On complementary spaces of the Lizorkin spaces

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§1. Introduction

Let $R^n$ be the $n$-dimensional Euclidean space. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $x = (x_1, \cdots, x_n) \in R^n$, we let

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad D^\alpha = \frac{\partial |\alpha|}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$ 

The Schwartz space $S(R^n)$ is defined to be the class of all $C^\infty$-functions $\varphi$ on $R^n$ such that

$$p_{\alpha,\beta}(\varphi) = \sup_{x \in R^n} |x^\alpha D^\beta \varphi(x)| < \infty$$

for all multi-indices $\alpha$ and $\beta$. We introduce two kinds of the Lizorkin spaces $\Phi_1(R^n)$ and $\Phi_2(R^n)$. The Lizorkin space $\Phi_1(R^n)$ of the first kind is defined to be the class of all functions $\varphi \in S(R^n)$ which satisfy

$$\int_{R^n} \varphi(x) x^\alpha dx = 0$$

for any multi-index $\alpha$. The Lizorkin space $\Phi_2(R^n)$ of the second kind is defined to be the class of all functions $\varphi \in S(R^n)$ which satisfy

$$\int_{-\infty}^\infty \varphi(x_1, \cdots, x_j, \cdots, x_n) x_j^\ell dx_j = 0$$

for $j = 1, \cdots, n$ and $\ell = 0, 1, 2, \cdots$. Clearly $\Phi_1(R^n) \supset \Phi_2(R^n)$.

An example of a function belonging to $\Phi_1(R^n)$ (resp. $\Phi_2(R^n)$) is $\mathcal{F}(e^{-|y|^2 - (1/|y|^2)})(x)$ (resp. $\mathcal{F}(e^{-|y|^2 - \sum_{j=1}^n 1/y_j^2})(x)$) where $\mathcal{F}\varphi$ is the Fourier transform of $\varphi$:

$$\mathcal{F}\varphi(x) = \int_{R^n} e^{-ixy} dy.$$
The Lizorkin spaces appeared in the theory of fractional derivatives, hypersingular integrals and Riesz potentials ([Sa2] and [SKM]). The properties of the Lizorkin spaces have studied by several authors. The denseness of the Lizorkin spaces in the Lebegue spaces was proved in O.I.Lizorkin [Li2] and S.G.Samko [Sa1]. Moreover P.I.Lizorkin [Li3] showed that the space $\Phi_1(R^n)$ is dense in the Sobolev spaces and T.Kurokawa [Ku] deals with the denseness of the space $\Phi_1(R^n)$ in the spaces of Beppo Levi type. The invariance of the space $\Phi_1(R^n)$ relative to Riesz potentials was noted by V.I.Semyanistyi [Se], P.I.Lizorkin [Li3] and S.Helgason [He]. T.Kurokawa [Ku] establish the invariance of the space $\Phi_1(R^n)$ relative to more general operators. In this note we are concerned with comlementary spaces of $\Phi_1(R^n)$ and $\Phi_2(R^n)$ in $S(R^n)$. For a subspace $V \subset S(R^n)$, if a subspace $W \subset S(R^n)$ satisfies the condition

$$S(R^n) = V \oplus W,$$

then we call $W$ a complementary space of $V$ in $S(R^n)$ where the symbol $\oplus$ indicates a direct sum. In section 2 as a preparation we introduce dual functions of polynomials and tensor product functions, and study their properties. In section 3 we sketch our plan to give comlementary spaces of $\Phi_1(R^n)$ and $\Phi_2(R^n)$ in $S(R^n)$.

