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Time Evolution with and without Remote Past

Kouji YANO and Yoichiro TAKAHASHI
Research Institute for Mathematical Sciences,
Kyoto University

1 Introduction

Time evolutions from an initial time to the future are usually discussed under certain conditions at the initial time. When we consider time evolutions from remote past, then we do not have an a priori way to impose conditions at the remote past. We know that a stationary stochastic process always admits remote past, but one may ask: "Does there exist any other process which admits remote past?"

In the recent paper [4] the authors have studied time evolutions whose transitions are governed by a given noise driven automorphism on compact abelian groups and showed how restrictive is the requirement for them to admit remote past.

The purpose of the present article is to explain our motivation to study time evolutions which admit remote past.

In the theory of stochastic differential equations, Tsirelson ([5]) has introduced a mysterious example which possesses a *non-strong solution.* For this, he considered the stochastic difference equation of the form

$$\eta_k = \xi_k + \eta_{k-1} \tag{1.1}$$

in discrete *negative* time. If a solution is non-strong, then it involves an extra randomness in addition to the past noise, which must come from the remote past. Yor [6] and Akahori–Uenishi–Yano [1] have studied the equation for general noise processes on compact groups to obtain necessary and sufficient conditions for exsistence of strong solutions and for uniqueness in law.

In the work [4] we studied the equation (1.1) on a compact abelian group G in the case where the noise process is stationary. Then we have succeeded in obtaining a complete description of solutions. It is shown that any possible solution is a mixture of a stationary process and a deterministic translation.

Now we may expect that, if a solution admits remote past, then a kind of stability comes from the remote past. The motivation of the work [4] is to confirm this expectation. We have studied the equation of the form

$$\eta_k = \xi_k + \varphi(\eta_{k-1}) \tag{1.2}$$

for a stationary noise process ξ_k and for an automorphism φ on G. We introduced stable sets in the direction of each character of G and showed that the stable set always has the probability one.

2 Non-strong solutions

2.1 Tsirelson's example

Consider, for instance, a stochastic differential equation on $\mathbb{R}$ of the following form:

$$dX_t = a(X_t)dB_t + b(X_t)dt,\ t \geq 0, \qquad X_0 = x. \tag{2.1}$$

If the coefficients are uniformly Lipschitz continuous and uniformly bounded, then we can construct a solution of (2.1) by successive approximation. Such a solution X is adapted to the noise process B: For any $t \geq 0$ there exists a functional F_t on $C([0,t],\mathbb{R})$ such that

$$X_t = F_t(B_s : s \in [0,t]) \qquad \text{a.s.} \tag{2.2}$$

A solution which enjoys the property (2.2) is called a *strong* solution.

Tsirelson ([5]) has considered a stochastic differential equation of the form

$$dX_t = dB_t + A_t(X_s : s \in [0,t])dt,\ t \geq 0, \qquad X_0 = x. \tag{2.3}$$

The drift coefficient $A_t(X_s : s \in [0,t])$ is a functional adapted to the process X defined as

$$A_t(X_s : s \in [0,t]) = \sum_{k \leq 0} \eta_{k-1} 1_{[t_{k-1},t_k)}(t) \tag{2.4}$$

with

$$\eta_k = \text{ the fractional part of } \frac{X_{t_k} - X_{t_{k-1}}}{t_k - t_{k-1}}, \qquad k \leq 0 \tag{2.5}$$

for a sequence $(t_k : k \leq 0)$ such that $t_k \searrow 0$ as $k \to -\infty$.

Theorem 2.1 (Tsirelson [5]). *Any solution of the stochastic differential equation* (2.3) *is non-strong.*

Tsirelson's idea for the proof of Theorem 2.1 is to reduce the problem to a stochastic equation in discrete time. For any solution of the equation (2.3), we set

$$\xi_k = \text{ the fractional part of } \frac{B_{t_k} - B_{t_{k-1}}}{t_k - t_{k-1}}, \qquad k \leq 0. \tag{2.6}$$

Then, under the identification of $[0,1)$ with the abelian group $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}^1$, the stochastic equation

$$\eta_k = \xi_k + \eta_{k-1}, \qquad k \leq 0 \tag{2.7}$$

holds where each ξ_k of the noise process $\xi = (\xi_k : k \leq 0)$ is independent of the past $(\eta_j, \xi_j : j \leq k-1)$. If the equation (2.3) possessed a strong solution, then the corresponding solution of the equation (2.7) should also be strong. Hence for the proof of Theorem 2.1 it suffices to show that the equation (2.7) does not have any strong solution.

