathfrak{a}^t)$ can be derived from "$\tau(\mathfrak{a}^t) = \widetilde{\tau}(\mathfrak{a}^t)$". See [10] for more details. So the following theorem (see also [2, 30]) is very important.

Theorem 2.7 ([11, Theorem 1.13]). *Let $(R, \mathfrak{m})$ be an excellent normal local domain with $d = \dim R \ge 1$. Let $J \subseteq R$ be a divisorial ideal of R such that* $\mathrm{ord}(cl(J)) < \infty$. *Then*

$$(0)_{H_\mathfrak{m}^d(J)}^{*\mathfrak{a}_\bullet} = (0)_{H_\mathfrak{m}^d(J)}^{*\mathfrak{a}_\bullet^{\mathrm{fg}}}.$$

If, in addition, R is $\mathbb{Q}$-Gorenstein, we have that $\widetilde{\tau}(\mathfrak{a}_\bullet) = \tau(\mathfrak{a}_\bullet)$.

Proof. One can prove this similarly as in the argument of the proof of [11, Theorem 1.13]. □

Remark 2.8. Lyubeznik and Smith [23] (see also Blickle [3]) proved another result: Let R be an $\mathbb{N}$-graded ring with the unique homogensou maximal ideal $\mathfrak{m}$ and $\mathfrak{a}$ a homogeneous ideal of R such that $\mathfrak{a} \cap R^o \neq \emptyset$. Put $\mathfrak{a}_\bullet = \{\mathfrak{a}^{\lceil tn \rceil}\}$ for some positive real number t. Then $(0)_E^{*\mathfrak{a}_\bullet} = (0)_E^{*\mathfrak{a}_\bullet^{fg}}$, where $E = E(R/\mathfrak{m})$. Namely, we have $\widetilde{\tau}(\mathfrak{a}_\bullet) = \tau(\mathfrak{a}_\bullet)$. How about arbitrary filtration?

3. Skoda's theorem

The following theorem, which is called *Skoda's theorem*, is an analogy of one of fundamental theorems with respect to multiplier ideals. See also Hara's report for more details.

Theorem 3.1 (Skoda's theorem). *Assume that R is a complete local ring. Let $\mathfrak{a}$, $\mathfrak{b}$ be ideals of R such that $\mathfrak{a} \cap R^o \neq \emptyset$ and $\mathfrak{b} \cap R^o \neq \emptyset$. Moreover, suppose that $\mathfrak{a}$ has a reduction generated by at most ℓ elements. Then*

$$\tau(\mathfrak{a}^\ell \mathfrak{b}^t) = \tau(\mathfrak{a}^{\ell-1}\mathfrak{b}^t)\mathfrak{a}.$$

Remark 3.2. A similar result holds for any F-finite $\mathbb{Q}$-Gorenstein normal local domain; See [10].

In order to prove this theorem, we need the next lemma.

Lemma 3.3. *Assume that $(R, \mathfrak{m})$ is complete. Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_n\}$ be a filtration of ideals in R, and let $\mathfrak{b}$ be an ideal of R. Then*

$$\tau(\mathfrak{a}_\bullet)\mathfrak{b} = \bigcap_M \operatorname{Ann}_R[(0)_M^{*\mathfrak{a}_\bullet} : \mathfrak{b}]_M,$$

where M runs through all finitely generated A-submodules of $E = E_R(R/\mathfrak{m})$.

Proof. Suppose that $z \in \operatorname{Ann}_E(\tau(\mathfrak{a}_\bullet)\mathfrak{b})$. Then $z\mathfrak{b} \subseteq \operatorname{Ann}_E(\tau(\mathfrak{a}_\bullet)) = \bigcup_M (0)_M^{*\mathfrak{a}_\bullet}$ by Matlis duality. Hence there exists a finitely generated A-submodule M of E such that $z\mathfrak{b} \subseteq (0)_M^{*\mathfrak{a}_\bullet} \subseteq (0)_{M+Rz}^{*\mathfrak{a}_\bullet}$. If necessary, replacing M with $M + Rz$, we may assume that $z \in M$. Then $z \in \bigcup_{M\subseteq E}[(0)_M^{*\mathfrak{a}_\bullet} : \mathfrak{b}]_M$. Since the opposite inclusion is also true, we have

$$\operatorname{Ann}_E(\tau(\mathfrak{a}_\bullet)\mathfrak{b}) = \bigcup_{M\subseteq E} [(0)_M^{*\mathfrak{a}_\bullet} : \mathfrak{b}]_M.$$

Therefore

$$\tau(\mathfrak{a}_\bullet)\mathfrak{b} = \operatorname{Ann}_R \operatorname{Ann}_E(\tau(\mathfrak{a}_\bullet)\mathfrak{b}) = \bigcap_{M\subseteq E} [(0)_M^{*\mathfrak{a}_\bullet} : \mathfrak{b}]_M,$$

as required. □

Sketch of the proof of Theorem 3.1. First we show that

$$(0)_M^{*\mathfrak{a}^\ell\mathfrak{b}^t} = (0)_M^{*\mathfrak{a}^{\ell-1}\mathfrak{b}^t} : \mathfrak{a}$$

for every finitely generated A-submodule M of E and for every $\ell \geq \mu(\mathfrak{a})$. Then by the above lemma, we get

$$\tau(\mathfrak{a}^\ell\mathfrak{b}^t) = \bigcap_{M \subseteq E} \operatorname{Ann}_R((0)_M^{*\mathfrak{a}^\ell\mathfrak{b}^t}) = \bigcap_{M \subseteq E} \operatorname{Ann}_R((0)_M^{*\mathfrak{a}^{\ell-1}\mathfrak{b}^t} : \mathfrak{a}) = \tau(\mathfrak{a}^{\ell-1}\mathfrak{b}^t)\mathfrak{a}.$$

□

Corollary 3.4. *Assume that $(R, \mathfrak{m})$ is complete. Put $d = \dim R \geq 1$. Then for any ideal $\mathfrak{a}$ of R such that $\mathfrak{a} \cap R^o \neq \emptyset$, we have*

$$\tau(\mathfrak{a}^t) = \tau(\mathfrak{a}^{d-1+\epsilon})\mathfrak{a}^{n-d+1},$$

where $n = \lfloor t \rfloor$ and $\epsilon = t - n$.

Remark 3.5. In the hypothesis of the above corollary, one can replace "$d = \dim R \geq 1$" with "$\mu(\mathfrak{a}) \geq 1$" or "$\mathfrak{a}$ has a reduction generated by at most d elements".

Applying Skoda's theorem to the ideal $\tau(\mathfrak{a}^t)$ in Example 2.3, we obtain the following result.

Example 3.6. Let $R = k[[x, y]]$ be a formal power series ring over a field k. Let $\mathfrak{m} = (x, y)R$ and $\mathfrak{a} = (x^3, xy, y^3)R$. For each real number $t \geq 0$, if we put $n = \lfloor t \rfloor$, then

$$\tau(\mathfrak{a}^t) = \begin{cases} R, & & (0 \leq t < 1) \\ \mathfrak{m}\mathfrak{a}^{n-1} & = \tau(\mathfrak{a}^n), & (n \leq t < n + \frac{1}{3}) \\ \mathfrak{m}^2\mathfrak{a}^{n-1} & = \mathfrak{a}^n : \mathfrak{m}, & (n + \frac{1}{3} \leq t < n + \frac{2}{3}) \\ \mathfrak{a}^n, & & (n + \frac{2}{3} \leq t < n + 1) \end{cases}$$

Briançon-Skoda's theorem for F-regular rings can be derived from Skoda's theorem. See below.