§2. Dual functions of polynomials and tensor product functions

Let $h \in C^\infty(R^1)$ be a function which satisfies the conditions $0 \leq h(t) \leq 1$, $h(-t) = h(t)$ and

$$h(t) = \begin{cases} 
1, & \text{for } |t| \leq 1/2 \\
0, & \text{for } |t| \geq 1.
\end{cases}$$

We fix the function $h(t)$. We denote by $A$ the set of sequences $a = \{a_j\}_{j=0,1,\ldots}$ which satify $0 < a_j \leq 1$ and $a_j \geq a_{j+1}$. For $a =$
\( \{a_j\}_{j=0,1,\ldots} \in \mathcal{A} \) we put
\[
\eta_j^a(t) = \frac{t^j}{j!} h\left(\frac{t}{a_j}\right), \quad j = 0, 1, 2, \ldots
\]
and
\[
\theta_j^a(t) = \frac{i^j}{2\pi} \mathcal{F} \eta_j^a(t), \quad j = 0, 1, 2, \ldots.
\]
Then \( \theta_j^a \in \mathcal{S}(R^1) \) and
\[
(2.1) \quad \int_{-\infty}^{\infty} \theta_j^a(t) t^k dt = \begin{cases} 1, & k = j \quad k, j = 0, 1, 2, \ldots, \\ 0, & k \neq j, \end{cases}
\]
Since \( \{\theta_j^a\}_{j=0,1,\ldots} \) satisfy (2.1), we call \( \{\theta_j^a\}_{j=0,1,\ldots} \) dual functions of polynomials associated with a sequence \( a \in \mathcal{A} \). For \( 1 \leq p \leq n \) we denote by \( M_p \) the set of subsets of \( \{1, 2, \ldots, n\} \) which have \( p \) elements. For \( \{i_1, i_2, \ldots, i_p\} \in M_p \) we always assume that \( i_1 < i_2 < \cdots < i_p \).
For multi-index \( \alpha = (\alpha_1, \cdots, \alpha_n) \) and \( \{i_1, \cdots, i_p\} \in M_p \) the notation \( (\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c) \) stands for
\[
(\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c) = (\alpha_{k_1}, \cdots, \alpha_{k_{n-p}})
\]
where \( \{k_1, \cdots, k_{n-p}\} = \{1, \cdots, n\} - \{i_1, \cdots, i_p\} \). Similarly, for \( x = (x_1, \cdots, x_n) \) we denote
\[
(\{x_{i_1}, \cdots, x_{i_p}\}^c) = (x_{k_1}, \cdots, x_{k_{n-p}}).
\]
Moreover we denote
\[
(\{x_{i_1}, \cdots, x_{i_p}\}^c)(\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c) = x_{k_1}^{\alpha_{k_1}} \cdots x_{k_{n-p}}^{\alpha_{k_{n-p}}},
\]
\[
(\{D_{i_1}, \cdots, D_{i_p}\}^c)(\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c) = D_{k_1}^{\alpha_{k_1}} \cdots D_{k_{n-p}}^{\alpha_{k_{n-p}}}.
\]
Let \( \alpha, \beta \) be multi-indices and \( \{i_1, \cdots, i_p\} \in M_p \). For a function \( \varphi(\{x_{i_1}, \cdots, x_{i_p}\}^c) \in \mathcal{S}(R^{n-p}) \) we define
\[
P(\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c)(\{\beta_{i_1}, \cdots, \beta_{i_p}\}^c)(\varphi)
= \sup_{(\{x_{i_1}, \cdots, x_{i_p}\}^c) \in R^{n-p}} |(\{x_{i_1}, \cdots, x_{i_p}\}^c)(\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c)(\{D_{i_1}, \cdots, D_{i_p}\}^c)(\{\beta_{i_1}, \cdots, \beta_{i_p}\}^c) \varphi(\{x_{i_1}, \cdots, x_{i_p}\}^c)|.
For \( \{i_1, \cdots, i_p\} \in M_p \) we denote by \( C_{i_1, \cdots, i_p}^a \) the set of \( p \)-multiple sequences of functions \( \{\varphi_{s_1 \cdots s_p} (\{x_{i_1}, \cdots, x_{i_p}\}^c)\}_{s_1, \cdots, s_p=0,1, \cdots} \subset S(R^{n-p}) \) which satisfy
\[
\sum_{s_1, \cdots, s_p=0}^{\infty} p((\alpha_{i_1}, \cdots, \alpha_{i_p})^c), ((\beta_{i_1}, \cdots, \beta_{i_p})^c) (\varphi_{s_1 \cdots s_p}) a_{s_1} \cdots a_{s_p} < \infty \]
for all multi-indices \( \alpha \) and \( \beta \). We note that the sequence
\[
\{\varphi_{s_1 \cdots s_n} (\{x_1, \cdots, x_n\}^c)\}_{s_1, \cdots, s_n=0,1, \cdots} \text{ is a } n\text{-multiple sequence of numbers } \{b_{s_1 \cdots s_n}\}_{s_1, \cdots, s_n=0,1, \cdots} \text{ and }
\]
\[
p((\alpha_1, \cdots, \alpha_n)^c), ((\beta_1, \cdots, \beta_n)^c) (b_{s_1 \cdots s_n}) = |b_{s_1 \cdots s_n}|.
\]
Therefore
\[
C_{1, \cdots, n}^a = \{\{b_{s_1 \cdots s_n}\}_{s_1, \cdots, s_n=0,1, \cdots} : \sum_{s_1, \cdots, s_n=0}^{\infty} |b_{s_1 \cdots s_n}| a_{s_1} \cdots a_{s_n} < \infty\}.
\]

