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Approximation for extinction probability of the contact process based on the Gröbner basis

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Abstract. In this note we give a new method for getting a series of approximations for the extinction probability of the one-dimensional contact process by using the Gröbner basis.

1 Introduction

Let $X = \{0,1\}^{\mathbb{Z}^d}$ denote a configuration space, where $\mathbb{Z}^d$ is the d-dimensional integer lattices. The contact process $\{\eta_t : t \geq 0\}$ is an X-valued continuous-time Markov process. The model was introduced by Harris in 1974 [1] and is considered as a simple model for the spread of a disease with the infection rate λ. In this setting, an individual at $x \in \mathbb{Z}^d$ for a configuration $\eta \in X$ is infected if $\eta(x) = 1$ and healthy if $\eta(x) = 0$. The formal generator is given by

$$\Omega f(\eta) = \sum_{x \in \mathbb{Z}^d} c(x,\eta)[f(\eta^x) - f(\eta)],$$

where $\eta^x \in X$ is defined by $\eta^x(y) = \eta(y)$ $(y \neq x)$, and $\eta^x(x) = 1 - \eta(x)$. Here for each $x \in \mathbb{Z}^d$ and $\eta \in X$, the transition rate is

$$c(x,\eta) = (1-\eta(x)) \times \lambda \sum_{y:|y-x|=1} \eta(y) + \eta(x),$$

with $|x| = |x_1| + \cdots + |x_d|$. In particular, the one-dimensional contact process is

$001 \to 011$	at rate	λ,
$100 \to 110$	at rate	λ,
$101 \to 111$	at rate	2λ,
$1 \to 0$	at rate	1.

Let $Y = \{A \subset \mathbb{Z}^d : |A| < \infty\}$, where $|A|$ is the number of elements in A. Let $\xi_t^A (\subset \mathbb{Z}^d)$ denote the state at time t of the contact process with $\xi_0^A = A$. There is a one-to-one correspondence between $\xi_t^A (\subset \mathbb{Z}^d)$ and $\eta_t \in X$ such that $x \in \xi_t^A$ if and only if $\eta_t(x) = 1$. For any $A \in Y$, we define the extinction probability of A by $\lim_{t \to \infty} P(\xi_t^A = \emptyset)$. Define $\nu_\lambda(A) = \nu_\lambda\{\eta : \eta(x) = 0$ for any $x \in A\}$, where ν_λ is an invariant measure of the process starting from a configuration: $\eta(x) = 1$ $(x \in \mathbb{Z}^d)$ and is called the *upper invariant measure*. In other words, let $\delta_1 S(t)$ denote the probability measure at time t for initial probability measure δ_i which is the pointmass $\eta \equiv i (i = 0, 1)$. Then $\nu_\lambda = \lim_{t \to \infty} \delta_1 S(t)$. Then self-duality of the process implies that $\nu_\lambda(A) = \lim_{t \to \infty} P(\xi_t^A = \emptyset)$. The correlation identities for $\nu_\lambda(A)$ can be obtained as follows:

Theorem 1.1 *For any $A \in Y$,*

$$\lambda \sum_{x \in A} \sum_{y:|y-x|=1} \Big[\nu_\lambda(A \cup \{y\}) - \nu_\lambda(A)\Big] + \sum_{x \in A} \Big[\nu_\lambda(A \setminus \{x\}) - \nu_\lambda(A)\Big] = 0.$$

From now on we consider the one-dimensional case. We introduce the following notation:

$$\nu_\lambda(\circ) = \nu_\lambda(\{0\}),\ \nu_\lambda(\circ\circ) = \nu_\lambda(\{0,1\}),\ \nu_\lambda(\circ \times \circ) = \nu_\lambda(\{0,2\}),\ \ldots .$$

By Theorem 1.1, we obtain

Corollary 1.2

(1) $\quad 2\lambda\nu_\lambda(\circ\circ) - (2\lambda + 1)\nu_\lambda(\circ) + 1 = 0,$

(2) $\quad \lambda\nu_\lambda(\circ\circ\circ) - (\lambda + 1)\nu_\lambda(\circ\circ) + \nu_\lambda(\circ) = 0,$

