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<th>Title</th>
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</thead>
<tbody>
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On complementary spaces of the Lizorkin spaces

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§1. Introduction

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we let

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$ 

The Schwartz space $S(\mathbb{R}^n)$ is defined to be the class of all $C^\infty$-functions $\varphi$ on $\mathbb{R}^n$ such that

$$p_{\alpha,\beta}(\varphi) = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \varphi(x)| < \infty$$

for all multi-indices $\alpha$ and $\beta$. We introduce two kinds of the Lizorkin spaces $\Phi_1(\mathbb{R}^n)$ and $\Phi_2(\mathbb{R}^n)$. The Lizorkin space $\Phi_1(\mathbb{R}^n)$ of the first kind is defined to be the class of all functions $\varphi \in S(\mathbb{R}^n)$ which satisfy

$$\int_{\mathbb{R}^n} \varphi(x)x^\alpha dx = 0$$

for any multi-index $\alpha$. The Lizorkin space $\Phi_2(\mathbb{R}^n)$ of the second kind is defined to be the class of all functions $\varphi \in S(\mathbb{R}^n)$ which satisfy

$$\int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_j, \cdots, x_n)x_j^\ell dx_j = 0$$

for $j = 1, \cdots, n$ and $\ell = 0, 1, 2, \cdots$. Clearly $\Phi_1(\mathbb{R}^n) \supset \Phi_2(\mathbb{R}^n)$. An example of a function belonging to $\Phi_1(\mathbb{R}^n)$ (resp. $\Phi_2(\mathbb{R}^n)$) is $\mathcal{F}(e^{-|y|^2-(1/|y|^2)})(x)$ (resp. $\mathcal{F}(e^{-|y|^2-\sum_{j=1}^{n}1/y_j^2})(x)$) where $\mathcal{F}\varphi$ is the Fourier transform of $\varphi$:

$$\mathcal{F}\varphi(x) = \int_{\mathbb{R}^n} e^{-ixy}dy.$$
The Lizorkin spaces appeared in the theory of fractional derivatives, hypersingular integrals and Riesz potentials ([Sa2] and [SKM]). The properties of the Lizorkin spaces have studied by several authors. The denseness of the Lizorkin spaces in the Lebegue spaces was proved in O.I.Lizorkin [Li2] and S.G.Samko [Sa1]. Moreover P.I.Lizorkin [Li3] showed that the space $\Phi_1(R^n)$ is dense in the Sobolev spaces and T.Kurokawa [Ku] deals with the denseness of the space $\Phi_1(R^n)$ in the spaces of Beppo Levi type. The invariance of the space $\Phi_1(R^n)$ relative to Riesz potentials was noted by V.I.Semyanistyi [Se], P.I.Lizorkin [Li3] and S.Helgason [He]. T.Kurokawa [Ku] establish the invariance of the space $\Phi_1(R^n)$ relative to more general operators. In this note we are concerned with comlementary spaces of $\Phi_1(R^n)$ and $\Phi_2(R^n)$ in $S(R^n)$. For a subspace $V \subset S(R^n)$, if a subspace $W \subset S(R^n)$ satisfies the condition

$$S(R^n) = V \oplus W,$$

then we call $W$ a complementary space of $V$ in $S(R^n)$ where the symbol $\oplus$ indicates a direct sum. In section 2 as a preparation we introduce dual functions of polynomials and tensor product functions, and study their properties. In section 3 we sketch our plan to give comlementary spaces of $\Phi_1(R^n)$ and $\Phi_2(R^n)$ in $S(R^n)$.

§2. Dual functions of polynomials and tensor product functions

Let $h \in C^\infty(R^1)$ be a function which satisfies the conditions $0 \leq h(t) \leq 1$, $h(-t) = h(t)$ and

$$h(t) = \begin{cases} 1, & \text{for } |t| \leq 1/2 \\ 0, & \text{for } |t| \geq 1. \end{cases}$$

We fix the function $h(t)$. We denote by $A$ the set of sequences $a = \{a_j\}_{j=0,1,\ldots}$ which satify $0 < a_j \leq 1$ and $a_j \geq a_{j+1}$. For $a =$

...
\{a_j\}_{j=0,1,\ldots} \in \mathcal{A} \text{ we put }
\eta_j^a(t) = \frac{t^j}{j!} h\left(\frac{t}{a_j}\right), \quad j = 0, 1, 2, \ldots

and
\theta_j^a(t) = \frac{i^j}{2\pi} \mathcal{F} \eta_j^a(t), \quad j = 0, 1, 2, \ldots.