The basic fact is

**Lemma 1.** Let \( \{i_1, \cdots, i_p\} \in M_p \). If a \( p \)-multiple sequence of functions \( \{\varphi_{s_1 \cdots s_p} (\{x_{i_1}, \cdots, x_{i_p}\}^c)\}_{s_1, \cdots, s_p=0,1, \cdots} \) belongs to \( C_{i_1, \cdots, i_p}^a \), then the \( p \)-multiple series
\[
\sum_{s_1, \cdots, s_p=0}^{\infty} \varphi_{s_1 \cdots s_p} (\{x_{i_1}, \cdots, x_{i_p}\}^c) \theta_{s_1}^a (x_{i_1}) \cdots \theta_{s_p}^a (x_{i_p})
\]
converges in \( S(R^n) \).

We introduce two kinds of tensor product functions associated with \( \{\theta_j^a\} \). If a function \( f \) has the following form
\[
(2.2) \quad f(x) = \sum_{s_1, \cdots, s_n=0}^{\infty} b_{s_1 \cdots s_n} \theta_{s_1}^a (x_1) \cdots \theta_{s_n}^a (x_n)
\]
where \( \{b_{s_1 \cdots s_n}\} \in C_{1, \cdots, n}^a \), then \( f \) is called a tensor product function of the first kind associated with \( \{\theta_j^a\} \). If a function \( f \) which has the
form

(2.3)
\[
f(x) = \sum_{p=1}^{n} (-1)^p \sum_{i_1, \ldots, i_p, s_1, \ldots, s_p} \sum_{\{x_1, \ldots, x_p\}^c} \lambda_{i_1, \ldots, i_p; s_1, \ldots, s_p} \theta_{s_1}^{\alpha}(x_{i_1}) \cdots \theta_{s_p}^{\alpha}(x_{i_p})
\]
satisfies the conditions

(i) \{\lambda_{i_1, \ldots, i_p; s_1, \ldots, s_p}\}_{s_1, \ldots, s_p=0,1,\ldots} \in C_{i_1, \ldots, i_p}^{\alpha},

(ii) for \(2 \leq p \leq n, \{i_1, \ldots, i_p\} \in M_p\) and \(s_1, \ldots, s_p \geq 0,\)

\[
\lambda_{i_1, \ldots, i_p; s_1, \ldots, s_p} \{x_1, \ldots, x_p\}^c = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \lambda_{i_1; s_1}(\{x_1\}^c) x_1^{s_1} \cdots x_p^{s_p} dx_1 \cdots dx_p
\]

where \(\ell = 1, \ldots, p,\) then we call \(f\) a tensor product function of the second kind associated with \(\{\theta_{j}^{\alpha}\}\) where the symbol \(\sim\) indicates that the variable underneath is deleted. We denote by \(T_1^{\alpha}(R^n)\) (resp. \(T_2^{\alpha}(R^n)\)) the class of all tensor product functions of the first kind (resp. the second kind) associated with \(\{\theta_{j}^{\alpha}\}\). By Lemma 1, we see that \(T_1^{\alpha}(R^n), T_2^{\alpha}(R^n) \subset S(R^n)\). A fundamental property of the tensor product functions is the following.