(3) $\quad 2\lambda\nu_\lambda(\circ\circ\circ\circ) + \nu_\lambda(\circ \times \circ) - (2\lambda + 3)\nu_\lambda(\circ\circ\circ) + 2\nu_\lambda(\circ\circ) = 0,$

(4) $\quad \lambda\nu_\lambda(\circ\circ\times\circ) - (2\lambda + 1)\nu_\lambda(\circ \times \circ) + \lambda\nu_\lambda(\circ\circ\circ) + \nu_\lambda(\circ) = 0.$

The detailed discussion concerning results in this section can be seen in Konno [2, 3]. If we regard $\lambda, \nu_\lambda(\circ), \nu_\lambda(\circ\circ), \nu_\lambda(\circ\ \circ\ \circ), \ldots$ as variables, then the left hand sides of the correlation identities by Theorem 1.1 are polynomials of degree at most two. In the next section, we give a new procedure for getting a series of approximations for extinction probabilities based on the Gröbner basis by using Corollary 1.2. As for the Gröbner basis, see [4], for example.

2 Our results

Put $x = \nu_\lambda(\circ)$, $y = \nu_\lambda(\circ\circ)$, $z = \nu_\lambda(\circ\ \circ\ \circ)$, $w = \nu_\lambda(\circ \times \circ)$, $s = \nu_\lambda(\circ\ \circ\ \circ\circ)$, $u = \nu_\lambda(\circ\ \circ \times\circ)$. Let $\prec$ denote the lexicographic order with $\lambda \prec x \prec y \prec w \prec z \prec u \prec s$. For $m = 1, 2, 3$, let I_m be the ideals of a polynomial ring $\mathbb{R}[x_1, x_2, \ldots, x_{n(m)}]$ over $\mathbb{R}$ as defined below. Here $x_1 = \lambda, x_2 = x, x_3 = y, x_4 = z, x_5 = w, x_6 = s, x_7 = u$ and $n(1) = 3; n(2) = 4, n(3) = 7$.

2.1 First approximation

We consider the following ideal based on Corollary 1.2 (1):

$$(5) \qquad I_1 = \langle\, 2\lambda y - 2\lambda x - x + 1,\ y - x^2 \,\rangle \subset \mathbb{R}[\lambda, x, y].$$

Here $y - x^2$ corresponds to the first (or mean-field) approximation: $\nu_\lambda^{(1)}(\circ\circ) = (\nu_\lambda^{(1)}(\circ))^2$. Then

$$(6) \qquad G_1 = \{(x-1)(2\lambda x - 1),\ y - x^2\}$$

is the reduced Gröbner basis for I_1 with respect to $\prec$. Therefore the solution except a trivial one $x (= y) = 1$ is $x = \nu_\lambda^{(1)}(\circ) = 1/(2\lambda)$. Remark that the trivial solution means that the invariant measure is δ_0. From this, we obtain the first approximation of the density of the particle, $\rho_\lambda = E_{\nu_\lambda}(\eta(x))$, as follows:

$$(7) \qquad \rho_\lambda^{(1)} = 1 - \nu_\lambda^{(1)}(\circ) = \frac{2\lambda - 1}{2\lambda},$$

for any $\lambda \geq 1/2$. This result gives the first lower bound $\lambda_c^{(1)}$ of the critical value λ_c of the one-dimensional contact process, that is, $\lambda_c^{(1)} = 1/2 \leq \lambda_c$. However it should be noted that the inequality is not proved in our approach. The estimated value of λ_c is about 1.649.

2.2 Second approximation

Consider the following ideal based on Corollary 1.2 (1) and (2):

$$I_2 = \langle\, 2\lambda y - 2\lambda x - x + 1,\ \lambda z - \lambda y - y + x,\ xz - y^2 \,\rangle \subset \mathbb{R}[\lambda, x, y, z].$$