Corollary 3.7. *Assume that $(R, \mathfrak{m})$ is a complete local ring of characteristic $p > 0$. Suppose that $\mathfrak{a}$ has a reduction generated by ℓ elements. Then*

$$\tau(R)\overline{\mathfrak{a}^{n+\ell-1}} \subseteq \mathfrak{a}^n,$$

where $\overline{\mathfrak{b}}$ denotes the integral closure of $\mathfrak{b}$.

Proof. Note that $\tau(\mathfrak{b}^n) = \tau(\overline{\mathfrak{b}^n})$ holds since $\{\mathfrak{b}^n\}$ is a reduction of $\{\overline{\mathfrak{b}^n}\}$. Then by Skoda's theorem and basic property of τ, we get

$$\tau(R)\overline{\mathfrak{a}^{n+\ell-1}} \subseteq \tau(\overline{\mathfrak{a}^{n+\ell-1}}) = \tau(\mathfrak{a}^{n+\ell-1}) = \tau(\mathfrak{a}^{\ell-1})\mathfrak{a}^n \subseteq \mathfrak{a}^n.$$

□

Remark 3.8. *Tight Closure Briançon–Skoda theorem* says that $\overline{\mathfrak{a}^{n+\ell-1}} \subseteq (\mathfrak{a}^n)^*$ under the same assumption in Corollary 3.7. (Precisely speaking, one does *not* need to assume that R is complete.)

On can also obtain the above corollary from Tight closure Briançon–Skoda theorem because $\tau(R)\mathfrak{b}^* \subseteq \mathfrak{b}$ by definition.

4. Howald type theorem–generalized test ideals of monomial ideals

Example 4.1. Let $R = k[x, y]$ be a polynomial ring over a field k. Let $\mathfrak{m} = (x, y)R$ and $\mathfrak{a} = (x^3, xy, y^3)R$. For each real number $t \geq 0$, if we put $n = \lfloor t \rfloor$, then

$$\tau(\mathfrak{a}^t) = \begin{cases} R, & (0 \leq t < 1) \\ \mathfrak{m}\mathfrak{a}^{n-1} = \tau(\mathfrak{a}^n), & (n \leq t < n + \frac{1}{3}) \\ \mathfrak{m}^2\mathfrak{a}^{n-1} = \mathfrak{a}^n : \mathfrak{m}, & (n + \frac{1}{3} \leq t < n + \frac{2}{3}) \\ \mathfrak{a}^n, & (n + \frac{2}{3} \leq t < n + 1) \end{cases}$$

I will explain how to determine $\tau(\mathfrak{a}^t)$ as above. Before doing that, we recall Howald's theorem, which gives a combinatorial description of the multiplier ideal $\mathcal{J}(\mathfrak{a}^t)$ of a monomial ideal in a polynomial ring over a field.

Theorem 4.2 (Howald). *Let $\mathfrak{a}$ be a monomial ideal in a polynomial ring $R = k[x_1, \ldots, x_d]$ over a field k. For a real number $t \geq 0$, we have*

$$\mathcal{J}(\mathfrak{a}^t) = (x^m \ : m + (1, 1, \ldots, 1) \in \operatorname{Int}(t \cdot \operatorname{Newt}(\mathfrak{a}))),$$

where the Newton polytope $\operatorname{Newt}(\mathfrak{a})$ *denotes the convex hull of $\{m \in \mathbb{N}^d : X^m \in \mathfrak{a}\}$ in $\mathbb{R}^d$, and* $\operatorname{Int}(X)$ *denotes the relative interior of a subset $X \in \mathbb{R}^d$.*

Question 4.3. *How about $\tau(\mathfrak{a}^t)$?*

The purpose of this section is to give an answer to this question. In fact, $\tau(\mathfrak{a}^t)$ admits a similar description in a more general situation.

Let $M = \mathbb{Z}^d$, $N = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ and denote the duality pairing of $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ with $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ by $\langle\ ,\ \rangle \colon M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$. Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone, and let $n_1, \ldots, n_s \in N$ be a primitive generators of σ. Then

$$\sigma^\vee = \{m \in M_{\mathbb{R}} \ : \ \langle m, n \rangle \geq 0 \ \text{ for all } n \in \sigma\} = \bigcap_{i=1}^{s} \{m \in M_{\mathbb{R}} \ : \ \langle m, n_i \rangle \geq 0\}.$$

Let $R = k[\sigma^\vee \cap M] \subseteq k[x_1^{\pm}, \ldots, x_d^{\pm}]$ denote the *toric ring* over a field k defined by σ. A toric ideal $\mathfrak{a} \subseteq R$ is an ideal of R generated by monomials in $x_1, \ldots, x_d$. Let $\mathfrak{a} \subseteq R$ be a toric ideal, and let $P = \operatorname{Newt}(\mathfrak{a}) \subset M_{\mathbb{R}}$ be the Newton polygon of $\mathfrak{a}$, that is, the convex hull of $\{m \in M \ : \ x^m \in \mathfrak{a}\}$ in $M_{\mathbb{R}}$. We denote the relative interior of P in $M_{\mathbb{R}}$ by $\operatorname{Int}(P)$. Note that R is $\mathbb{Q}$-Gorenstein if and only if there exists $w \in M_{\mathbb{R}}$ such that $\langle w, n_i \rangle = 1$ for $i = 1, \ldots, s$.

Example 4.4.
1. Put $M = \mathbb{Z}^d$, $\sigma = \mathbb{R}_+(1, 0, \ldots, 0) + \cdots + \mathbb{R}_+(0, \ldots, 0, 1)$. Then $R = k[\sigma^\vee \cap M] = k[x_1, \ldots .x_d]$ is a polynomial ring, and one can take $w = (1, \ldots, 1)$.
2. Put $M = \mathbb{Z}^2$. Put $n_1 = (3, -1)$, $n_2 = (0, 1)$ and $\sigma = \mathbb{R}_+ n_1 + \mathbb{R}_+ n_2$. Then $k[\sigma^\vee \cap M] = k[s, st, st^2, st^3] \cong k[x^3, x^2y, xy^2, y^3]$. If we take $w = (\frac{2}{3}, 1)$, then $\langle w, n_1 \rangle = \langle w, n_2 \rangle = 1$.

The main result in this section is the following theorem.

Theorem 4.5. *Let M, N, σ be as above, and let $n_1, \dots, n_s \in N$ be a primitive generators of σ. Let $\mathfrak{a} \subseteq R$ be a toric ideal (monomial ideal) in R. Assume that R is $\mathbb{Q}$-Gorenstein, that is, there exists $w \in M_{\mathbb{R}}$ such that $\langle w, n_i \rangle = 1$ for all $i = 1, \dots, s$. Then for any real number $t \geq 0$, we have*

$$\tau(\mathfrak{a}^t) = \{x^m \in R \,:\, m \in M,\, m + w \in \mathrm{Int}(t \cdot \mathrm{Newt}(\mathfrak{a}))\}.$$

Corollary 4.6. *Let $R = k[x_1, \dots, x_d]$ or $k[[x_1, \dots, x_d]]$, and let $\mathfrak{a}$ be a monomial ideal of R. Then $\tau(\mathfrak{a}^t) = \mathcal{J}(\mathfrak{a}^t)$ is integrally closed for each $t \in \mathbb{R}$.*

Remark 4.7. The above theorem has been generalized to the class of non-$\mathbb{Q}$-Gorenstein toric rings by Blickle [3].