**Lemma 2.** (i) Let \(f\) be a tensor product function of the first kind with the form (2.2). Then

\[
\int_{R^n} f(x_1, \ldots, x_n) x_1^{t_1} \cdots x_n^{t_n} dx_1 \cdots dx_n = b_{t_1 \cdots t_n}
\]

for \(t_1, \ldots, t_n \geq 0.\)

(ii) Let \(f\) be a tensor product function of the second kind with the form (2.3). Then

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n) x_{k_1}^{t_1} \cdots x_{k_q}^{t_q} dx_{k_1} \cdots dx_{k_q} = \lambda_{k_1, \ldots, k_q; t_1, \ldots, t_q} \{x_{k_1}, \ldots, x_{k_q}\}^c
\]
for $1 \leq q \leq n, \{k_1, \ldots, k_q\} \in M_q$ and $t_1, \ldots, t_q \geq 0$.

§3. Complementary spaces of the Lizorkin spaces

For $\{i_1, \ldots, i_p\} \in M_p, s_1, \ldots, s_p \geq 0$ and $\varphi \in S(R^n)$, we define

$$
\mu_{i_1, \ldots, i_p; s_1, \ldots, s_p}(\varphi)(\{x_{i_1}, \ldots, x_{i_p}\})^c)
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(x_1, \ldots, x_n) x_{i_1}^{s_1} \cdots x_{i_p}^{s_p} dx_{i_1} \cdots dx_{i_p}.
$$

Moreover, for $a \in \mathcal{A}$ and $\{i_1, \ldots, i_p\} \in M_p$ we set $S_{i_1, \ldots, i_p}^a = \{\varphi \in S(R^n) : \{\mu_{i_1, \ldots, i_p; s_1, \ldots, s_p}(\varphi)(\{x_{i_1}, \ldots, x_{i_p}\})^c\}_{s_1, \ldots, s_p = 0, 1, \ldots} \in C_{i_1, \ldots, i_p}^a\}$ and

$$
S^a(R^n) = \cap_{p=1}^{n} \cap_{\{i_1, \ldots, i_p\} \in M_p} S_{i_1, \ldots, i_p}^a(R^n).
$$

If $\varphi \in \Phi_1(R^n)$, then $\mu_{1, \ldots, n; s_1, \ldots, s_n}(\varphi) = 0$ for $s_1, \ldots, s_n \geq 0$. Hence $\Phi_1(R^n) \subset S_{1, \ldots, n}^a$ for any $a \in \mathcal{A}$. If $\varphi \in \Phi_2(R^n)$, then $\mu_{i_1, \ldots, i_p; s_1, \ldots, s_p}(\varphi) = 0$ for $1 \leq p \leq n, \{i_1, \ldots, i_p\} \in M_p$ and $s_1, \ldots, s_p \geq 0$. Hence $\Phi_2(R^n) \subset S^a(R^n)$ for any $a \in \mathcal{A}$. Moreover, By Lemma 2 (i), (ii) and the definitions of $T_1^a, T_2^a$ we see that $T_1^a(R^n) \subset S_{1, \ldots, n}^a$ and $T_2^a(R^n) \subset S^a(R^n)$. We introduce some operators. For $\varphi \in S_{1, \ldots, n}^a(R^n)$, we define

$$
T_{1, \ldots, n}^a \varphi(x) = \sum_{s_1, \ldots, s_n = 0}^{\infty} \mu_{1, \ldots, n; s_1, \ldots, s_n}(\varphi) \theta_{s_1}^a(x_1) \cdots \theta_{s_n}^a(x_n)
$$

and

$$
U_{1, \ldots, n}^a \varphi = \varphi - T_{1, \ldots, n}^a \varphi.
$$

Further, for $\varphi \in S^a(R^n)$ we define

$$
T_j^a \varphi(x) = \sum_{s=0}^{\infty} \mu_{j, s}(\varphi)(\{x_j\})^c) \theta_s^a(x_j), \quad j = 1, \ldots, n.
$$
and
\[ U^a_j \varphi = \varphi - T^a_j \varphi. \]

