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Continuity properties and vanishing exponential integrability of Riesz potentials of Orlicz functions

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1 Introduction and statement of results

For $0 < \alpha < n$ and a locally integrable function $f$ on $\mathbb{R}^n$, we define the Riesz potential $U_\alpha f$ of order $\alpha$ by

$$U_\alpha f(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y) dy.$$ 

Here it is natural to assume that $U_\alpha |f| \not\equiv \infty$, which is equivalent to

$$\int_{\mathbb{R}^n} (1 + |y|)^{\alpha-n} |f(y)| dy < \infty. \tag{1.1}$$

In the present paper, we treat functions $f$ satisfying an Orlicz condition:

$$\int_{\mathbb{R}^n} \Phi_{p,\varphi}(|f(y)|) dy < \infty. \tag{1.2}$$

Here $\Phi_{p,\varphi}(r)$ is a positive nondecreasing function on the interval $(0, \infty)$ of the form

$$\Phi_{p,\varphi}(r) = r^p \varphi(r),$$

where $p > 1$ and $\varphi(r)$ is a positive monotone function on $[0, \infty)$ which is of logarithmic type; that is, there exists $c_1 > 0$ such that

$$c_1^{-1} \varphi(r) \leq \varphi(r^2) \leq c_1 \varphi(r) \quad \text{whenever } r > 0. \tag{\varphi 1}$$

We set

$$\Phi_{p,\varphi}(0) = 0,$$

because we will see in the proof of Lemma 2.1 below that

$$\lim_{r \to 0^+} \Phi_{p,\varphi}(r) = 0 = \Phi_{p,\varphi}(0).$$

For an open set $G \subset \mathbb{R}^n$, we denote by $L^{\Phi_{p,\varphi}}(G)$ the family of all locally integrable functions $g$ on $G$ such that

$$\int_G \Phi_{p,\varphi}(|g(x)|) dx < \infty,$$
and define
\[ \| g \|_{\Phi_{p,\varphi}} = \| g \|_{\Phi_{p,\varphi},G} = \inf \left\{ \lambda > 0 : \int_{G} \Phi_{p,\varphi}(\| g(x) \| / \lambda) \, dx \leq 1 \right\} \].

This is a quasi-norm in \( L^{\Phi_{p,\varphi}}(G) \).

Our first aim in the present paper is to establish integral inequalities for Riesz potentials of functions in \( L^{\Phi_{p,\varphi}} \). For this purpose, if \( 1 < p < n/\alpha \), then we set
\[ \varphi_{p}^{*}(r) = \left[ \int_{0}^{r} \{ t^{\alpha p-n} \varphi(t) \}^{-p'/p} t^{-1} dt \right]^{1/p'} \]
for \( r \geq 0 \), where \( 1/p + 1/p' = 1 \); if \( p = n/\alpha > 1 \), then we set
\[ \varphi_{p}^{*}(r) = \left[ \int_{1}^{r} \{ \varphi(t) \}^{-p'/p} t^{-1} dt \right]^{1/p'} \]
for \( r \geq 2 \), and extend it to be a (strictly) increasing continuous function on \([0, \infty)\) such that \( \varphi_{p}^{*}(t) = (t/2) \varphi_{p}^{*}(2) \) for \( t \in [0,2) \).

Following Alberico and Cianchi [3], we consider the Sobolev conjugate \( \Psi_{p,\varphi} \) of \( \Phi_{p,\varphi} \) defined by
\[ \Psi_{p,\varphi}(r) = (\psi_{n} \circ (\varphi_{p}^{*})^{-1})(r) \]
for \( r \geq 0 \), where \( \psi_{n}(r) = r^{n} \) and \( (\varphi_{p}^{*})^{-1} \) is the inverse of the function \( \varphi_{p}^{*} \). Note that \( \Psi_{p,\varphi}(r) \) is continuous on \([0, \infty)\) and \( \Psi_{p,\varphi}(0) = 0 \).

As an extension of Alberico and Cianchi [3, Theorem 2.3], we state our first result in the following.