Sketch of the proof of Theorem 4.5. We now consider the Frobenius action on $E = H^d_{\mathfrak{m}}(R) = \mathrm{Hom}_k(R, k)$. Note that

$$E = \bigoplus_{\langle m, n_i \rangle \leq 0} kX^m, \qquad \mathbb{F}^e(E) = H^d_{\mathfrak{m}}(\omega_R^{(q)}) = \bigoplus_{\langle m, n_i \rangle \leq q-1} kX^m.$$

Then e-times Frobenius map is defined by

$$F^e \colon E \to \mathbb{F}^e(E) \ (X^m \mapsto X^{qm}),$$

where $q = p^e$.

In the following, we may assume that R is F-finite. We show that if $X^m \in \tau(\mathfrak{a}^t)$, then $m + w \in \mathrm{Int}(t \cdot \mathrm{Newt}(\mathfrak{a}))$. (The converse will be proved by a similar method.) Suppose that $X^m \in \tau(\mathfrak{a}^t)$. By assumption, $\tau(\mathfrak{a}^t) = \mathrm{Ann}_R(0)^{*\mathfrak{a}^t}_E$. Then one can easily see that $X^m \in \tau(\mathfrak{a}^t) \Longleftrightarrow X^{-m} \in (0)^{*\mathfrak{a}^t}_E$. Since 1 is a $\mathfrak{a}^t$-test element, there exists $q = p^e$ such that $X^{-qm}\mathfrak{a}^{\lceil tq \rceil} \neq (0)$ in $\mathbb{F}^e$. For such a rational number q,

$$(-qm + \lceil tq \rceil \cdot \mathrm{Newt}(\mathfrak{a})) \cap M \cap \{u \in M \,:\, \langle u, n_i \rangle \leq q - 1 (i = 1, \dots, s)\} \neq \emptyset.$$

Take $u \in M$ and $v \in \mathrm{Newt}(\mathfrak{a})$ such that $u = qm + \lceil tq \rceil v$ and $\langle u, n_i \rangle \leq q - 1$ for every $i = 1, \dots, s$. Since $\langle w - \frac{1}{q}u, n_i \rangle \geq \frac{1}{q} > 0$, we have

$$m + w = \left(t + \frac{\epsilon}{q}\right)v + \left(w - \frac{1}{q}u\right) \mathrm{Int}(t \cdot \mathrm{Newt}(\mathfrak{a})),$$

where $\epsilon = \lceil tq \rceil - tq$. The proof can be easily see the proof of this theorem. □

5. Discreteness and rationality of jumping exponents

In the previous section, we gave a way to compute the test ideals with respect to monomial ideals, but it is not so easy to compute the test ideals with respect to other ideals. In this section, we will introduce another way by Blickle, Mustaţă and Simth in [BMS]. As its application, we can show that discreteness and rationality of jumping exponents.

Throughout this section, let R be a Noetherian ring of characteristic $p > 0$ such that R is a finitely generated free R^p-module. For example, any regular local ring whose residue field is perfect (i.e., $k = k^p$) and any polynomial ring over a perfect field $k = k^p$ satisfies this condition. Conversely, any Noetherian ring which satisfies this condition is an F-finite regular ring.

Definition 5.1. A non-negative real number t is a *jumping exponent* for $\mathfrak{a}$ if $\tau(\mathfrak{a}^t) \neq \tau(\mathfrak{a}^{t-\epsilon})$ whenever $\epsilon > 0$.

Suppose that R is weakly F-regular, that is, $\tau(R) = R$. Then we put

$$\mathrm{fpt}(\mathfrak{a}) = \max\{t \in \mathbb{R}_{\geq 0} \,:\, \tau(\mathfrak{a}^t) = R\}.$$

We call it the *F-threshold for* $\mathfrak{a}$. Note that $\mathrm{fpt}(\mathfrak{a})$ is equal to the minimum jumping exponent.

Example 5.2. Let $R = k[x,y]$ (or $k[[x,y]]$) be a polynomial ring (a formal power series ring) over a field $k = k^p$. Let $\mathfrak{m} = (x,y)R$ and $\mathfrak{a} = (x^3, xy, y^3)R$. For each real number $t \geq 0$, if we put $n = \lfloor t \rfloor$, then

$$\tau(\mathfrak{a}^t) = \begin{cases} R, & (0 \leq t < 1) \\ \mathfrak{m}\mathfrak{a}^{n-1} \quad = \tau(\mathfrak{a}^n), & (n \leq t < n + \frac{1}{3}) \\ \mathfrak{m}^2\mathfrak{a}^{n-1} \quad = \mathfrak{a}^n : \mathfrak{m}, & (n + \frac{1}{3} \leq t < n + \frac{2}{3}) \\ \mathfrak{a}^n, & (n + \frac{2}{3} \leq t < n+1) \end{cases}$$

Thus the set of jumping exponents for $\mathfrak{a}$ is

$$\left\{ n,\, n + \frac{1}{3},\, n + \frac{2}{3} \ (n = 1, 2, \ldots) \right\}.$$

In fact, we will show that the set of jumping exponents for an ideal is discrete, and that the jumping exponent is a rational number.

5.1. **A variant of $\tau(\mathfrak{a}^t)$.** We first introduce the notion of $\tau'(\mathfrak{a}^t)$ as a variant of $\tau(\mathfrak{a}^t)$.

Lemma 5.3. *Let $e, r \geq 0$ be integers. Take a system of generators $h_1, \ldots, h_s$ of $\mathfrak{a}^r$. Take a free basis $e_1, \ldots, e_N$ of R over R^{p^e}, for each i we can write h_i as follows:*

$$h_i = \sum_{j=1}^{N} a_{ij}^{p^e} e_j$$

for some $a_{ij} \in R$. If we put

$$I_{r,e}(\mathfrak{a}) = (a_{ij} \,:\, 1 \leq i \leq s,\, 1 \leq j \leq N),$$

then $I_{r,e}(\mathfrak{a})$ is equal to the smallest ideal J for which $\mathfrak{a}^r \subseteq J^{[p^e]}$ holds.

In particular, $I_{r,e}(\mathfrak{a})$ is independent of the choice of free basis of R over R^{p^e} and generators of $\mathfrak{a}^r$.

