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Author(s)	HIRAYAMA, MICHIHIRO; SUMI, NAOYA
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ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR EXPANSIVE DIFFEOMORPHISMS OF THE 2-TORUS

MICHIHIRO HIRAYAMA (平山 至大) AND NAOYA SUMI (鷲見 直哉)

ABSTRACT. We establish an equivalent criterion for certain expansive diffeomorphisms of the 2-torus to admit an invariant Borel probability measure that is absolutely continuous with respect to the Riemannian volume. Our result is closely related to the well known Livšic-Sinai theorem for Anosov diffeomorphisms.

1. INTRODUCTION

Let $g\colon M \to M$ be a transitive C^2 Anosov diffeomorphism on a compact Riemannian manifold M. A celebrated work of Livšic and Sinai [6] says that g admits an invariant Borel probability measure that is absolutely continuous with respect to the Riemannian volume on M if and only if $|\mathrm{Jac}(D_p g^n)| = 1$ holds for every periodic point $p \in \mathrm{Fix}(g^n)$ and $n \in \mathbb{N}$, where Jac stands for the Jacobian and $\mathrm{Fix}(g) = \{x \in M \colon g(x) = x\}$. We refer the reader to [2] for more precise. Our aim here is to further the study of relations of this type for certain expansive diffeomorphisms.

Let $f\colon M \to M$ be a $C^{1+\alpha}(\alpha > 0)$ diffeomorphism of a compact Riemannian manifold M preserving a hyperbolic Borel probability measure μ. In Corollary 5.6 of [5] Ledrappier proved that the following (A) and (B) are equivalent.

Property (A). The measure μ is absolutely continuous with respect to the volume on M.

Property (B). The measure μ is absolutely continuous with respect to both stable and unstable laminations (see the definition in the next section).
It follows from the Pesin entropy formula ([8]) that (B) is equivalent to the following:

Property (C). μ is absolutely continuous with respect to unstable lamination and

$$\int \log |\mathrm{Jac}(D_x f)|\, d\mu(x) = 0. \tag{1.1}$$

Moreover we can derive (C) from the following:

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Property (D). μ is absolutely continuous with respect to unstable lamination and $|\mathrm{Jac}(D_p f^n)| = 1$ holds for $p \in \mathrm{Fix}(f^n)$ and $n \in \mathbb{N}$.

The Livšic-Sinai theorem could be reformulated in this context as the properties (A) and (D) are equivalent for Anosov diffeomorphisms. It then asserts that all properties above are equivalent, particularly that (C) implies (D). Little seems to be known for this implication in the broader context beyond Anosov. It is to this problem that we would turn.

To state the result we recall the following notion. Let $x \in M$ and $\delta > 0$. Define the *local stable* and *local unstable sets* at x by

$$\mathcal{W}^s_\delta(x) = \{y \in M : d(f^n(x), f^n(y)) \leq \delta \ (n \geq 0)\},$$
$$\mathcal{W}^u_\delta(x) = \{y \in M : d(f^{-n}(x), f^{-n}(y)) \leq \delta \ (n \geq 0)\},$$

where d is the distance on M induced by the Riemannian metric.

Theorem 1.1. *Let $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ be an expansive C^2 diffeomorphism of the 2-torus preserving a hyperbolic Borel probability measure μ. Assume that for all $x \in \mathbb{T}^2$ the local stable and unstable sets at x form C^1 curves and they intersect transversally at x in the sense that $T_x\mathbb{T}^2 = T_x\mathcal{W}^s_\delta(x) \oplus T_x\mathcal{W}^u_\delta(x)$. Then the following two assertions are equivalent:*

(1) *μ is absolutely continuous with respect to the Riemannian volume on $\mathbb{T}^2$.*
(2) *μ is absolutely continuous with respect to unstable lamination and $|Jac(D_p f^n)| = 1$ for $p \in Fix(f^n)$ and $n \in \mathbb{N}$.*

As an immediate corollary of this theorem we conclude, under the same assumption as in Theorem 1.1, all the properties (A), (B), (C) and (D) are equivalent for expansive C^2 diffeomorphisms on the 2-torus preserving a hyperbolic Borel probability measure.

