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<th>Title</th>
<th>Approximation for extinction probability of the contact process based on the Grobner basis (Theory of Biomathematics and its Applications III)</th>
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<td>Author(s)</td>
<td>Konno, Norio</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2007), 1551: 57-62</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2007-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/80894">http://hdl.handle.net/2433/80894</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
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<td>Textversion</td>
<td>publisher</td>
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Approximation for extinction probability of the contact process based on the Gröbner basis

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Abstract. In this note we give a new method for getting a series of approximations for the extinction probability of the one-dimensional contact process by using the Gröbner basis.

1 Introduction

Let $X = \{0, 1\}^{\mathbb{Z}^d}$ denote a configuration space, where $\mathbb{Z}^d$ is the $d$-dimensional integer lattices. The contact process $\{\eta_t : t \geq 0\}$ is an $X$-valued continuous-time Markov process. The model was introduced by Harris in 1974 [1] and is considered as a simple model for the spread of a disease with the infection rate $\lambda$. In this setting, an individual at $x \in \mathbb{Z}^d$ for a configuration $\eta \in X$ is infected if $\eta(x) = 1$ and healthy if $\eta(x) = 0$. The formal generator is given by

$$\Omega f(\eta) = \sum_{x \in \mathbb{Z}^d} c(x, \eta)[f(\eta^x) - f(\eta)],$$

where $\eta^x \in X$ is defined by $\eta^x(y) = \eta(y)$ ($y \neq x$), and $\eta^x(x) = 1 - \eta(x)$. Here for each $x \in \mathbb{Z}^d$ and $\eta \in X$, the transition rate is

$$c(x, \eta) = (1 - \eta(x)) \times \lambda \sum_{y:|y-x|=1} \eta(y) + \eta(x),$$
with $|x| = |x_1| + \cdots + |x_d|$. In particular, the one-dimensional contact process is

\[
\begin{align*}
001 &\rightarrow 011 \quad \text{at rate } \lambda, \\
100 &\rightarrow 110 \quad \text{at rate } \lambda, \\
101 &\rightarrow 111 \quad \text{at rate } 2\lambda, \\
1 &\rightarrow 0 \quad \text{at rate } 1.
\end{align*}
\]

Let $Y = \{A \subset \mathbb{Z}^d : |A| < \infty\}$, where $|A|$ is the number of elements in $A$. Let $\xi_t^A(\subset \mathbb{Z}^d)$ denote the state at time $t$ of the contact process with $\xi_0^A = A$. There is a one-to-one correspondence between $\xi_t^A(\subset \mathbb{Z}^d)$ and $\eta \in X$ such that $x \in \xi_t^A$ if and only if $\eta_t(x) = 1$. For any $A \in Y$, we define the extinction probability of $\Lambda$ by $\mathbb{P}(\xi_t^A = \emptyset)$.

Let $\nu_\lambda(A) = \nu_\lambda(\{\eta : \eta(x) = 0 \text{ for any } x \in A\})$, where $\nu_\lambda$ is an invariant measure of the process starting from a configuration: $\eta(x) = 1 (x \in \mathbb{Z}^d)$ and is called the \textit{upper invariant measure}. In other words, let $\delta_1 S(t)$ denote the probability measure at time $t$ for initial probability measure $\delta_1$ which is the pointmass $\eta \equiv i(i = 0, 1)$. Then $\nu_\lambda = \lim_{t \to \infty} \delta_1 S(t)$. Then self-duality of the process implies that $\nu_\lambda(A) = \lim_{t \to \infty} \mathbb{P}(\xi_t^A = \emptyset)$. The correlation identities for $\nu_\lambda(A)$ can be obtained as follows:

**Theorem 1.1** For any $A \in Y$,

\[
\lambda \sum_{x \in A} \sum_{y : |y-x|=1} \left[ \nu_\lambda(A \cup \{y\}) - \nu_\lambda(A) \right] + \sum_{x \in A} \left[ \nu_\lambda(A \setminus \{x\}) - \nu_\lambda(A) \right] = 0.
\]

From now on we consider the one-dimensional case. We introduce the following notation:

\[
\nu_\lambda(\circ) = \nu_\lambda(\{0\}), \quad \nu_\lambda(\circ \circ) = \nu_\lambda(\{0, 1\}), \quad \nu_\lambda(\circ \times \circ) = \nu_\lambda(\{0, 2\}), \ldots
\]