Then \(\theta_j^a \in \mathcal{S}(\mathbb{R}^1)\) and
\[
\int_{-\infty}^{\infty} \theta_j^a(t) t^k dt = \begin{cases} 1, & k = j, k, j = 0, 1, 2, \ldots, \\
0, & k \neq j, \end{cases}
\]

Since \(\{\theta_j^a\}_{j=0,1,\ldots}\) satisfy (2.1), we call \(\{\theta_j^a\}_{j=0,1,\ldots}\) dual functions of polynomials associated with a sequence \(a \in \mathcal{A}\). For \(1 \leq p \leq n\) we denote by \(M_p\) the set of subsets of \(\{1, 2, \ldots, n\}\) which have \(p\) elements.

For \(\{i_1, i_2, \ldots, i_p\} \in M_p\) we always assume that \(i_1 < i_2 < \cdots < i_p\).
For multi-index \(\alpha = (\alpha_1, \cdots, \alpha_n)\) and \(\{i_1, \cdots, i_p\} \in M_p\) the notation \(\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c\) stands for

\(\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c = (\alpha_{k_1}, \cdots, \alpha_{k_{n-p}})\)

where \(\{k_1, \cdots, k_{n-p}\} = \{1, \cdots, n\} - \{i_1, \cdots, i_p\}\). Similarly, for \(x = (x_1, \cdots, x_n)\) we denote

\(\{x_{i_1}, \cdots, x_{i_p}\}^c = (x_{k_1}, \cdots, x_{k_{n-p}})\).

Moreover we denote
\[
\{(x_i, \cdots, x_i)^c\}^{(\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c)} = x_{k_1}^{\alpha_{k_1}} \cdots x_{k_{n-p}}^{\alpha_{k_{n-p}}}.
\]
\[
\{(D_i, \cdots, D_i)^c\}^{(\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c)} = D_{k_1}^{\alpha_{k_1}} \cdots D_{k_{n-p}}^{\alpha_{k_{n-p}}}.
\]

Let \(\alpha, \beta\) be multi-indices and \(\{i_1, \cdots, i_p\} \in M_p\). For a function \(\varphi(\{x_{i_1}, \cdots, x_{i_p}\}^c) \in \mathcal{S}(\mathbb{R}^{n-p})\) we define
\[
P(\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c, (\{\beta_{i_1}, \cdots, \beta_{i_p}\}^c)(\varphi)
= \sup_{\{(x_{i_1}, \cdots, x_{i_p})^c\} \in \mathbb{R}^{n-p}} |(\{(x_{i_1}, \cdots, x_{i_p})^c\}^{(\{\alpha_{i_1}, \cdots, \alpha_{i_p}\}^c)}
\{(D_{i_1}, \cdots, D_{i_p})^c\}^{(\{\beta_{i_1}, \cdots, \beta_{i_p}\}^c)}\varphi(\{x_{i_1}, \cdots, x_{i_p}\}^c)|.
For \( \{i_1, \cdots, i_p\} \in M_p \) we denote by \( C_{i_1, \cdots, i_p}^a \) the set of \( p \)-multiple sequences of functions \( \{ \varphi_{s_1 \cdots s_p}(\{x_{i_1}, \cdots, x_{i_p}\}^c) \}_{s_1, \cdots, s_p=0,1, \cdots} \subset S(R^{n-p}) \) which satisfy
\[
\sum_{s_1, \cdots, s_p=0}^{\infty} p((\alpha_{i_1}, \cdots, \alpha_{i_p})^\circ, (\beta_{i_1}, \cdots, \beta_{i_p})^\circ)(\varphi_{s_1 \cdots s_p})a_{s_1} \cdots a_{s_p} < \infty
\]
for all multi-idices \( \alpha \) and \( \beta \). We note that the sequence \( \{ \varphi_{s_1 \cdots s_n}(\{x_1, \cdots, x_n\}^c) \}_{s_1, \cdots, s_n=0,1, \cdots} \) is a \( n \)-multiple sequence of numbers \( \{b_{s_1 \cdots s_n}\}_{s_1, \cdots, s_n=0,1, \cdots} \) and
\[
p((\alpha_1, \cdots, \alpha_n)^\circ, (\beta_1, \cdots, \beta_n)^\circ)(b_{s_1 \cdots s_n}) = |b_{s_1 \cdots s_n}|.
\]
Therefore
\[
C_{1, \cdots, n}^a = \{ \{b_{s_1 \cdots s_n}\}_{s_1, \cdots, s_n=0,1, \cdots} : \sum_{s_1, \cdots, s_n=0}^{\infty} |b_{s_1 \cdots s_n}|a_{s_1} \cdots a_{s_n} < \infty \}.
\]
The basic fact is