The implication that (1) follows from (2) would be given with no assumption on local manifolds as in the theorem. More precisely we establish the implication for *every* diffeomorphism on a Riemannian manifold preserving a hyperbolic measure.

Proposition 1.2. *Let $f\colon M \to M$ be a $C^{1+\alpha}(\alpha > 0)$ diffeomorphism of a compact Riemannian manifold M preserving a hyperbolic Borel probability measure μ. If the measure μ is absolutely continuous with respect to the $\mathcal{W}^u$-lamination and $|Jac(D_p f^n)| = 1$ for $p \in Fix(f^n)$ and $n \in \mathbb{N}$, then it is absolutely continuous with respect to the Riemannian volume on M.*

We emphasize the assumption in Theorem 1.1 may not guarantee the existence of hyperbolic absolutely continuous invariant probability measures. After the construction of a diffeomorphism of a compact surface with nonzero Lyapunov exponents which is not Anosov due to Katok [4], a diffeomorphism of $\mathbb{T}^2$ admitting no hyperbolic absolutely continuous invariant probability measures is given as follows. Start with the hyperbolic linear automorphism g of $\mathbb{T}^2$ having positive eigenvalues $\alpha^{-1} < 1 < \alpha$.

Proposition 1.3. *There is a one-parameter family $\{g_a\}_{a\in[0,1]}$ of C^∞ diffeomorphisms of $\mathbb{T}^2$ with $g_0 = g$ satisfying the following:*

(1) *for each $a \in [0,1)$ the diffeomorphism g_a is Anosov and admits an invariant probability measure which is absolutely continuous with respect to the Riemannian volume on $\mathbb{T}^2$;*

(2) *the diffeomorphism g_1 admits no hyperbolic absolutely continuous invariant probability measures while it satisfies the assumption as in Theorem 1.1 and $|Jac(D_q g_1^n)| = 1$ for $q \in Fix(g_1^n)$ and $n \in \mathbb{N}$.*

We refer the reader to [3] for the complete description of this work.

2. DEFINITIONS

(2A) Let M be a compact C^∞ manifold with a Riemannian norm $\|\cdot\|$, $f : M \to M$ a $C^{1+\alpha}(\alpha > 0)$ diffeomorphism of M and $Df : TM \to TM$ the derivative of f. Let also μ be a Borel probability measure invariant under f. The point $x \in M$ is said to be *Lyapunov regular* if there exist real numbers $\chi_1(x) > \cdots > \chi_{r(x)}(x)$ and a D_xf-invariant decomposition $T_xM = E_1(x) \oplus \cdots \oplus E_{r(x)}(x)$ such that for each $i = 1, 2, \ldots, r(x)$

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \chi_i(x) \qquad (v \in E_i(x) \setminus \{0\})$$

exists, and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |\mathrm{Jac}(D_x f^n)| = \sum_{i=1}^{r(x)} \chi_i(x) \dim E_i(x).$$

We denote by Γ the set of Lyapunov regular points. By the multiplicative ergodic theorem ([7]) Γ has full μ-measure. The numbers $\chi_i(x)$ are called the *Lyapunov exponents* of f at the point x. The functions $x \mapsto \chi_i(x)$, $r(x)$ and $\dim E_i(x)$ are Borel measurable and f-invariant. We call a measure μ *hyperbolic* if none of the Lyapunov exponents for μ vanish and there exist Lyapunov exponents with different signs for μ-almost every $x \in M$.