By Theorem 1.1, we obtain

**Corollary 1.2**

\[
\begin{align*}
(1) &\quad 2\lambda \nu_\lambda(\circ \circ) - (2\lambda + 1) \nu_\lambda(\circ) + 1 = 0, \\
(2) &\quad \lambda \nu_\lambda(\circ \circ \circ) - (\lambda + 1) \nu_\lambda(\circ \circ) + \nu_\lambda(\circ) = 0, \\
(3) &\quad 2\lambda \nu_\lambda(\circ \circ \circ) + \nu_\lambda(\circ \times \circ) - (2\lambda + 3) \nu_\lambda(\circ \circ \circ) + 2\nu_\lambda(\circ \circ) = 0, \\
(4) &\quad \lambda \nu_\lambda(\circ \times \circ) - (2\lambda + 1) \nu_\lambda(\circ \times \circ) + \lambda \nu_\lambda(\circ \circ \circ) + \nu_\lambda(\circ) = 0.
\end{align*}
\]
The detailed discussion concerning results in this section can be seen in Konno [2, 3]. If we regard $\lambda, \nu_\lambda(o), \nu_\lambda(oo), \nu_\lambda(o \circ o), \ldots$ as variables, then the left hand sides of the correlation identities by Theorem 1.1 are polynomials of degree at most two. In the next section, we give a new procedure for getting a series of approximations for extinction probabilities based on the Gröbner basis by using Corollary 1.2. As for the Gröbner basis, see [4], for example.

2 Our results

Put $x = \nu_\lambda(o), y = \nu_\lambda(oo), z = \nu_\lambda(o \circ o), w = \nu_\lambda(o \times o), s = \nu_\lambda(o \circ oo), u = \nu_\lambda(o \circ o \circ o)$. Let $<$ denote the lexicographic order with $\lambda < x < y < w < z < u < s$. For $m = 1, 2, 3$, let $I_m$ be the ideals of a polynomial ring $\mathbb{R}[x_1, x_2, \ldots, x_{n(m)}]$ over $\mathbb{R}$ as defined below. Here $x_1 = \lambda, x_2 = x, x_3 = y, x_4 = z, x_5 = w, x_6 = s, x_7 = u$ and $n(1) = 3, n(2) = 4, n(3) = 7$.

2.1 First approximation

We consider the following ideal based on Corollary 1.2 (1):

\begin{equation}
I_1 = (2\lambda y - 2\lambda x - x + 1, y - x^2) \subset \mathbb{R}[\lambda, x, y].
\end{equation}

Here $y - x^2$ corresponds to the first (or mean-field) approximation: $\nu_\lambda^{(1)}(oo) = (\nu_\lambda^{(1)}(o))^2$. Then

\begin{equation}
G_1 = \{(x - 1)(2\lambda x - 1), y - x^2\}
\end{equation}

is the reduced Gröbner basis for $I_1$ with respect to $<$. Therefore the solution except a trivial one $x(= y) = 1$ is $x = \nu_\lambda^{(1)}(o) = 1/(2\lambda)$. Remark that the trivial solution means that the invariant measure is $\delta_0$. From this, we obtain the first approximation of the density of the particle, $\rho_\lambda = E_{\nu_\lambda}(\eta(x))$, as follows:

\begin{equation}
\rho_\lambda^{(1)} = 1 - \nu_\lambda^{(1)}(o) = \frac{2\lambda - 1}{2\lambda},
\end{equation}

for any $\lambda \geq 1/2$. This result gives the first lower bound $\lambda^{(1)}_c$ of the critical value $\lambda_c$ of the one-dimensional contact process, that is, $\lambda^{(1)}_c = \frac{1}{2} \leq \lambda_c$. However it should be noted that the inequality is not proved in our approach. The estimated value of $\lambda_c$ is about 1.649.
2.2 Second approximation

Consider the following ideal based on Corollary 1.2 (1) and (2):

\[ I_2 = \langle 2\lambda y - 2\lambda x - x + 1, \lambda z - \lambda y - y + x, xz - y^2 \rangle \subset \mathbb{R}[\lambda, x, y, z]. \]