**Theorem A.** Let \( \alpha p \leq n \) and \( G \) be a bounded open set in \( \mathbb{R}^{n} \). Then there exists \( \varepsilon_{0} > 0 \) such that
\[ \int_{G} \Psi_{p,\varphi}(\varepsilon_{0} U_{\alpha} |f|(x)) \, dx \leq 1 \]
whenever \( f \) is a locally integrable function on \( G \) such that \( \| f \|_{\Phi_{p,\varphi}} \leq 1 \).

Cianchi [2, Theorem 2] gave a necessary and sufficient condition that the operator \( f \mapsto U_{\alpha} f \) is bounded from one Orlicz space \( L^{\Phi} \) to another Orlicz space \( L^{\Psi} \); but our statement is straightforward and simple. Further Edmunds and Evans [4, Theorems 3.6.10, 3.6.16] discussed the boundedness of Bessel potentials in Lorenz-Karamata space setting.

Since our function \( \Phi_{p,\varphi} \) may not be convex, for the reader’s convenience, we give a proof of Theorem A different from Cianchi [2] in the next section.

**Remark 1.1** Theorem A implies that
\[ \| U_{\alpha} f \|_{\Psi_{p,\varphi}} \leq \varepsilon_{0}^{-1} \| f \|_{\Phi_{p,\varphi}} \]
whenever \( f \in L^{\Phi_{p,\varphi}}(G) \), where the quasi-norm \( \| \cdot \|_{\Phi_{p,\varphi}} \) is defined in the same way as \( \| \cdot \|_{\Phi_{p,\varphi}} \).
\textbf{Example 1.2} Consider $\Phi_{p,q}(r) = r^p (\log r)^q$ for large $r > 0$, where $p = n/\alpha > 1$ and $q \leq p - 1$. If $q < p - 1$, then
\[ \Psi_{p,q}(r) = C \exp(n r^{p/(p-1-q)}) \]
and if $q = p - 1$, then
\[ \Psi_{p,q}(r) = C \exp(n \exp(r^{p'})) \]
for $r \geq 1$. Hence we have the exponential integrability obtained by Edmunds, Gurka and Opic [5, Theorem 4.6], [6, Theorems 3.1 and 3.2] and the authors [12, Theorems A and B].

\textbf{Corollary 1.3} Let $\alpha p = n$ and $G$ be a bounded open set in $\mathbb{R}^n$. Let $\Phi_{p,q}(r) = r^p (\log r)^q$ for large $r > 0$.

(1) If $q < p - 1$, then there exists $\varepsilon_0 > 0$ such that
\[ \int_G \{ \exp(\varepsilon_0 U_\alpha |f|(x)^{\beta}) - 1 \} dx \leq 1 \]
whenever $f$ is a locally integrable function on $G$ such that $\| f \|_{\Phi_{p,q}} \leq 1$, where $\beta = p/(p - 1 - q)$.

(2) If $q = p - 1$, then there exists $\varepsilon_0 > 0$ such that
\[ \int_G \{ \exp(\exp(\varepsilon_0 U_\alpha |f|(x)^{\beta}) - e) \} dx \leq 1 \]
whenever $f$ is a locally integrable function on $G$ such that $\| f \|_{\Phi_{p,q}} \leq 1$, where $\beta = p/(p - 1)$.

In the case $q > p - 1$, $U_\alpha f$ is shown to be continuous in $G$; see Remark 1.5.

Denote by $p^\#$ the Sobolev conjugate of $p$ which is defined by
\[ \frac{1}{p^\#} = \frac{1}{p} - \frac{\alpha}{n} > 0. \]
We also obtain Sobolev's type inequality for Riesz potentials in the following:

\textbf{Corollary 1.4} Let $\alpha p < n$. Then
\[ \int_{\mathbb{R}^n} \{ U_\alpha |f|(x) \varphi(U_\alpha |f|(x)^{1/p}) \}^{p^\#} dx \leq C \]
whenever $f$ is a locally integrable function on $\mathbb{R}^n$ such that $\| f \|_{\Phi_{p,q}} \leq 1$, where $C$ is a positive constant independent of $f$. 
For a measurable function $u$ on $\mathbb{R}^n$, we define the integral mean over a measurable set $E \subset \mathbb{R}^n$ of positive measure by

$$\int_E u(x) \, dx = \frac{1}{|E|} \int_E u(x) \, dx.$$ 

As an application of Theorem A, we discuss continuity properties for Riesz potentials of functions in $L^{\Phi_{p,\varphi}}(\mathbb{R}^n)$, as an extension of Adams and Hurri-Syrjänen [1, Theorem 1.6] and the authors [14, Theorems A and B].