Proof. By definition, $\mathfrak{a}^r \subseteq I_{r,e}(\mathfrak{a})^{[p^e]}$. So it is enough to show that if $\mathfrak{a}^r \subseteq J^{[p^e]}$, then $I_{r,e}(\mathfrak{a}) \subseteq J$. Suppose that $\mathfrak{a}^r \subseteq J^{[p^e]}$. Put $J = (g_1, \ldots, g_m)$. Since $\mathfrak{a}^r \subseteq J^{[p^e]}$, one can write as follows:

$$h_i = \sum_{\ell=1}^{m} g_\ell^{p^e} b_\ell$$

for some $b_\ell \in R$. Let $e_1^*, \ldots, e_N^* \in \mathrm{Hom}_{R^{p^e}}(R, R^{p^e})$ be a dual basis of $e_1, \ldots, e_N$. That is, $e_j^*(e_i) = \delta_{ij}$ for $i, j = 1, \ldots, N$. Then

$$a_{ij}^{p^e} = e_j^*(h_i) = \sum_{\ell=1}^{m} g_\ell^{p^e} e_j^*(b_\ell) \in J^{[p^e]}.$$

Thus $a_{ij} \in J$ because the Frobenius map is injective. Hence $I_{r,e}(\mathfrak{a}) \subseteq J$, as required. □

Lemma 5.4. *Let $\mathfrak{a}$, $\mathfrak{b}$ ideals of R, and let r, $r' \geq 0$ be integers.*

(1) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $I_{r,e}(\mathfrak{a}) \subseteq I_{r,e}(\mathfrak{b})$.*
(2) *If $r \leq r'$, then $I_{r,e}(\mathfrak{a}) \supseteq I_{r',e}(\mathfrak{a})$.*
(3) *$I_{r,e}(\mathfrak{a}) \subseteq I_{rp,e+1}(\mathfrak{a})$.*
(4) *If $e \leq e'$, $\dfrac{r}{p^e} \geq \dfrac{r'}{p^{e'}}$, then $I_{r,e}(\mathfrak{a}) \subseteq I_{r',e'}(\mathfrak{a})$.*
(5) *For a real number $t \geq 0$, we have $I_{\lceil tp^e \rceil,e}(\mathfrak{a}) \subseteq I_{\lceil tp^{e+1} \rceil,e+1}(\mathfrak{a})$.*

Proof. (1) By definition, $\mathfrak{a}^r \subseteq \mathfrak{b}^r \subseteq I_{r,e}(\mathfrak{b})^{[p^e]}$. Hence the minimality of $I_{r,e}(\mathfrak{a})$ yields that $I_{r,e}(\mathfrak{a}) \subseteq I_{r,e}(\mathfrak{b})$.

(2) By definition, $\mathfrak{a}^{r'} \subseteq \mathfrak{a}^r \subseteq I_{r,e}(\mathfrak{a})^{[p^e]}$. Hence the minimality of $I_{r',e}(\mathfrak{a})$ yields that $I_{r',e}(\mathfrak{a}) \subseteq I_{r,e}(\mathfrak{a})$.

(3) Since $(\mathfrak{a}^r)^{[p]} \subseteq \mathfrak{a}^{rp} \subseteq I_{rp,e+1}(\mathfrak{a})^{[p^{e+1}]}$, we have $\mathfrak{a}^r \subseteq I_{rp,e+1}(\mathfrak{a})^{[p^e]}$. Hence $I_{r,e}(\mathfrak{a}) \subseteq I_{rp,e+1}(\mathfrak{a})$.

(4) The assumption yields $rp^{e'-e} \geq r'$. Thus $I_{r,e}(\mathfrak{a}) \subseteq I_{rp^{e'-e},e'}(\mathfrak{a}) \subseteq I_{r',e'}(\mathfrak{a})$ by (3) and (2).

(5) Write $tp^e = N - \epsilon$, where $N \in \mathbb{Z}$ and $0 \leq \epsilon < 1$. Then since $tp^{e+1} = Np - \epsilon p$, we get

$$\frac{\lceil tp^{e+1} \rceil}{p^{e+1}} = \frac{Np - \lfloor \epsilon p \rfloor}{p^{e+1}} = \frac{N}{p^e} - \frac{\lfloor \epsilon p \rfloor}{p^{e+1}} \leq \frac{N}{p^e} = \frac{\lceil tp^e \rceil}{p^e}.$$

Therefore (5) follows from (4). □

Let us define a variant of "$\tau(\mathfrak{a}^t)$"as a limit of $I_{\lceil tp^e, \rceil, e}(\mathfrak{a})$.

Definition 5.5 ([4]). Let R be a Noetherian ring of characteristic $p > 0$. Suppose that R is a finitely generated free R^p-module. Then for any ideal $\mathfrak{a}$ of R and a real number $t \geq 0$, we define

$$\tau'(\mathfrak{a}^t) = \bigcup_{e=1}^{\infty} I_{\lceil tp^e \rceil,e}(\mathfrak{a}).$$

Remark 5.6. By Lemma 5.4, $\{I_{\lceil tp^e \rceil,e}(\mathfrak{a})\}$ forms an increasing sequence of ideals in R, and it stabilizes. Hence $\tau'(\mathfrak{a}^t)$ is equal to $I_{\lceil tp^e \rceil,e}(\mathfrak{a})$ for sufficiently large $e \gg 0$.

Moreover, $\tau'(\mathfrak{a}^t)$ commutes with completion or localization.

For any F-finite regular local ring, this $\tau'(\mathfrak{a}^t)$ coincides the original one. Moreover, the similar result is true for polynomial rings via localization.

Proposition 5.7. *Assume that $(R,\mathfrak{m})$ is an F-finite regular local ring. Then for any ideal $\mathfrak{a}$ of R and any real number $t \geq 0$, we have $\tau'(\mathfrak{a}^t) = \widetilde{\tau}(\mathfrak{a}^t) = \tau(\mathfrak{a}^t)$.*

Proof. Since an F-finite regular local ring is an excellent $\mathbb{Q}$-Gorenstein normal local domain, we have that $\widetilde{\tau}(\mathfrak{a}^t) = \tau(\mathfrak{a}^t)$. So it is enough to show that $\tau'(\mathfrak{a}^t) = \mathrm{Ann}_R(0)_E^{*\mathfrak{a}^t}$, where $E = E_R(R/\mathfrak{m})$.

Put $Z_{r,e}(\mathfrak{a}) = \{\xi \in E : \mathfrak{a}^r \subseteq \mathrm{Ann}_R(\xi^{p^e})\}$ for all integers r, $e \geq 0$. We claim that $Z_{r,e}(\mathfrak{a}) = \mathrm{Ann}_E I_{r,e}(\mathfrak{a})$. In fact, for any $\xi \in E \cong H^d_{\mathfrak{m}}(R)$, we can write $\xi = \left[\frac{z}{(x_1\cdots x_d)^n}\right]$ for some $z \in R$ and $n \in \mathbb{N}$, where $\mathfrak{m} = (x_1,\dots,x_d)$ and $d = \dim R$. Then since $\mathrm{Ann}_R(\xi) = (x_1^n,\dots,x_d^n) : z$, we get

$$\mathrm{Ann}_R\, \xi^{p^e} = (x_1^{np^e},\dots,x_d^{np^e}) : z^{p^e} = (\mathrm{Ann}_R(\xi))^{[p^e]}.$$

Hence $Z_{r,e}(\mathfrak{a}) = \{\xi \in E : \mathfrak{a}^r \subseteq \mathrm{Ann}_R(\xi)^{[p^e]}\} = \{\xi \in E : I_{r,e}(\mathfrak{a}) \subseteq \mathrm{Ann}_R(\xi)\}$. That is, $Z_{r,e}(\mathfrak{a}) = \mathrm{Ann}_E(I_{r,e}(\mathfrak{a}))$, as required.