2.2 Non-strong solutions for stochastic equations in discrete time

Let G be a compact abelian group and consider the stochastic equation in discrete time:

$$\eta_k = \xi_k + \eta_{k-1}, \qquad k \leq 0 \tag{2.8}$$

Let $\mu = (\mu_k : k \leq 0)$ be a sequence of probability measures μ_k on G.

Definition 2.2. A solution of the equation (2.8) with noise law μ is a probability measure $\boldsymbol{P}$ on $G^{-\mathbf{N}}$ such that if $\eta = (\eta_k : k \leq 0)$ is the coordinate process then the process ξ defined by $\xi_k = \eta_k - \eta_{k-1}$ satisfies the following:

(i) Each ξ_k is independent of the past $(\eta_j, \xi_j : j \leq k-1)$;

(ii) The law of each ξ_k is μ_k.

Let $\mathcal{F}_k^\eta$ (resp. $\mathcal{F}_k^\xi$) denote the σ-field generated by $(\eta_j : j \leq k)$ (resp. $(\xi_j : j \leq k)$). It is obvious by definition that $\mathcal{F}_k^\xi \subset \mathcal{F}_k^\eta$ for any k.

Definition 2.3. A solution $\boldsymbol{P}$ is called *strong* if $\mathcal{F}_k^\eta \subset \mathcal{F}_k^\xi$ for any k $\boldsymbol{P}$-a.s., that is, for any k there exists a sequence of functions F_k on $G^{-\mathbf{N}}$ such that

$$\eta_k = F_k(\xi_j : j \leq k), \quad k \leq 0 \qquad \boldsymbol{P}\text{-a.s.} \tag{2.9}$$

For a solution $\boldsymbol{P}$ of the equation (2.8) with noise law μ, let λ_k for $k \leq 0$ denote the law of η_k under $\boldsymbol{P}$. Then the family of marginal laws $\lambda = (\lambda_k : k \leq 0)$ satisfies the convolution equation

$$\lambda_k = \mu_k * \lambda_{k-1}. \tag{2.10}$$

Conversely, it is easy to see (cf. [1]) by Kolmogorov extension theorem that, for a solution λ of the convolution equation (2.10), there exists a unique solution $\boldsymbol{P}$ of the stochastic equation (2.8) under which the law of each η_k coincides with λ_k.

Let ν_G stand for the normalized Haar measure of G. Note that for any probability measure μ on G it holds that $\nu_G = \mu * \nu_G$. Therefore, for any given noise law $\mu = (\mu_k : k \leq 0)$, the equation (2.8) always possesses a solution $\boldsymbol{P}_\mu^*$ whose marginal laws satisfy $\lambda_k = \nu_G$, $k \leq 0$.

Lemma 2.4 ([6], [1]). *Suppose that $G \neq \{0\}$. Then the solution $\boldsymbol{P}_\mu^*$ is non-strong. In particular, the equation* (2.8) *with any given noise law μ possesses a non-strong solution.*

We denote the set of solutions of the equation (2.8) with noise law μ by $\mathcal{P}_\mu$. Then the set $\mathcal{P}_\mu$ is a non-empty closed convex subset of the compact convex space which consists of probability measures on $G^{-\mathbf{N}}$ equipped with the topology of weak convergence. We denote the extremal points of $\mathcal{P}_\mu$ by $\mathrm{ex}(\mathcal{P}_\mu)$.

Set $\mathcal{F}_{-\infty}^\eta = \cap_k \mathcal{F}_k^\eta$, which may be understood as the information of the remote past. It is well-known that $\boldsymbol{P} \in \mathrm{ex}(\mathcal{P}_\mu)$ if and only if $\mathcal{F}_{-\infty}^\eta$ is $\boldsymbol{P}$-trivial. In particular, since $\mathcal{F}_{-\infty}^\xi$ is always $\boldsymbol{P}$-trivial by Kolmogorov's 0-1 law, a strong solution is always an extremal point of $\mathcal{P}_\mu$.

2.3 A criterion for existence of a strong solution

In this subsection we state a version in our settings of a criterion for existence of a strong solution obtained by Yor [6] in the case $G = \mathbb{T}^1$ and generalized by Akahori–Uenishi–Yano [1].