Our main result is now stated as follows:

**Theorem B.** Let $f$ be a locally integrable function on $\mathbb{R}^n$ satisfying (1.1) and (1.2). Set

$$E_\infty = \{x \in \mathbb{R}^n : \int_{\mathbb{R}^n} |x - y|^{\alpha-n} |f(y)| \, dy = \infty\},$$

$$E_* = \{x \in \mathbb{R}^n : \limsup_{r \to 0} r^{\alpha p-n} \varphi(r^{-1})^{-1} \int_{B(x,r)} \Phi_{p,\varphi}(|f(y)|) \, dy > 0\},$$

$$E^* = \{x \in \mathbb{R}^n : \limsup_{r \to 0} r^{\alpha p-n} \int_{B(x,r)} \Phi_{p,\varphi}(|f(y)|) \, dy > 0\}.$$ 

If $x_0 \in \mathbb{R}^n \setminus (E_\infty \cup E_* \cup E^*)$, then

$$\lim_{r \to 0+} \int_{B(x_0,r)} \Psi_{p,\varphi}(A|U_\alpha f(x) - U_\alpha f(x_0)|) \, dx = 0$$

(1.3)

holds for all $A > 0$.

We discuss the size of the exceptional sets after proving this theorem, in the final section.

**Remark 1.5** Suppose

$$\int_1^\infty \{t^{\alpha p-n} \varphi(t)\}^{1/p'} t^{-1} \, dt < \infty$$

(1.4)

and set

$$\varphi_p(r) = \left(\int_r^\infty \{t^{\alpha p-n} \varphi(t)\}^{1/p'} t^{-1} \, dt\right)^{1/p'}.$$ 

Then it is known (see [9, Theorem 1] and [10, Corollary 3.1]) that $U_\alpha f$ is continuous on $\mathbb{R}^n$ and

$$|U_\alpha f(x) - U_\alpha f(x_0)| = o(\varphi_p(1/|x - x_0|)) \quad \text{as } x \to x_0$$

for all $x_0 \in \mathbb{R}^n$, whenever $f$ satisfies (1.1) and (1.2). On the contrary, if (1.4) does not hold, then we can find an $f$ satisfying (1.1) and (1.2) such that $U_\alpha f$ is not continuous (see [13, Remark 3.3]).
2 Proof of Theorem A

In spite of the fact that $\Phi_{p,\varphi}$ may not be convex, Theorem A must be a consequence of Cianchi [2] in spirit. But we here give a proof of Theorem A, because our method is straightforward and several materials are also needed for a proof of our main Theorem B. In fact, our proof is based on the boundedness of maximal functions, by use of the methods in the paper by Hedberg [7].

Throughout this paper, let $C$, $C_1$, $C_2$, ... denote various constants independent of the variables in question.

First we collect properties which follow from condition (\varphi 1) (see [11] and [13]).

(\varphi 2) $\varphi$ satisfies the doubling condition, that is, there exists $c > 1$ such that

$$c^{-1}\varphi(r) \leq \varphi(2r) \leq c\varphi(r)$$

whenever $r > 0$.

(\varphi 3) For each $\gamma > 0$, there exists $c = c(\gamma) \geq 1$ such that

$$c^{-1}\varphi(r) \leq \varphi(r^\gamma) \leq c\varphi(r)$$

whenever $r > 0$.

(\varphi 4) If $\gamma > 0$, then there exists $c = c(\gamma) \geq 1$ such that

$$s^\gamma\varphi(s) \leq ct^\gamma\varphi(t)$$

whenever $0 < s < t$.

(\varphi 5) If $\gamma > 0$, then there exists $c = c(\gamma) \geq 1$ such that

$$t^{-\gamma}\varphi(t) \leq cs^{-\gamma}\varphi(s)$$

whenever $0 < s < t$.