Since 1 is an $\mathfrak{a}^t$-test element and

$$(0)_E^{*\mathfrak{a}^t\mathrm{fg}} = (0)_E^{*\mathfrak{a}^t},$$

we have $(0)_E^{*\mathfrak{a}^t} = \bigcap_{e=1}^{\infty} Z_{\lceil tp^e\rceil,e}(\mathfrak{a})$. Thus we get

$$\begin{aligned}
\mathrm{Ann}_R((0)_E^{*\mathfrak{a}^t}) &= \mathrm{Ann}_R\Big(\bigcap_{e=1}^{\infty} Z_{\lceil tp^e\rceil,e}(\mathfrak{a})\Big)\\
&= \mathrm{Ann}_R\Big(\bigcap_{e=1}^{\infty} \mathrm{Ann}_E\, I_{\lceil tp^e\rceil,e}(\mathfrak{a})\Big)\\
&= \mathrm{Ann}_R\,\mathrm{Ann}_E\, I_{\lceil tp^e\rceil,e}(\mathfrak{a}) \quad (e \gg 0)\\
&= \mathrm{Ann}_R\,\mathrm{Ann}_E\, \tau'(\mathfrak{a}^t)\\
&= \tau'(\mathfrak{a}^t).
\end{aligned}$$

This completes the proof. □

This method enables us to compute $\tau(f^t)$ for any element f in an F-finite regular local ring, although it may be not so useful to compute $\tau(\mathfrak{a}^t)$ for monomial ideals. In fact, we can obtain the following, which provides an example of $\tau(f^c)$ where $c = \mathrm{fpt}(f)$ such that $\tau(f^c)$ is not integrally closed.

Example 5.8. Let $R = k[[x,y,z]]$ be a formal power series ring over a prime field $k = \mathbb{F}_p$. Put $f = x^p + y^{2p+1} + z^{2p+1}$. Then

(1) $\mathrm{fpt}(f) = \frac{1}{p}$.

(2) $\tau(f^{1/p}) = (x,y^2,z^2)$. This ideal is not integrally closed. In fact, $\overline{(x,y^2,z^2)} = (x,y^2,yz,z^2)$.

Proof. First we prove (2). In fact, since $f^{\lceil (1/p)p^e\rceil} = f^{p^{e-1}} = x^{p^e}\cdot 1 + y^{p^e}\cdot y^{p^{e-1}} + z^{p^e}\cdot z^{p^{e-1}}$ and 1, $y^{p^{e-1}}$, $z^{p^{e-1}}$ forms part of a free basis of R over R^{p^e}, we have that $I_{\lceil (1/p)p^e\rceil,e}(fR) = (x,y^2,z^2)$ for each $e \geq 1$ by definition. Hence $\tau(f^{1/p}) = (x,y^2,z^2)$. Moreover, it is clear that $\overline{(x,y^2,z^2)} = (x,y^2,yz,z^2)$.

Next we give a sketch of the proof that $\mathrm{fpt}(fR) = \frac{1}{p}$. In order to do that, it is enough to show that if $0 \le t < \frac{1}{p}$ then $\tau(f^t) = R$. Put $t = \frac{1}{p} - \epsilon$ for $0 < t < \frac{1}{p}$. Take a sufficiently large integer e for which $\epsilon \cdot p^e \ge 1$ holds. Then since the expansion of $f^{\lceil tp^e \rceil}$ contains a term $x^{\lceil tp^e \rceil}$, which is part of a free basis of R over R^{p^e}, we conclude that $1 \in I_{\lceil tp^e \rceil, e}(fR)$. Hence $\tau(f^t) = R$. $\square$

Remark 5.9. More generally, we can show that *any ideal is obtained in this form in an F-finite regular local ring* in some sense.

5.2. **Discreteness and rationality.** In this subsection, we give a sketch of the proof of the following theorem (see [4] for more general form) based on the idea of Blickle, Mustaţă and Smith. Note that a similar result is also known for multiplier ideals.

Theorem 5.10. *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field $k = k^p$. Let $\mathfrak{a}$ be an ideal of R. Then*

(1) *The set of jumping exponents for $\mathfrak{a}$ is discrete, that is, in every finite interval, there are only finitely many such numbers.*
(2) *Any jumping exponent for $\mathfrak{a}$ is a rational number.*

Remark 5.11. In [4], Blickle, Mustaţă and Smith proved a similar result in the above theorem for essentially of finite type a field of k, but it remains still open for a formal power series ring over a field k.

In this talk, Professor Ambro asked me whether the set of jumping exponents for $\mathfrak{a}$ is periodic. Although I think that it is probably true, but I do not have a proof.

Now let us strat our proof of the above theorem with giving the following lemma, which plays a key role.

Lemma 5.12. *There exists $\epsilon > 0$ such that $\tau(\mathfrak{a}^t) = I_{r,e}(\mathfrak{a})$ whenever $t < \frac{r}{p^e} < t + \epsilon$. In particular, $\tau(\mathfrak{a}^t) = \tau(\mathfrak{a}^{t'})$ for every $t' \in [t, t+\epsilon)$.*

Proof. Suppose that there exists integers r_m, $e_m \ge 1$ for which $I_{r_m, e_m}(\mathfrak{a}) \ne I_{r_{m+1}, e_{m+1}}(\mathfrak{a})$ for each $m \ge 1$ and a decreasing sequence $\frac{r_m}{e_m}$ converges t. We may assume that $e_m \le e_{m+1}$ without loss of generality. Then $I_{r_m, e_m}(\mathfrak{a}) \subsetneq I_{r_{m+1}, e_{m+1}}(\mathfrak{a})$ by Lemma 5.4. But this contradicts the assending chain condition for ideals in R. Hence there exists $\epsilon > 0$ and an ideal I in R such that $I_{r,e}(\mathfrak{a}) = I$ for $t < \frac{r}{p^e} < t + \epsilon$.

In order to prove the lemma, it suffices to show that $I = \tau(\mathfrak{a}^t)$ in the above situation. Take a large enough $e \in \mathbb{N}$ such that $\tau(\mathfrak{a}^t) = I_{\lceil tp^e \rceil, e}(\mathfrak{a})$ and $\frac{\lceil tp^e \rceil}{p^e} < t + \epsilon$. If $tp^e \notin \mathbb{Z}$, then $t < \frac{\lceil tp^e \rceil}{p^e}$. Hence $I = \tau(\mathfrak{a}^t)$ by the choice of I and ϵ. So we may assume that $tp^e \in \mathbb{Z}$ and $t + \frac{1}{p^e} < t + \epsilon$. Then $I = I_{tp^e+1, e}(\mathfrak{a}) \subseteq I_{tp^e, e}(\mathfrak{a}) = \tau(\mathfrak{a}^t)$. Thus we want to show that $\mathfrak{a}^{tp^e} \subseteq I^{[p^e]}$. Indeed, this implies that $I_{tp^e, e}(\mathfrak{a}) \subseteq I$. Now let $z \in \mathfrak{a}^{tp^e}$. If $e' \ge e$, then since $t < \frac{tp^{e'}+1}{p^{e'}} < t + \epsilon$, we have that $\mathfrak{a}^{tp^{e'}+1} \subseteq I^{[p^{e'}]}$. Fix $0 \ne d \in \mathfrak{a}$. Then since $du^{p^{e'-e}} \in \mathfrak{a}^{tp^e+1} \subseteq (I^{[p^e]})^{[p^{e'-e}]}$, we obtain that $u \in (I^{[p^e]})^* = I^{[p^e]}$, as required. $\square$

We now introduce the notion of the (generalized) F-threshold for $\mathfrak{a}$. Let $\mathfrak{a}$, J be ideals of R such that $\mathfrak{a} \subseteq \sqrt{J}$. Put $\nu_{\mathfrak{a}}^J(p^e) = \max\{r \in \mathbb{Z} : \mathfrak{a}^r \not\subseteq J^{[p^e]}\}$. If such an integer r does not exist, then we put $\nu_{\mathfrak{a}}^J(p^e) = 0$. One can easily see that $\{\frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}\}$ is an increasing sequence and bounded above. So we define

$$c^J(\mathfrak{a}) := \sup_{e \in \mathbb{N}} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e} = \lim_{e \to \infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}$$

and call it the *F-threshold for $\mathfrak{a}$ with respect to J.* Note that the original F-threshold for $\mathfrak{a}$ is equal to $c^{\mathfrak{m}}(\mathfrak{a})$. The next theorem gives a relationship between jumping exponents for $\mathfrak{a}$ and F-thresholds for $\mathfrak{a}$.