We keep the settings of Section 2.2. Let Γ denote the character group of G. For any noise law $\mu = (\mu_k : k \le 0)$, we define

$$\Gamma_\mu = \left\{ \chi \in \Gamma : \prod_{j \le k} |\mu_j(\chi)| > 0 \quad \text{for some } k \right\}. \tag{2.11}$$

Set

$$G_\mu = \{ g \in G : \chi(g) = 1 \quad \text{for any } \chi \in \Gamma_\mu \}. \tag{2.12}$$

It is obvious by definition that G_μ is a closed normal subgroup of G.

Remark 2.5. Γ_μ is a subgroup of the character group Γ (See [4, Proposition 3.5]). Hence the relation between the subgroup Γ_μ and the quotient group G/G_μ is in Pontryagin duality.

Theorem 2.6 ([6, Theorem 4], [1, Theorem 1.4]). *The equation* (2.8) *with a given noise law* μ *possesses a strong solution if and only if* $G_\mu = \{0\}$.

Example 2.7. Let $G = \mathbb{T}^1 \cong [0,1)$ and let ξ_k be defined as in (2.6). Then, for $\chi_n(g) = e^{2\pi i n g}$ with $n \in \mathbb{Z}$, we have

$$\mu_k(\chi_n) = \int_{-\infty}^{\infty} e^{2\pi i n x/\sqrt{t_k - t_{k-1}}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = \exp\left\{ -\frac{(2\pi n)^2}{2(t_k - t_{k-1})} \right\}. \tag{2.13}$$

Since $\sum_k 1/(t_k - t_{k-1}) = \infty$, we see that $\chi_n \in \Gamma_\mu$ if and only if $n = 0$. This means that $\Gamma_\mu = \{1\}$ and hence that $G_\mu = G$. By Theorem 2.6, we conclude that the equation (2.8) with the noise law given by (2.6) does not possess a strong solution.

2.4 A criterion for uniqueness in law

We also state a criterion for uniqueness of solutions. The following result has been obtained by Yor [6] in the case $G = \mathbb{T}^1$ and generalized by Akahori–Uenishi–Yano [1].

Theorem 2.8 ([6, Theorem 3], [1, Theorem 1.4]). *The solution of the equation* (2.8) *with a given noise law* μ *is unique if and only if* $G_\mu = G$.

The theorem is an immediate consequence of the following theorem. The group G acts on the product group $G^{-\mathbf{N}}$ as $(g\eta)_k = g\eta_k$ for $k \le 0$ where $\eta = (\eta_k : k \le 0)$. Then the point action of G on $G^{-\mathbf{N}}$ induces an action on the space of continuous functions on $G^{-\mathbf{N}}$ as $(gf)(h) = f(gh)$ for $g, h \in G$, and therefore induces an action on the space of probability measures on $G^{-\mathbf{N}}$ as $(g\boldsymbol{P})(f) = \boldsymbol{P}(gf)$.

Theorem 2.9 ([1, Theorems 1.3 and 1.5]). *The set of extremal solutions* $\mathrm{ex}(\mathcal{P}_\mu)$ *is isomorphic to the quotient space* G/G_μ. *That is,*

(i) For $\boldsymbol{P}^1, \boldsymbol{P}^2 \in \mathrm{ex}(\mathcal{P}_\mu)$, *there exists an element* $g \in G$ *such that* $g\boldsymbol{P}^1 = \boldsymbol{P}^2$;

(ii) For $\boldsymbol{P} \in \mathrm{ex}(\mathcal{P}_\mu)$, $g\boldsymbol{P} = \boldsymbol{P}$ *if and only if* $g \in G_\mu$.

3 The stochastic equation with stationary noise

We keep the notations in Section 2.2. Hence the theorems in Sections 2.3 and 2.4 also hold.

In this section we assume that the noise process is identity: $\mu_k = \mu$ for any $k \le 0$ for an arbitrary probability measure μ on G. In this case the definition (2.11) of the Γ_μ implies

$$\Gamma_\mu = \{\chi \in \Gamma : |\mu(\chi)| = 1\}. \tag{3.1}$$

Recall

$$G_\mu = \{g \in G : \chi(g) = 1 \text{ for any } \chi \in \Gamma_\mu\}. \tag{3.2}$$

Consider the stochastic equation

$$\eta_k = \xi_k + \eta_{k-1}, \qquad k \le 0 \tag{3.3}$$

and recall that this is equivalent to the convolution equation

$$\lambda_k = \mu * \lambda_{k-1}, \qquad k \le 0. \tag{3.4}$$

Theorem 3.1 **([4, Theorem 1.1]).** *There exists an element $\alpha(\mu) \in G/G_\mu$ such that any solution $(\lambda_k : k \le 0)$ of the equation (3.4) satisfies the following:*

(i) Each λ_k is G_μ-invariant;

(ii) The projections $\widehat{\lambda}_k$ of λ_k on G/G_μ evolve by the Weyl transformation:

$$\widehat{\lambda}_k = \alpha(\mu)\widehat{\lambda}_{k-1}, \qquad k \le 0. \tag{3.5}$$

We can restate the theorem in terms of the process of the stochastic equation (3.3).