**Lemma 2.1** Let $1 < p_1 < p < p_2$. Then there exists $C > 1$ such that

$$C^{-1}A^{p_1}\Phi_{p,\varphi}(r) \leq \Phi_{p,\varphi}(Ar) \leq CA^{p_2}\Phi_{p,\varphi}(r)$$

whenever $r > 0$ and $A > 1$.

**Corollary 2.2** Let $\alpha p \leq n$ and $1 < p_1 < p < p_2$. Let $G$ be a bounded open set in $\mathbb{R}^n$. Then there exists a positive constant $C$ such that

$$C^{-1}\left\{\|f\|_{\Phi_{p,\varphi}}\right\}^{p_2} \leq \int_G \Phi_{p,\varphi}(|f(y)|)dy \leq C\left\{\|f\|_{\Phi_{p,\varphi}}\right\}^{p_1}$$

whenever $f$ is a locally integrable function on $G$ such that $\|f\|_{\Phi_{p,\varphi}} \leq 1$. 
Lemma 2.3 (cf. [13, Lemma 2.5]) Let $G$ be a bounded open set in $\mathbb{R}^n$ and $\varepsilon > 0$. Let $p_0$ be given so that $p_0 = p$ if $\varphi$ is nondecreasing, and $1 < p_0 < p$ if $\varphi$ is nonincreasing. If $x \in G$, $\delta > 0$ and $f$ is a nonnegative measurable function on $G$, then

$$
\int_{G - B(x, \delta)} |x - y|^{\alpha - n} f(y) dy \leq C \varphi_p^{*}(\delta^{-1}) \left\{ \varepsilon + c(\varepsilon) \left( \int_{G} \Phi_{p, \varphi}(f(y)) dy \right)^{1/p_0} \right\},
$$

where $C$ and $c(\varepsilon)$ are positive constants such that $C$ is independent of $\varepsilon$ but $c(\varepsilon)$ may depend on $\varepsilon$. In case $\alpha p < n$,

$$
\int_{\mathbb{R}^n - B(x, \delta)} |x - y|^{\alpha - n} f(y) dy \leq C \varphi_p^{*}(\delta^{-1}) \left\{ 1 + \left( \int_{\mathbb{R}^n} \Phi_{p, \varphi}(f(y)) dy \right)^{1/p_0} \right\},
$$

for all $x \in \mathbb{R}^n$ and nonnegative measurable functions $f$ on $\mathbb{R}^n$.

For a locally integrable function $f$ on $\mathbb{R}^n$, define the maximal function by

$$
Mf(x) = \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{G \cap B(x, r)} |f(y)| dy,
$$

where $|B(x, r)|$ denotes the $n$-dimensional Lebesgue measure of the ball $B(x, r)$ centered at $x$ of radius $r > 0$.

We denote by $c(\varepsilon)$ various constants which may depend on $\varepsilon$.

Lemma 2.4 Let $\alpha p = n$ and $G$ be a bounded open set in $\mathbb{R}^n$. Then, for each $\eta > 0$, there exist $\varepsilon_0 > 0$ and $c(\varepsilon_0) > 0$ such that

$$
(\varphi_p^{*})^{-1}(U_\alpha f(x)) \leq c(\varepsilon_0) \{ \Phi_{p, \varphi}(Mf(x)) \}^{1/n} + \eta
$$

for all nonnegative measurable functions $f$ on $G$ satisfying $\int_{G} \Phi_{p, \varphi}(f(y)) dy \leq \varepsilon_0$.

In case $\alpha p < n$, we find from $(\varphi 4)$ and $(\varphi 5)$ that

$$
C^{-1} r^{(n - \alpha p)/p} \{ \varphi(r) \}^{-1/p} \leq \varphi_p^{*}(r) \leq C r^{(n - \alpha p)/p} \{ \varphi(r) \}^{-1/p}, \quad (2.1)
$$

so that

$$
C^{-1} r^{p/(n - \alpha p)} \{ \varphi(r) \}^{1/(n - \alpha p)} \leq (\varphi_p^{*})^{-1}(r) \leq C r^{p/(n - \alpha p)} \{ \varphi(r) \}^{1/(n - \alpha p)} \quad (2.2)
$$

for $r > 0$.