Here $xz - y^2$ corresponds to the second (or pair) approximation: $\nu_\lambda^{(2)}(\circ)\nu_\lambda^{(2)}(\circ\circ\circ) = (\nu_\lambda^{(2)}(\circ\circ))^2$. Then

$$G_2 = \{(x-1)((2\lambda-1)x-1),\ 1 + 2\lambda(y-x) - x,\\ -y - yx + 2x^2, -z - y(2+y) + 4x^2\}$$

is the reduced Gröbner basis for I_2 with respect to $\prec$. Therefore the solution except a trivial one $x(=y=z) = 1$ is $x = \nu_\lambda^{(2)}(\circ) = 1/(2\lambda - 1)$. As in a similar way of the first approxaimation, we get the second approximation of the density of the particle:

$$\rho_\lambda^{(2)} = \frac{2(\lambda-1)}{2\lambda-1},$$

for any $\lambda \geq 1$. This result implies the second lower bound $\lambda_c^{(2)} = 1$. We should remark that if we take

$$I_2' = \langle\, 2\lambda y - 2\lambda x - x + 1,\ \lambda z - \lambda y - y + x,\ y - x^2,\ z - x^3 \,\rangle \subset \mathbb{R}[\lambda, x, y, z],$$

then we have

$$G_2' = \{z - 1,\ y - 1,\ x - 1\}$$

is the reduced Gröbner basis for I_2' with respect to $\prec$. Here $y - x^2$ and $z - x^3$ correspond to an approximation: $\nu_\lambda^{(2')}(\circ\circ) = (\nu_\lambda^{(2')}(\circ))^2$ and $\nu_\lambda^{(2')}(\circ\circ\circ) = (\nu_\lambda^{(2')}(\circ))^3$, respectively. Then we have only trivial solution: $x = y = z = 1$.

2.3 Third approximation

Consider the following ideal based on Corollary 1.2 (1)–(4):

$$I_3 = \langle\, 2\lambda y - 2\lambda x - x + 1,\ \lambda z - \lambda y - y + x,\\ 2\lambda s + w - (2\lambda+3)z + 2y,\ \lambda u - (2\lambda+1)w + \lambda z + x,\\ ys - z^2,\ xu - yw \,\rangle \subset \mathbb{R}[\lambda, x, y, z, w, s, u].$$

Here $ys-z^2$ and $xu-yw$ correspond to the third approximation: $\nu_\lambda^{(3)}(\circ\circ)\nu_\lambda^{(3)}(\circ\circ\ \circ\circ) = (\nu_\lambda^{(3)}(\circ\circ\circ))^2$ and $\nu_\lambda^{(3)}(\circ)\nu_\lambda^{(3)}(\circ\circ\times\circ) = \nu_\lambda^{(3)}(\circ\circ)\nu_\lambda^{(3)}(\circ\times\circ)$, respectively. Then

$$G_3 = \{(x-1)((12\lambda^3-5\lambda-1)x^2-2\lambda(2\lambda+3)x-\lambda+1), \ldots\}$$

is the reduced Gröbner basis for I_3 with respect to $\prec$. Therefore the solution except a trivial one $x=1$ is $x=\nu_\lambda^{(3)}(\circ) = (\lambda(2\lambda+3)+\sqrt{D})/(12\lambda^3-5\lambda-1)$, where $D = 16\lambda^4+4\lambda^2+4\lambda+1$. Then we obtain the third approximation of the density of the particle:

$$\rho_\lambda^{(3)} = \frac{4\lambda(3\lambda^2-\lambda-3)}{12\lambda^3-2\lambda^2-8\lambda-1+\sqrt{D}}, \tag{8}$$

for any $\lambda \geq (1+\sqrt{37})/6$. This result corresponds to the third lower bound $\lambda_c^{(3)} = (1+\sqrt{37})/6 \approx 1.180$.

3 Summary

We obtain the first, second, and third approximations for the extinction probability, the density of the particle, and the lower bound of the one-dimensional contact process by using the Gröbner basis with respect to a suitable term order. These results coincide with results given by the Harris lemma (more precisely, the Katori-Konno method, see [3]) or the BFKL inequality [5] (see also [3]). As we saw, the generators of I_m in Section 2 have degree at most two in $x_1, x_2, \ldots$, such as $2\lambda y - 2\lambda x - x + 1$, $ys - z^2$ in the case of I_3. We expect that this property will lead to get the higher order approximations of the process (and other interacting particle systems having a similar property) effectively.

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