Theorem 5.13. *Let $\mathfrak{a}$ be an ideal of R. Now consider the two maps*

$$\begin{array}{rcccl} \tau(\mathfrak{a}^\bullet) & : & \mathbb{R}_{\geq 0} & \to & \{\textit{ideals of } R\} \quad (t \mapsto \tau(\mathfrak{a}^t)), \\ c^\bullet(\mathfrak{a}) & : & \{\textit{ideals of } R\} & \to & \mathbb{R}_{\geq 0} \qquad (J \mapsto c^J(\mathfrak{a})). \end{array}$$

Then

(1) *$c^J(\mathfrak{a})$ is a jumping exponent for $\mathfrak{a}$.*
(2) *If t is a jumping exponent for $\mathfrak{a}$, then $t = c^{\tau(\mathfrak{a}^t)}(\mathfrak{a})$.*

Proof. We omit it here. The proof could be seen at Watanabe's report. $\square$

We are now ready to prove Theorem 5.10.

Proof of Theorem 5.10. (1) Suppose that a sequence $\{\alpha_m\}$ of jumping exponents such that $\lim \alpha_m$ exists (say, α). By Lemma 5.12, we may assume that $\alpha_m \uparrow \alpha$.

Fix integers r, $e \geq 1$. Assume that $\mathfrak{a}$ is generated by polynomials of degree at most d, and that $e_1, \dots, e_N$ be a free basis of R over R^{p^e}. Let $h_1, \dots, h_s$ be a minimal system of generators of $\mathfrak{a}^r$. If we write $h_i = \sum_{j=1}^N a_{ij}^{p^e} e_j$ for some $a_{ij} \in R$ $(i = 1, \dots, s)$, then $I_{r,e}(\mathfrak{a}) = (a_{ij} : i = 1, \dots, s,\ j = 1, \dots, N)$ and $\deg(a_{ij}^{p^e}) \leq \deg h_i \leq rd$. Thus $I_{r,e}(\mathfrak{a})$ can be generated by polynomials of degree at most $\frac{rd}{p^e}$. Hence $\tau(\mathfrak{a}^{\alpha_m})$ can be generated by polynomials of degree at most $\lfloor \alpha_m d \rfloor \leq \lfloor \alpha d \rfloor$. On the othere hand, since $\{\tau(\mathfrak{a}^{\alpha_m})\}$ form a strictly decreasing sequence of ideals, there exists a strictly increasing sequence of subspaces of $k[x_1, \dots, x_n]_{\lfloor \alpha d \rfloor}$. This is a contradiction. Hence we get (1).

(2) Let $\alpha > 0$ be a jumping exponent for $\mathfrak{a}$. Then we can write as $\alpha = c^J(\mathfrak{a})$ for some ideal J of R. Since $\nu_{\mathfrak{a}}^{J^{[p]}}(p^e) = \nu_{\mathfrak{a}}^J(p^{e+1})$, we have $p\alpha = c^{J^{[p]}}(\mathfrak{a})$ is also jumping exponent for $\mathfrak{a}$. So is $p^e\mathfrak{a}$. Put $m = \mu(\mathfrak{a})$, and let $p^{e_0}\alpha > m$. For every $e \geq e_0$, Skoda's theorem implies that $\{p^e\alpha\} + m - 1$ is a jumping exponent for α, where $\{\beta\} = \beta - \lfloor \beta \rfloor$ is the fractional part of $\beta \in \mathbb{R}$. Note that these jumping exponents are in the interval $[m-1, m)$. By (1), there exists a pair (e_1, e_2) such that $e_0 \leq e_1 < e_2$ and $\{p^{e_1}\alpha\} + m - 1 = \{p^{e_2}\alpha\} + m - 1$. This means that $(p^{e_1} - p^{e_2})\alpha \in \mathbb{Z}$. Hence α is a rational number. $\square$

Remark 5.14. Watanabe and the author [35] have investigated the *Hilbert–Kunz multiplicities* of local rings. For a d-dimensional Noetherian local ring of characteristic $p > 0$, the Hilbert–Kunz multiplicity $e_{\mathrm{HK}}(R)$ is defined by

$$e_{\mathrm{HK}}(R) = \lim_{e\to\infty} \frac{l_A(A/\mathfrak{m}^{[p^e]})}{p^{ed}}.$$

Then the rationality of $e_{\mathrm{HK}}(R)$ remains open in general.

6. Restriction theorem

In this section, we state so-called *Restriction theorem* with respect to the generalized test ideal. A Similar result for multiplier ideals is known.

Before stating the theorem, let us recall a brief background. Now let $(R, \mathfrak{m})$ be an excellent normal local domain of characteristic $p > 0$. Let $x \in \mathfrak{m}$ be a non-zero element. Then

(1) If R/xR is F-rational, then R is F-rational.
(2) If R/xR is weakly F-regular and R is $\mathbb{Q}$-Gorenstein, then R is (weakly) F-regular.

In (2), the $\mathbb{Q}$-Gorensteinness is *not* superfluous. In fact, Singh [28] constructed the following example.

Example 6.1 ([28]). Let $S = k[A, B, C, D, E]$ be a polynomial ring over a field k with $\mathrm{char}(k) = p > 0$. Let m, n be integers with $(p, m) = 1$ and $m - \frac{m}{n} > 2$. Put

$$I = I_2\begin{pmatrix} A^2 + T^m & B & D \\ C & A^2 & B^2 - D \end{pmatrix}$$

Then R/TR is F-regular, but R is not F-regular (even not F-pure).

The following theorem (Restriction theorem) generalizes the above result (under some extra condition). In fact, if $\tau(S) = S = R/xR$, then the theorem implies that $\tau(R) = R$.

Theorem 6.2 (Restriction Theorem). *Let $(R, \mathfrak{m})$ be a normal $\mathbb{Q}$-Gorenstein complete local domain of characteristic $p > 0$. Assume that S is a $\mathbb{Q}$-Gorenstein normal local domain. Then for any filtration of ideals $\mathfrak{a}_\bullet$ in R, we have*

$$\tau(\mathfrak{a}_\bullet S) \subseteq \tau(\mathfrak{a}_\bullet)S.$$

Proof. One can prove this by a similar argument as in the proof of [11, Theorem 4.1]. □

In [11], we proved so-called *Subadditivity theorem* using the idea of "restriction to the diagonal". But we can prove it more directly following Takagi's idea.