Lemma 2.5 Let $\alpha p < n$. Then

$$
(\varphi_p^{*})^{-1}(U_\alpha f(x)) \leq C \{ \Phi_{p, \varphi}(Mf(x)) \}^{1/n}
$$

for all nonnegative measurable functions $f$ on $\mathbb{R}^n$ satisfying $\int_{\mathbb{R}^n} \Phi_{p, \varphi}(f(y)) dy \leq 1$. 
Note that
\[
C^{-1} \frac{\Phi_{p,\varphi}(t)}{t} \leq \int_{0}^{t} s^{-1} d\Phi_{p,\varphi}(s) \leq C \frac{\Phi_{p,\varphi}(t)}{t}
\]
for all \( t > 0 \) by (\( \varphi 4 \)) and (\( \varphi 5 \)).

The next lemma is an extension of Stein [15, Chapter 1], whose proof will be done along the same lines as in Stein [15, Chapter 1].

**Lemma 2.6** For a locally integrable function \( f \) on \( \mathbb{R}^n \),
\[
\int \Phi_{p,\varphi}(Mf(x)) \, dx \leq C \int \Phi_{p,\varphi}(|f(x)|) \, dx.
\]

**Proof of Theorem A.** We give a proof of Theorem A only in case \( \alpha p = n \). With the aid of Lemma 2.4, for \( \eta > 0 \) we find \( \epsilon_1 > 0 \) such that
\[
(\varphi_p^*)^{-1}(U\alpha f(x)) \leq C(\epsilon_1)\{\Phi_{p,\varphi}(Mf(x))\}^{1/n} + \eta
\]
for all nonnegative measurable functions \( f \) on \( G \) satisfying \( \int \Phi_{p,\varphi}(f(y)) dy \leq \epsilon_1 \). Hence, in view of Lemma 2.6, we obtain
\[
\int_G \Phi_{p,\varphi}(U\alpha f(x)) \, dx \leq C(\epsilon_1) \int_G \Phi_{p,\varphi}(Mf(x)) \, dx + C\eta^n |G|
\]
for all nonnegative measurable functions \( f \) on \( G \) satisfying \( \int \Phi_{p,\varphi}(f(y)) dy \leq \epsilon_1 \). Now, letting \( C\eta^n |G| \leq 1/2 \) and using Corollary 2.2, we find \( 0 < \epsilon_0 < \epsilon_1 \) such that
\[
\int_G \Psi_{p,\varphi}(U\alpha f(x)) \, dx \leq 1
\]
for all nonnegative measurable functions \( f \) on \( G \) satisfying \( \|f\|_{\Phi_{p,\varphi}} \leq \epsilon_0 \). This implies that
\[
\int_G \Psi_{p,\varphi}(\epsilon_0 U\alpha f(x)) \, dx \leq 1
\]
for all nonnegative measurable functions \( f \) on \( G \) satisfying \( \|f\|_{\Phi_{p,\varphi}} \leq 1 \). Now the proof is completed.

\[\square\]

**3 Proof of Theorem B**

For a proof of Theorem B, we prepare a series of lemmas.
LEMMA 3.1 Let $\alpha p \leq n$. Then there exist $\beta > 1$ and $C > 0$ such that 
$$\varphi_p^*(Ar) \leq CA^\beta \varphi_p^*(r)$$
for all $r > 0$ and $A > 1$.

With the aid of Lemma 3.1, we establish the following result.

LEMMA 3.2 There exist $C > 1$ and $0 < \epsilon_0 < 1$ such that 
$$\int_G \Psi_p\Phi(U_\alpha|f|(y))dy \leq C\{\|f\|_{\Phi_p,\varphi}\}^{n/\beta}$$
whenever $f$ is a locally integrable function on $G$ such that $\|f\|_{\Phi_p,\varphi} \leq \epsilon_0$, where $\beta$ is given in Lemma 3.1.

We further need the following result.