Corollary 3.2. *Let $\boldsymbol{P}$ be a solution of the equation (3.3). Let $a \in \alpha(\mu)$ be fixed. Then there exists a random variable $\gamma \in G$ and a process $\zeta = (\zeta_k : k \le 0)$ such that the following hold:*

(i) $\eta_k = \gamma + ka + \zeta_k$ for $k \le 0$;

(ii) ζ is a stationary process with independent increments whose stationary law is the normalized Haar measure on G_μ such that $\zeta_k - \zeta_{k-1} = \xi_k - a$.

(iii) γ is independent of $(\zeta_k : k \le 0)$.

Example 3.3. Let $G = \mathbb{T}^1 \cong [0,1)$ and let $\mu = p\delta_{x_1} + (1-p)\delta_{x_2}$ with $0 < p < 1$ and $x_1, x_2 \in [0,1)$, $0 \le x_1 < x_2 < 1$. For $\chi_n(g) = e^{2\pi i n g}$ with $n \in \mathbb{Z}$, we see that $\chi_n \in \Gamma_\mu$ if and only if $n(x_2 - x_1) \in \mathbb{Z}$.

(i) If $x_2 - x_1$ is rational, then we can take two positive integers r, p such that r and p are mutually coprime and $x_2 - x_1 = r/p$. Then $\Gamma_\mu = \{\chi_n : n \in p\mathbb{Z}\}$ and hence $G_\mu = \mathbb{Z}_p := \{m/p : m = 0, 1, \ldots, p-1\}$ and $\alpha(\mu) = x_1 + \mathbb{Z}_p (= x_2 + \mathbb{Z}_p)$. In this case the solutions of the equation (3.3) is not unique and any solution is non-strong.

(ii) If $x_2 - x_1$ is irrational, then $\Gamma_\mu = \{1\}$ and $G_\mu = G$, and hence the solution of the equation (3.3) is unique (and non-strong).

Example 3.4. Consider the backward heat equation

$$\frac{\partial p}{\partial t} = -\frac{1}{2}\frac{\partial^2 p}{\partial x^2} \qquad t > 0,\ x \in \mathbb{R} \tag{3.6}$$

with periodic boundary condition:

$$p(t, x+1) = p(t, x), \qquad t > 0,\ x \in \mathbb{R}. \tag{3.7}$$

Then what initial conditions are allowed for a classical solution to exist on the whole time $(0, \infty)$? The answer is that the only possible initial condition is that $p(0, x)$ is a constant function. To prove this, we assume that there exists a function $p(t, x) \in C^{1,2}((0, \infty) \times \mathbb{R})$ such that (3.6) and (3.7) hold. Since the value of the integral $\int_0^1 p(t, x)dx$ is constant for any $t > 0$, we may assume that it is constant one. Let $\lambda_k(dx) = p(-k, x)dx$ for $k \leq 0$. Then the backward heat equation (3.6) with the periodic boundary condition (3.7) yields the convolution equation (3.4) for $G = [0, 1) \cong \mathbb{T}^1$ where

$$\mu(dx) = \sum_{n \in \mathbb{Z}} e^{-(x+n)^2/2} \frac{dx}{\sqrt{2\pi}}, \qquad \text{on } [0, 1). \tag{3.8}$$

Since $\mu(\chi_n) = \exp(-(2\pi n)^2/2)$, we see that $\Gamma_\mu = \{0\}$ and $G_\mu = G$. Hence we conclude that the convolution equation has the only solution such that $\lambda_0(dx)$ is the Lebesgue measure on $[0, 1)$.

Remark 3.5. In Furstenberg's theory (cf., e.g., [3]) he has studied stationary processes on a G-space M. In the case where $M = G$, a μ-process in Furstenberg's sense is a solution of the stochastic equation (3.3) which is a stationary process with stationary measure μ. Then it is easy to see that his definition that a μ-process is *proper* coincides with our definition that it is strong. In our settings, there exists a solution of the stochastic equation (3.3) which is a proper μ-process in Furstenberg's sense if and only if $G_\mu = \{0\}$ and $a(\mu) = 0$; This means that μ is the point mass at 0.