Theorem 6.3 (Subadditivity Theorem). *Let $(R, \mathfrak{m})$ be a complete or an F-finite regular local ring of characteristic $p > 0$. Let $\mathfrak{a}$, $\mathfrak{b}$ ideals in R. Then*

(1) $\tau(\mathfrak{a}\mathfrak{b}) \subseteq \tau(\mathfrak{a})\tau(\mathfrak{b})$.

(2) *For any real numbers $t, s \geq 0$, $\tau(\mathfrak{a}^t\mathfrak{b}^s) \subseteq \tau(\mathfrak{a}^t)\tau(\mathfrak{b}^s)$.*

We state the sketch of the proof of (2) (after Takagi). Let $(R, \mathfrak{m})$ be a complete local ring. We first show that

$$[(0)^{*\mathfrak{a}^t\mathfrak{b}^s} : \tau(\mathfrak{a}^t)^{*\mathfrak{a}^t}]_M \supseteq [(0)^{*\mathfrak{b}^s}_M : \tau(\mathfrak{a}^t)]_M.$$

for any finitely generated A-module $M \subseteq E = E_R(R/\mathfrak{m})$. Then by Lemma 3.3, we have

$$\tau(\mathfrak{a}^t)^{*\mathfrak{a}^t}\tau(\mathfrak{a}^t\mathfrak{b}^s) \subseteq \tau(\mathfrak{a}^t)\tau(\mathfrak{b}^s).$$

Hence it suffices to show that $\tau(\mathfrak{a}^t)^{*\mathfrak{a}^t} = R$ if R is regular. See also Takagi's report for more details.

Remark 6.4. In the above theorem, (1) holds in the case 2-dimensional F-finite F-regular local domain. In higher dimensional case, there is a counterexample to (1).

On the other hand, (2) is sometimes said to be *Strong Subadditivity Theorem.* This is not necessary true for Gorenstein F-regular local domains even in 2-dimensional case.

Example 6.5. Let $R = k[[x, y]]$ be a formal power series ring over a field k. Put $\mathfrak{a} = (x^3, xy, y^3)$. Let $t, s \geq 0$ be real numbers, and put $n = \lfloor t \rfloor$, $m = \lfloor s \rfloor$. Then

$$\tau(\mathfrak{a}^t) = \begin{cases} R, & (0 \leq t < 1) \\ \mathfrak{m}\mathfrak{a}^{n-1}, & (n \leq t < n + \frac{1}{3}) \\ \mathfrak{m}^2\mathfrak{a}^{n-1}, & (n + \frac{1}{3} \leq t < n + \frac{2}{3}) \\ \mathfrak{a}^n, & (n + \frac{2}{3} \leq t < n + 1) \end{cases}$$

Now let us confirm the Strong Subadditivity Theorem in this case. If $n \leq t < n + \frac{1}{3}$, then since $n + m \leq t + s$, we have

$$\tau(\mathfrak{a}^{t+s}) \subseteq \mathfrak{m}\mathfrak{a}^{n+m-1} = (\mathfrak{m}\mathfrak{a}^{n-1})\mathfrak{a}^m \subseteq \tau(\mathfrak{a}^t)\tau(\mathfrak{a}^s).$$

If $n + \frac{1}{3} \leq t < n + 1$ and $m + \frac{1}{3} \leq s < m + 1$, then since $n + m + \frac{2}{3} \leq t + s$, we have

$$\tau(\mathfrak{a}^{t+s}) \subseteq \mathfrak{a}^{n+m} = \mathfrak{a}^n\mathfrak{a}^m \subseteq \tau(\mathfrak{a}^t)\tau(\mathfrak{a}^s).$$

7. How to compute $\tau(\mathfrak{a})$ in a 2-dimensional Gorenstein rational local domain

Throughout this section, let $(R, \mathfrak{m})$ be a 2-dimensional excellent local domain of characteristic $p > 0$ with infinite residue field k. Then one can define the multiplier ideal $\mathcal{J}(\mathfrak{a})$ as well as $\tau(\mathfrak{a})$ for any $\mathfrak{m}$-primary ideal $\mathfrak{a}$. So it is natural to ask the following question.

Question 7.1. *Let R be as above, and let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal of R. When does $\tau(\mathfrak{a}) = \mathcal{J}(\mathfrak{a})$ hold? In particular, is $\tau(\mathfrak{a})$ integrally closed?*

In the following, we take a restrict our attention to Gorenstein *rational* local domains. In this case, $\mathfrak{a}$-primary ideals enjoy many good properties. Let us summarize them.

Lemma 7.2. *Assume that $(R, \mathfrak{m})$ be a 2-dimensional excellent Gorenstein rational local domain, so-called rational double point (RDP). Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary integrally closed ideal of R. Then*

(1) *There exist a minimal reduction $\mathfrak{b}$ of $\mathfrak{a}$ (i.e., $\mathfrak{b}$ is a parameter ideal which is contained in $\mathfrak{a}$ such that $\mathfrak{a}^{r+1} = \mathfrak{b}\mathfrak{a}^r$ for some $r \geq 0$) such that $\mathfrak{a}^2 = \mathfrak{b}\mathfrak{a}$.*
(2) *$\overline{\mathfrak{a}^n} = \mathfrak{a}^n$ for every $n \geq 1$.*
(3) *$\mathcal{J}(\mathfrak{a}) = \mathfrak{b} : \mathfrak{a}$ for every minimal reduction $\mathfrak{b}$ of $\mathfrak{a}$.*

In the above lemma, the Rees algebra $R(\mathfrak{a}) = R[\mathfrak{a}t] = \bigoplus_{n \geq 0} \mathfrak{a}^n t^n$, which is a subalgebra of $R[t]$, is a Cohen–Macaulay normal domain. In fact, if R contains a $\mathbb{Q}$, then such an algebra has also rational singularity (due to Lipman [20]). So one expects that "*$R(\mathfrak{a})$ must be F-rational*" in our situation (R contains $\mathbb{F}_p$). However, it is *not* necessarily true. The situation is a little bit complicated in our case.

Question 7.3. *Let $(R, \mathfrak{m})$ and $\mathfrak{a} = \overline{\mathfrak{a}}$ be as above. When is $R(\mathfrak{a})$ F-rational?*

In higher dimensional case, we do not have any satisfactory answer. But in 2-dimensional case, we have the following criterion; see [12].

Proposition 7.4. *Let $(R, \mathfrak{m})$ be a 2-dimensional excellent Gorenstein rational local domain. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary integrally closed ideal, and $\mathfrak{b} = (x, y)$ a minimal reduction of $\mathfrak{a}$. Then*

$$R(\mathfrak{a}) \text{ is } F\text{-rational} \iff (x^\ell, y^\ell)^{*\mathfrak{b}} = \mathfrak{a}^{2\ell-1} + (x^\ell, y^\ell) \text{ for any } \ell \geq 2.$$

The F-rationality of the Rees algebra $R(\mathfrak{a})$ is closely related to the first question. Using this criterion, we can show the following.