LEMMA 3.3 Let $\alpha p \leq n$. For a nonnegative measurable function $f$ on $\mathbb{R}^n$ satisfying (1.2), set 
$$E_* = \left\{ x \in \mathbb{R}^n : \lim_{r \to 0^+} \sup_{r \to 0^+} r^{\alpha p-n} \varphi(r^{-1})^{-1} \int_{B(x,r)} \Phi_p\Phi(f(y))dy > 0 \right\}$$
and 
$$E^* = \left\{ x \in \mathbb{R}^n : \lim_{r \to 0^+} \sup_{r \to 0^+} r^{\alpha p-n} \int_{B(x,r)} \Phi_p\Phi(f(y))dy > 0 \right\}.$$
If $x_0 \in \mathbb{R}^n \setminus (E_* \cup E^*)$, then 
$$\lim_{r \to 0} r^{-n} \int_{B(x_0,r)} \Phi_p\Phi(r^\alpha f(y)) dy = 0.$$

For $x_0 \in \mathbb{R}^n$ and $r > 0$, set $f_{x_0,r}(w) = r^\alpha f(x_0 + rw)\chi_E(0,1)$, where $\chi_E$ denotes the characteristic function of $E$. Then note that 
$$\int_{B(x_0,r)} |x-y|^\alpha f(y)dy = \int_{B(0,1)} |z-w|^\alpha f(x_0 + rw) dw$$
$$= U_\alpha f_{x_0,r}(z) \tag{3.1}$$
for $x = x_0 + rz$.

We are now ready to prove our main Theorem B.

PROOF OF THEOREM B. For a nonnegative measurable function $f$ on $\mathbb{R}^n$ satisfying (1.1) and (1.2), it suffices to show that (1.3) holds for $x_0 \in \mathbb{R}^n \setminus (E_* \cup E_* \cup E^*)$. Write 
$$U_\alpha f(x) - U_\alpha f(x_0) = \int_{B(x_0,2|x-x_0|)} |x-y|^\alpha f(y)dy$$
$$+ \int_{\mathbb{R}^n \setminus B(x_0,2|x-x_0|)} |x-y|^\alpha f(y)dy - U_\alpha f(x_0)$$
$$= U_1(x) + U_2(x).$$
If \( y \in \mathbb{R}^n - B(x_0, 2|x - x_0|) \), then \(|x_0 - y| \geq 2|x - y|\), so that, since \( U_\alpha f(x_0) < \infty \), we can apply Lebesgue’s dominated convergence theorem to obtain

\[
\lim_{x \to x_0} U_2(x) = 0. \tag{3.2}
\]

Since \((\varphi_p^*)^{-1}\) is nondecreasing, we have

\[
(\varphi_p^*)^{-1}(A|U_\alpha f(x) - U_\alpha f(x_0)|) \leq (\varphi_p^*)^{-1}(A U_1(x) + A|U_2(x)|) \\
\leq (\varphi_p^*)^{-1}(2AU_1(x)) + (\varphi_p^*)^{-1}(2A|U_2(x)|),
\]

so that

\[
\Psi_{p,\varphi}(A|U_\alpha f(x) - U_\alpha f(x_0)|) \leq C\psi_n((\varphi_p^*)^{-1}(2AU_1(x))) + C\psi_n((\varphi_p^*)^{-1}(2A|U_2(x)|)) \\
= C\Psi_{p,\varphi}(2AU_1(x)) + C\Psi_{p,\varphi}(2A|U_2(x)|).
\]

In view of (3.2), we have

\[
\lim_{x \to x_0} \Psi_{p,\varphi}(2A|U_2(x)|) = 0.
\]

Note that

\[
U_1(x) \leq \int_{B(x_0, r)} |x - y|^\alpha f(y) \, dy = U_\alpha f_r(x)
\]

for \( x \in B(x_0, r/2) \), where \( f_r = f\chi_{B(x_0, r)} \). Hence, we have only to show that

\[
\lim_{r \to 0+} \int_{B(x_0, r)} \Psi_{p,\varphi}(2AU_\alpha f_r(x)) \, dx = 0.
\]