Let $x \in \Gamma$. We define the *stable* and the *unstable manifolds* at x as

$$\mathcal{W}^s(x) = \left\{ y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^n x, f^n y) < 0 \right\},$$

$$\mathcal{W}^u(x) = \left\{ y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^{-n} x, f^{-n} y) < 0 \right\}.$$

Then $\mathcal{W}^s(x)$ and $\mathcal{W}^u(x)$ are injectively immersed manifolds satisfying

$$T_x\mathcal{W}^s(x) = E^s(x),\ T_x\mathcal{W}^u(x) = E^u(x),$$

where $E^s(x) = \oplus_{i:\chi_i(x)<0} E_i(x)$ and $E^u(x) = \oplus_{i:\chi_i(x)>0} E_i(x)$ ([1]). Both $\mathcal{W}^s(x)$ and $\mathcal{W}^u(x)$ inherit a Riemannian structure from M and hence a Riemannian volume and a distance. We write the volume and the distance on $\mathcal{W}^\tau(x)$ as m_x^τ and d_x^τ, respectively ($\tau = s, u$).

(2B) We call

$$e_f(\mu) = -\int \log|\mathrm{Jac}(D_x f)|\, d\mu(x)$$

the *entropy production* for μ (in the sense of Ruelle [10]). It is easy to see that the entropy production is independent of the choice of Riemannian metrics and the multiplicative ergodic theorem asserts

$$e_f(\mu) = -\int \sum_{i=1}^{r(x)} \chi_i(x) \dim E_i(x)\, d\mu(x).$$

We refer the reader to [10, 11] for more precise. Note that the equation (1.1) says the entropy production for μ vanishes.

(2C) Let $\mathcal{B}$ be the Borel σ-algebra of M completed with respect to μ and ξ a partition of M. We say a subset $A \subset M$ *ξ-set* if it is unions of elements of ξ. A countable system $\{A_i\}_{i\geq 1} \subset \mathcal{B}$ of measurable ξ-sets is said to be a *basis* of ξ if for any two distinct elements C_1, C_2 of ξ, there exists A_{i_0} such that, up to sets of measure zero, either $C_1 \subset A_{i_0}$ and $C_2 \not\subset A_{i_0}$ or $C_1 \not\subset A_{i_0}$ and $C_2 \subset A_{i_0}$. A partition with a basis is said to be *measurable*. Denote by $\mathcal{B}_\xi$ the sub σ-algebra of $\mathcal{B}$ whose elements are unions of elements of ξ. We denote by $C_\xi(x)$ the element of ξ containing $x \in M$. We write $\eta \leq \xi$ if η is, up to sets of measure zero, a sub-partition of ξ.

For a measurable partition ξ of M, there exists a *canonical system of conditional measures*: for μ-almost every $x \in M$ there is a probability measure μ_x^ξ defined on $C_\xi(x)$ such that the function $x \mapsto \mu_x^\xi(A)$ is $\mathcal{B}_\xi$-measurable and $\mu(A) = \int \mu_x^\xi(A) d\mu(x)$ for every $A \in \mathcal{B}$. See [9] for more details.

Let $\mathcal{W}^u = \{\mathcal{W}^u(x) \colon x \in \Gamma\}$ be the *unstable lamination* and ξ^u a measurable partition of M. We say that ξ^u is *subordinate* to the $\mathcal{W}^u$-lamination if for μ-almost every $x \in M$, $C_{\xi^u}(x) \subset \mathcal{W}^u(x)$ and $C_{\xi^u}(x)$ contains an open neighborhood of x in $\mathcal{W}^u(x)$. The measure μ is said to be *absolutely continuous with respect to the $\mathcal{W}^u$-lamination* if for every measurable partition ξ^u subordinate to the $\mathcal{W}^u$-lamination, $\mu_x^{\xi^u}$ is absolutely continuous with respect to m_x^u for μ-almost every $x \in M$. The measurable partition subordinate to the $\mathcal{W}^s$-lamination and the absolute continuity with respect to the $\mathcal{W}^s$-lamination are defined similarly.

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Department of Mathematics, Graduate School of Science, Hiroshima University, Higashi-Hiroshima 739-8526, Japan

E-mail address: `hirayama@math.sci.hiroshima-u.ac.jp`

Department of Mathematics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan

E-mail address: `nsumi@math.titech.ac.jp`