Moreover
\[ U^a \varphi = U^a_1 \cdots U^a_n \varphi. \]

We see that
\[ U^a \varphi = \varphi - \sum_{p=1}^{n} (-1)^{p+1} \sum_{\{i_1, \ldots, i_p\} \in M_p} T^a_{i_1, \ldots, i_p} \varphi \]

where
\[ T^a_{i_1, \ldots, i_p} \varphi(x) = \sum_{s_1, \ldots, s_p=0}^{\infty} \mu_{i_1, \ldots, i_p; s_1, \ldots, s_p}(\varphi)(\{x_{i_1}, \ldots, x_{i_p}\}^c)\theta_{s_1}^a(x_{i_1}) \cdots \theta_{s_p}^a(x_{i_p}). \]

We put
\[ T^a = \sum_{p=1}^{n} (-1)^{p+1} \sum_{\{i_1, \ldots, i_p\} \in M_p} T^a_{i_1, \ldots, i_p}. \]

We establish properties of these operators which are necessary for decompositions of \( S^a_{1, \ldots, n}(R^n) \) and \( S^a(R^n) \). About ranges of these operators we have

**Lemma 3.** (i) If \( \varphi \in S^a_{1, \ldots, n}(R^n) \), then \( T^a_{1, \ldots, n} \varphi, U^a_{1, \ldots, n} \varphi \in S^a_{1, \ldots, n}(R^n) \).
(ii) If \( \varphi \in S^a(R^n) \), then \( T^a \varphi, U^a \varphi \in S^a(R^n) \).

**Lemma 4.** (i) \( \varphi \in S^a_{1, \ldots, n}(R^n) \), then \( U^a_{1, \ldots, n} \varphi \in \Phi_1(R^n) \).
(ii) If \( \varphi \in S^a(R^n) \), then \( U^a \varphi \in \Phi_2(R^n) \).

**Lemma 5.** (i) \( \varphi \in S^a_{1, \ldots, n}(R^n) \), then \( T^a_{1, \ldots, n} \varphi \in T^a_1(R^n) \).
(ii) If \( \varphi \in S^a(R^n) \), then \( T^a \varphi \in T^a_2(R^n) \).

These operators become the identity operators on each proper subspace. In fact we have

**Lemma 6.** (i) \( \varphi \in \Phi_1(R^n) \), then \( U^a_{1, \ldots, n} \varphi = \varphi \).
(ii) If $\varphi \in \Phi_2(R^n)$, then $U^a\varphi = \varphi$

**LEMMA 7.** $\varphi \in T_1^a(R^n)$, then $T_1^a\varphi = \varphi$.

(ii) If $\varphi \in T_2^a(R^n)$, then $T_2^a\varphi = \varphi$.

Now we give decompositions of $S_{1,\ldots,n}^a(R^n)$ and $S^a(R^n)$.

**THEOREM 8.**

(i) $S_{1,\ldots,n}^a(R^n) = \Phi_1(R^n) \oplus T_1^a(R^n)$.

(ii) $S^a(R^n) = \Phi_2(R^n) \oplus T_2^a(R^n)$.

In order to give a decomposition of $S(R^n)$, we need a relation between $S(R^n)$ and $S^a(R^n)$ (or $S_{1,\ldots,n}^a(R^n)$). We have

**LEMMA 9.** $S(R^n) = \bigcup_{a \in A} S^a(R^n)$, $S(R^n) = \bigcup_{a \in A} S_{1,\ldots,n}^a(R^n)$.

Taking Lemma 9 into account we put

$T_1(R^n) = \bigcup_{a \in A} T_1^a(R^n)$, $T_2(R^n) = \bigcup_{a \in A} T_2^a(R^n)$.

Then we have

**THEOREM 10.**

(i) $S(R^n) = \Phi_1(R^n) \oplus T_1(R^n)$.

(ii) $S(R^n) = \Phi_2(R^n) \oplus T_2(R^n)$.

**References**


[Li3] P.I.Lizorkin, Operators connected with fractional differentiation and classes of differentiable functions, ibid. 117(1972), 251-286.