Note that \( U_\alpha(f_r)(x) = U_\alpha(f_{r,x_0,r})(x) \) for \( x = x_0 + rz \) and

\[
\int_{B(0,1)} \Phi_{p,\varphi}((f_{r,x_0,r}(w)) \, dw = r^{-n} \int_{B(x_0, r)} \Phi_{p,\varphi}(r^\alpha f(y)) \, dy
\]

which tends to zero as \( r \to +0 \) by Lemma 3.3. Hence we have by Lemma 3.2 and Corollary 2.2

\[
\int_{B(x_0, r)} \Psi_{p,\varphi}(2AU_1(x)) \, dx \leq \int_{B(0,1)} \Psi_{p,\varphi}(U_\alpha(2A(f_{r,x_0,r}))(z)) \, dz \\
\leq C\{\|2A(f_{r,x_0,r})\|_{\Phi_{p,\varphi}}\}^{n/\beta} \\
\leq C(2A)^{n/\beta} \left( \int_{B(0,1)} \Phi_{p,\varphi}((f_{r,x_0,r}(z)) \, dz \right)^{n/(p_2 \beta)} \\
\leq C(2A)^{n/\beta} \left( r^{-n} \int_{B(x_0, r)} \Phi_{p,\varphi}(r^\alpha f(y)) \, dy \right)^{n/(p_2 \beta)}.
\]

Consequently it follows from Lemma 3.3 that the left hand side tends to zero as \( r \to 0+ \). Thus the proof is completed. \( \square \)
4 Size of exceptional sets

To evaluate the size of exceptional sets in Theorem B, we introduce the notion of capacity. For a set \( E \subset \mathbb{R}^n \) and an open set \( G \subset \mathbb{R}^n \), we define

\[
C_{\alpha, \phi_{p, \varphi}}(E; G) = \inf_f \int_G \Phi_{p, \varphi}(f(y)) dy
\]

where the infimum is taken over all nonnegative measurable functions \( f \) on \( \mathbb{R}^n \) such that \( f \) vanishes outside \( G \) and \( U_\alpha f(x) \geq 1 \) for every \( x \in E \) (cf. Meyers [8] and the first author [11]). When \( \varphi \equiv 1 \), we write \( C_{\alpha, p} \) for \( C_{\alpha, \Phi_{p, \varphi}} \). We say that \( E \) is of \( C_{\alpha, \Phi_{p, \varphi}} \)-capacity zero, written as \( C_{\alpha, \Phi_{p, \varphi}}(E) = 0 \), if

\[
C_{\alpha, \Phi_{p, \varphi}}(E \cap G; G) = 0 \quad \text{for every bounded open set } G.
\]

The following can be obtained readily from the definition of \( C_{\alpha, \Phi_{p, \varphi}} \); see [11, Theorem 1.1, Chapter 2].

**Lemma 4.1** For a nonnegative measurable function \( f \) on \( \mathbb{R}^n \) satisfying (1.1) and (1.2), set

\[
E_\infty = \{x \in \mathbb{R}^n : \int |x-y|^{\alpha-n}f(y) dy = \infty\}.
\]

Then

\[
C_{\alpha, \Phi_{p, \varphi}}(E_\infty) = 0.
\]

As in the proof of Lemma 7.3 and Corollary 7.2 in [10], we can prove the following results.

**Lemma 4.2** Let \( \alpha p \leq n \). For a nonnegative measurable function \( f \) on \( \mathbb{R}^n \) satisfying (1.2), set

\[
E_* = \{x \in \mathbb{R}^n : \limsup_{r \to 0} r^{\alpha p-n}\varphi(r^{-1})^{-1} \int_{B(x, r)} \Phi_{p, \varphi}(f(y)) dy > 0\}.
\]

Then \( C_{\alpha, \Phi_{p, \varphi}}(E_*) = 0 \).

**Lemma 4.3** For a nonnegative measurable function \( f \) in \( L^p(\mathbb{R}^n) \), set

\[
E^* = \{x \in \mathbb{R}^n : \limsup_{r \to 0} r^{\alpha p-n} \int_{B(x, r)} f(y)^p dy > 0\}.
\]

If \( \alpha p < n \), then \( C_{\alpha, p}(E^*) = 0 \); and if \( \alpha p = n \), then \( E^* \) is empty.