Here \( xz - y^2 \) corresponds to the second (or pair) approximation: \( \nu_\lambda^{(2)}(o) \nu_\lambda^{(2)}(oo \circ o) = (\nu_\lambda^{(2)}(oo))^{2} \). Then

\[ G_2 = \{(x - 1)((2\lambda - 1)x - 1), 1 + 2\lambda(y - x) - x, 
-y - yx + 2x^2, -z - y(2 + y) + 4x^2\} \]

is the reduced Gröbner basis for \( I_2 \) with respect to \( \prec \). Therefore the solution except a trivial one \( x(= y = z) = 1 \) is \( x = \nu_\lambda^{(2)}(o) = 1/(2\lambda - 1) \). As in a similar way of the first approximation, we get the second approximation of the density of the particle:

\[ \rho_\lambda^{(2)} = \frac{2(\lambda - 1)}{2\lambda - 1}, \]

for any \( \lambda \geq 1 \). This result implies the second lower bound \( \lambda_c^{(2)} = 1 \). We should remark that if we take

\[ I_2' = \langle 2\lambda y - 2\lambda x - x + 1, \lambda z - \lambda y - y + x, y - x^2, z - x^3 \rangle \subset \mathbb{R}[\lambda, x, y, z], \]

then we have

\[ G_2' = \{z - 1, y - 1, x - 1\} \]

is the reduced Gröbner basis for \( I_2' \) with respect to \( \prec \). Here \( y - x^2 \) and \( z - x^3 \) correspond to an approximation: \( \nu_\lambda^{(2')}(oo) = (\nu_\lambda^{(2')}(o))^{2} \) and \( \nu_\lambda^{(2')}(o \circ o) = (\nu_\lambda^{(2')}(o))^{3} \), respectively. Then we have only trivial solution: \( x = y = z = 1 \).

2.3 Third approximation

Consider the following ideal based on Corollary 1.2 (1)–(4):

\[ I_3 = \langle 2\lambda y - 2\lambda x - x + 1, \lambda z - \lambda y - y + x, 
2\lambda s + w - (2\lambda + 3)z + 2y, \lambda u - (2\lambda + 1)w + \lambda z + x, 
ys - z^2, xu - yw \rangle \subset \mathbb{R}[\lambda, x, y, z, w, s, u]. \]
Here $ys - z^2$ and $xu - yw$ correspond to the third approximation: $\nu_\lambda^{(3)}(\circ \circ)\nu_\lambda^{(3)}(\circ \circ \circ) = (\nu_\lambda^{(3)}(\circ \circ \circ))^2$ and $\nu_\lambda^{(3)}(\circ)\nu_\lambda^{(3)}(\circ \circ \circ) = \nu_\lambda^{(3)}(\circ \circ \circ)\nu_\lambda^{(3)}(\circ \circ \circ)$, respectively. Then

$$G_3 = \{(x - 1)((12\lambda^3 - 5\lambda - 1)x^2 - 2\lambda(2\lambda + 3)x - \lambda + 1), \ldots\}$$

is the reduced Gröbner basis for $I_3$ with respect to $\prec$. Therefore the solution except a trivial one $x = 1$ is $x = \nu_\lambda^{(3)}(\circ) = (\lambda(2\lambda + 3) + \sqrt{D})/(12\lambda^3 - 5\lambda - 1)$, where $D = 16\lambda^4 + 4\lambda^2 + 4\lambda + 1$. Then we obtain the third approximation of the density of the particle:

$$\rho_\lambda^{(3)} = \frac{4\lambda(3\lambda^2 - \lambda - 3)}{12\lambda^3 - 2\lambda^2 - 8\lambda - 1 + \sqrt{D}},$$

for any $\lambda \geq (1 + \sqrt{37})/6$. This result corresponds to the third lower bound $\lambda_c^{(3)} = (1 + \sqrt{37})/6 \approx 1.180$.

### 3 Summary

We obtain the first, second, and third approximations for the extinction probability, the density of the particle, and the lower bound of the one-dimensional contact process by using the Gröbner basis with respect to a suitable term order. These results coincide with results given by the Harris lemma (more precisely, the Katori-Konno method, see [3]) or the BFKL inequality [5] (see also [3]). As we saw, the generators of $I_m$ in Section 2 have degree at most two in $x_1, x_2, \ldots$, such as $2\lambda y - 2\lambda x - x + 1$, $ys - z^2$ in the case of $I_3$. We expect that this property will lead to get the higher order approximations of the process (and other interacting particle systems having a similar property) effectively.

**Acknowledgment.** The author thanks Takeshi Kajiwara for valuable discussions and comments.

**References**