Theorem 7.5. *Let $(R, \mathfrak{m})$ be a 2-dimensional excellent Gorenstein rational local domain. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary integrally closed ideal, and $\mathfrak{b} = (x, y)$ a minimal reduction of $\mathfrak{a}$. Then*

(1) *$\tau(\mathfrak{a}) \subseteq \mathcal{J}(\mathfrak{a})$.*
(2) *$\mathcal{R} = R(\mathfrak{a})$ is F-rational if and only if $\tau(\mathfrak{a}) = \mathcal{J}(\mathfrak{a})$.*
When this is the case, the graded canonical module $\omega_{\mathcal{R}}$ is given by

$$\omega_{\mathcal{R}} = \bigoplus_{n \geq 1} H^0(Y, \mathfrak{a}^n \mathcal{O}_Y) = \bigoplus_{n \geq 1} \tau(\mathfrak{a}^n),$$

where $Y = \operatorname{Proj}(R(\mathfrak{a}))$.
(3) *If $(R, \mathfrak{m})$ is F-rational, then so is $R(\mathfrak{a})$.*

Proof. (1) Put $\mathfrak{b} = (x, y)$. We first show that $\mathfrak{a} = \mathfrak{b}^{*\mathfrak{b}}$. For each $q = p^e$, since

$$\mathfrak{a}^q \mathfrak{b}^q = \mathfrak{b}^{2q-1}\mathfrak{a} \subseteq \mathfrak{b}^{2q-1} \subseteq \mathfrak{b}^{[q]},$$

we have $\mathfrak{a} \subseteq \mathfrak{b}^{*\mathfrak{b}}$. Conversely, let $z \in \mathfrak{b}^{*\mathfrak{b}}$. Then there exists $c \neq 0$ such that $cz^q\mathfrak{b}^q \subseteq \mathfrak{b}^{[q]}$ for all $q = p^e$. Hence $cz^q \in (x^q, y^q) : (x, y)^q = (x, y)^{q-1}$ because x, y forms a regular sequence. This implies that $z \in \overline{\mathfrak{b}} = \mathfrak{a}$.

Next, we show that $\tau(\mathfrak{b}) \subseteq \mathfrak{b} : \mathfrak{a}$. In fact,

$$\tau(\mathfrak{b}) = \bigcap_{\ell=1}^{\infty} (x^\ell, y^\ell) : (x^\ell, y^\ell)^{*\mathfrak{b}} \subseteq (x, y) : (x, y)^{*\mathfrak{b}} = \mathfrak{b} : \mathfrak{a}.$$

Hence $\tau(\mathfrak{a}) = \tau(\mathfrak{b}) \subseteq \mathfrak{b} : \mathfrak{a} = \mathcal{J}(\mathfrak{a})$.

For (2),(3), we omit the proof here. □

As a corollary, we have

Corollary 7.6. *Let $(R, \mathfrak{m})$ be a 2-dimensional Gorenstein F-rational local domain. Then for any $\mathfrak{m}$-primary integrally closed ideal $\mathfrak{a}$ and for any integer $n \geq 1$, we have that $\tau(\mathfrak{a}^n) = \mathcal{J}(\mathfrak{a}^n)$. In particular, $\tau(\mathfrak{a}^n)$ is integrally closed.*

Remark 7.7. Using the canonical cover trick, one can generalize the above corollary to 2-dimensional strongly F-regular local rings.

We believe that $\tau(\mathfrak{a}^t)$ is integrally closed for any real number $t \geq 0$, but we have no proof. We will give a more direct proof of Corollary 7.6 here. Namely we have:

Lemma 7.8. *Let $(R, \mathfrak{m})$ and $\mathfrak{a}$ be as in Corollary 7.6. Take a minimal reduction $\mathfrak{b}$ of $\mathfrak{a}$. Then for any real number t with $0 < t \leq 1$, we have*

$$\tau(\mathfrak{a}^t) = \tau(\mathfrak{b}^t) = \mathfrak{b} : \mathfrak{b}^{*\mathfrak{b}^t}.$$

Proof. Put $\mathfrak{b} = (x, y)$ and $\mathfrak{b}^{[\ell]} = (x^\ell, y^\ell)$ for each $\ell \geq 1$. Since R is an excellent Gorenstein local ring, we know that

$$\tau(\mathfrak{a}^t) = \tau(\mathfrak{b}^t) = \bigcap_{\ell=1}^{\infty} \mathfrak{b}^{[\ell]} : (\mathfrak{b}^{[\ell]})^{*\mathfrak{b}^t}.$$

So it is enough to show that $\mathfrak{b}^{[\ell]} : (\mathfrak{b}^{[\ell]})^{*\mathfrak{b}^t} = \mathfrak{b} : \mathfrak{b}^{*\mathfrak{b}^t}$ for $\ell \geq 2$. In order to do that, we prove the following claim:

Claim: $(\mathfrak{b}^{[\ell]})^{*\mathfrak{b}^t} = (x^\ell, y^\ell) + \mathfrak{b}^{*\mathfrak{b}^t}(xy)^{\ell-1}$.

Suppose that $z \in (\mathfrak{b}^{[\ell]})^{*\mathfrak{b}^t}$. By definition, there exists $c \in R^o$ such that $cz^q\mathfrak{b}^{\lceil tq \rceil} \subseteq (x^{\ell q}, y^{\ell q})$ for all $q = p^e$. Since $0 < t \leq 1$, we have

$$cz^q \in (x^{\ell q}, y^{\ell q}) : (x, y)^q \subseteq (x, y)^{(2\ell-1)q-1} + (x^{\ell q}, y^{\ell q}) \subseteq ((xy)^{(\ell-1)q}, x^{\ell q}, y^{\ell q}).$$

This implies that $z \in ((xy)^{\ell-1}, x^\ell, y^\ell)^* = ((xy)^{\ell-1}, x^\ell, y^\ell)$. Hence we can write as $z = w(xy)^{\ell-1} \pmod{(x^\ell, y^\ell)}$. Then $cw^q\mathfrak{b}^{\lceil tq \rceil} \subseteq (x^{\ell q}, y^{\ell q}) : (xy)^{(\ell-1)q} = (x^q, y^q)$. Thus $w \in \mathfrak{b}^{*\mathfrak{b}}$, as required.

By Claim, we get

$$\mathfrak{b}^{[\ell]} : (\mathfrak{b}^{[\ell]})^{*\mathfrak{b}^t} = \mathfrak{b}^{[\ell]} : (\mathfrak{b}^{[\ell]} + \mathfrak{b}^{*\mathfrak{b}^t}(xy)^{\ell-1}) = (\mathfrak{b} : (xy)^{\ell-1}) : \mathfrak{b}^{*\mathfrak{b}^t} = \mathfrak{b} : \mathfrak{b}^{*\mathfrak{b}^t}.$$

□

Remark 7.9. When $t = 1$, we have $\tau(\mathfrak{a}) = \tau(\mathfrak{b}) = \mathfrak{b} : \mathfrak{a}$; see the proof of Theorem 7.5(1).

Example 7.10. Let k be a field of characteristic 2.

(1) Put $R = k[[x,y,z]]/(x^2+y^3+z^3)$. Then R is a rational double point of type (D_4), but not F-rational. However, $R(\mathfrak{m})$ is F-rational. Indeed, $\tau(\mathfrak{m}) = \mathcal{J}(\mathfrak{m}) = \mathfrak{m}$.

(2) Put $R = k[[x,y,z]]/(x^2+y^3+z^5)$. Then R is a rational double point of type (E_8), but not F-rational. Moreover, $R(\mathfrak{m})$ is *not* F-rational. Indeed, $\tau(\mathfrak{m}) \subsetneq (x,y,z^2)$ and $\mathcal{J}(\mathfrak{m}) = \mathfrak{m}$.

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