Finally, in view of Theorem B and Lemmas 4.1 - 4.3, we establish the following result.
COROLLARY 4.4 Let $\alpha p \leq n$. If $f$ is a locally integrable function on $\mathbb{R}^n$ satisfying (1.1) and (1.2), then
\[
\lim_{r \to 0^+} \int_{B(x_0, r)} \nabla \cdot (A|U_{\alpha}f(x) - U_{\alpha}f(x_0)|) \, dx = 0
\]
holds for all $A > 0$ and all $x_0 \in \mathbb{R}^n \setminus E$, where $C_{\alpha, \Phi_{p, \varphi}}(E) = 0$ when $\alpha p = n$ or $\varphi$ is nonincreasing and $C_{\alpha, p}(E) = 0$ when $\alpha p < n$ and $\varphi$ is nondecreasing.

COROLLARY 4.5 Let $\alpha p = n$ and $\varphi(r)$ be of the form $(\log r)^{q_1}(\log \log r)^{q_2}$ for large $r > 0$, where $q_1$ and $q_2$ are real numbers. Set $\Phi(r) = \Phi_{p, \varphi}(r) = r^p \varphi(r)$. Suppose $f$ is a locally integrable function on $\mathbb{R}^n$ satisfying (1.1) and (1.2).

1. If $q_1 < p - 1$, then
\[
\lim_{r \to 0^+} \int_{B(x_0, r)} \{\exp(A|U_{\alpha}f(x) - U_{\alpha}f(x_0)|^{\beta_1}(\log(1+|U_{\alpha}f(x) - U_{\alpha}f(x_0)|)^{\beta_2}) - 1\} \, dx = 0
\]
for every $A > 0$ and every $x_0 \in \mathbb{R}^n$ except in a set of $C_{\alpha, \Phi_{p, \varphi}}$-capacity zero, where $\beta_1 = p/(p-1-q_1)$ and $\beta_2 = q_2/(p-1-q_1)$.

2. If $q_1 > p - 1$, then $U_{\alpha}f$ is continuous on $\mathbb{R}^n$ and
\[
|U_{\alpha}f(x) - U_{\alpha}f(x_0)| = o((\log(1/|x-x_0|))^{1/\beta_1}(\log \log(1/|x-x_0|))^{-q_2/p}) \quad \text{as } x \to x_0
\]
for every $x_0 \in \mathbb{R}^n$.

For the continuity of $U_{\alpha}f$ (case (2)), see Remark 1.5. The case $q_1 = p - 1$ is treated as follows:

COROLLARY 4.6 Let $\alpha p = n$, $\varphi(r) = \varphi_{p-1,q}(r) = (\log r)^{p-1}(\log \log r)^q$ for large $r > 0$ and $\Phi_{p, \varphi}(r) = r^p \varphi(r)$. Suppose $f$ is a locally integrable function on $\mathbb{R}^n$ satisfying (1.1) and (1.2).

1. If $q < p - 1$, then
\[
\lim_{r \to 0^+} \int_{B(x_0, r)} \{\exp(\exp(A|U_{\alpha}f(x) - U_{\alpha}f(x_0)|^{\beta})) - e\} \, dx = 0
\]
for every $A > 0$ and every $x_0 \in \mathbb{R}^n$ except in a set of $C_{\alpha, \Phi_{p, \varphi}}$-capacity zero, where $\beta = p/(p-1-q)$.

2. If $q = p - 1$, then
\[
\lim_{r \to 0^+} \int_{B(x_0, r)} \{\exp(\exp(A|U_{\alpha}f(x) - U_{\alpha}f(x_0)|^{\beta})) - e^e\} \, dx = 0
\]
for every $A > 0$ and every $x_0 \in \mathbb{R}^n$ except in a set of $C_{\alpha, \Phi_{p, \varphi}}$-capacity zero, where $\beta = p/(p-1)$.

3. If $q > p - 1$, then $U_{\alpha}f$ is continuous on $\mathbb{R}^n$ and
\[
|U_{\alpha}f(x) - U_{\alpha}f(x_0)| = o((\log(\log(1/|x-x_0|)))^{(p-1-q)/p}) \quad \text{as } x \to x_0
\]
for every $x_0 \in \mathbb{R}^n$. 

References