Remark 3.6. Brossard and Leuridan [2] has studied the existence and uniqueness problem of solutions and recurrence or transience problem in Markov chains in discrete negative time. They assumed that the transition probability has a density with respect to a measure. However, the transition probability corresponding to the time evolution by the Weyl transformation never possesses a density.

4 The stochastic equation as the iteration by a noise driven automorphism

Let G be a compact abelian group and consider the stochastic equation

$$\eta_k = \xi_k + \varphi(\eta_{k-1}), \qquad k \leq 0 \tag{4.1}$$

where the noise ξ is stationary with stationary measure μ and φ is an automorphism on G.

If we set $\eta_k^\varphi = \varphi^{-k}\eta_k$ and $\xi_k^\varphi = \varphi^{-k}\xi_k$, then we obtain

$$\eta_k^\varphi = \xi_k^\varphi + \eta_{k-1}^\varphi, \qquad k \le 0. \tag{4.2}$$

This is exactly the stochastic equation (2.8) with noise law given by $\mu_k^\varphi = \varphi^{-k}\mu$. Now we denote the subgroup Γ_{μ^φ} for $\mu^\varphi = (\mu_k^\varphi : k \le 0)$ as is defined in (2.11) simply by Γ_μ. We say that $\boldsymbol{P}$ is a solution of the equation (4.1) with law μ if the law of $(\eta_k^\varphi : k \le 0)$ under $\boldsymbol{P}$ is a solution of the equation (4.2) with law μ^φ.

Proposition 4.1 ([4, Proposition 4.3]). *The following statements hold:*

(i) Γ_μ is φ-invariant.

(ii) G_μ is φ-invariant.

(iii) Let $\boldsymbol{P}$ be a solution of the equation (4.1). Then the law of η_k under $\boldsymbol{P}$ is G_μ-invariant.

The above proposition is an immediate consequence of the following characterization of the subgroup Γ_μ.

Lemma 4.2 ([4, Lemma 3.4]). *A character χ belongs to Γ_μ if and only if*

$$\sum_{n=0}^{\infty} \int_G \int_G \mu(dx)\mu(dy)|\chi(\varphi^n x) - \chi(\varphi^n y)|^2 < \infty. \tag{4.3}$$

For $\chi \in \Gamma$, we define

$$W_2^s(\chi, \varphi) = \left\{(x, y) \in G \times G : \sum_{n=0}^{\infty} |\chi(\varphi^n x) - \chi(\varphi^n y)|^2 < \infty\right\} \tag{4.4}$$

and, for $x \in G$,

$$W_2^s(x; \chi, \varphi) = \{y \in G : (x, y) \in W_2^s(\chi, \varphi)\} \tag{4.5}$$

$$= \left\{y \in G : \sum_{n=0}^{\infty} |\chi(\varphi^n x) - \chi(\varphi^n y)|^2 < \infty\right\}. \tag{4.6}$$

We call the set $W_2^s(x; \chi, \varphi)$ the *ℓ^2-stable set of x in the direction χ with respect to φ*. It is obvious that the set $W_2^s(x; \chi, \varphi)$ is contained in the stable set

$$W^s(x; \chi, \varphi) = \left\{y \in G : \lim_{n\to\infty} \frac{\chi(\varphi^n x)}{\chi(\varphi^n y)} = 1\right\}. \tag{4.7}$$

Proposition 4.3 ([4, Proposition 4.3]). *Assume that G has a countable base. If $\chi \in \Gamma_\mu$, then $(\mu \otimes \mu)(W_2^s(\chi, \varphi)) = 1$.*

Theorem 4.4 ([4, Theorem 1.3]). *Assume that G has a countable base. There exists an element $\alpha(\mu) \in G/G_\mu$ such that $\mu(\cap_{\chi\in\Gamma_\mu} W^s(a, \chi, \varphi)) = 1$ for any $a \in \alpha(\mu)$.*

Remark 4.5. If φ is the identity, then it is the case in Section 3. In this case, we have

$$W_2^s(x;\chi,\varphi) = W^s(x;\chi,\varphi) = \{y \in G : \chi(x) = \chi(y)\} \tag{4.8}$$

and

$$\bigcap_{\chi\in\Gamma_\mu} W_2^s(x;\chi,\varphi) = \bigcap_{\chi\in\Gamma_\mu} W^s(x;\chi,\varphi) = x + G_\mu. \tag{4.9}$$

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