**Lemma 1.** Let \( \{i_1, \cdots, i_p\} \in M_p \). If a \( p \)-multiple sequence of functions \( \{ \varphi_{s_1 \cdots s_p}(\{x_{i_1}, \cdots, x_{i_p}\}^c) \}_{s_1, \cdots, s_p=0,1, \cdots} \) belongs to \( C_{i_1, \cdots, i_p}^a \), then the \( p \)-multiple series
\[
\sum_{s_1, \cdots, s_p=0}^{\infty} \varphi_{s_1 \cdots s_p}(\{x_{i_1}, \cdots, x_{i_p}\}^c)\theta_{s_1}^a(x_{i_1}) \cdots \theta_{s_p}^a(x_{i_p})
\]
converges in \( S(R^n) \).

We introduce two kinds of tensor product functions associated with \( \{\theta_j^a\} \). If a function \( f \) has the following form
\[
(2.2) \quad f(x) = \sum_{s_1, \cdots, s_n=0}^{\infty} b_{s_1 \cdots s_n} \theta_{s_1}^a(x_1) \cdots \theta_{s_n}^a(x_n)
\]
where \( \{b_{s_1 \cdots s_n}\} \in C_{1, \cdots, n}^a \), then \( f \) is called a tensor product function of the first kind associated with \( \{\theta_j^a\} \). If a function \( f \) which has the
(2.3)\[ f(x) = \sum_{p=1}^{n} (-1)^{p} \sum_{\{i_1, \cdots, i_p\} \in M_p} \sum_{s_1, \cdots, s_p=0}^{\infty} \lambda_{i_1, \cdots, i_p; s_1, \cdots, s_p} \left(\{x_{i_1}, \cdots, x_{i_p}\}^c\right) \]

\[ \theta_{s_1}^{a}(x_{i_1}) \cdots \theta_{s_p}^{a}(x_{i_p}) \]

satisfies the conditions

(i) \( \{\lambda_{i_1, \cdots, i_p; s_1, \cdots, s_p}\}_{s_1, \cdots, s_p=0,1,\cdots} \in C_{i_1, \cdots, i_p}^{a} \),

(ii) for \( 2 \leq p \leq n \), \( \{i_1, \cdots, i_p\} \in M_p \) and \( s_1, \cdots, s_p \geq 0 \),

\[ \lambda_{i_1, \cdots, i_p; s_1, \cdots, s_p} \left(\{x_{i_1}, \cdots, x_{i_p}\}^c\right) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \lambda_{i; s_1}(\{x_{i_1}\}^c)x_{i_1}^{s_1} \cdots \lambda_{i; s_p}(\{x_{i_p}\}^c)x_{i_p}^{s_p} dx_{i_1} \cdots dx_{i_p} \]

where \( \ell = 1, \cdots, p \), then we call \( f \) a tensor product function of the second kind associated with \( \{\theta_{j}^{a}\} \) where the symbol \( \sim \) indicates that the variable underneath is deleted. We denote by \( T_1^{a}(R^n) \) (resp. \( T_2^{a}(R^n) \)) the class of all tensor product functions of the first kind (resp. the second kind) associated with \( \{\theta_{j}^{a}\} \). By Lemma 1, we see that \( T_1^{a}(R^n), T_2^{a}(R^n) \subset S(R^n) \). A fundamental property of the tensor product functions is the following.

**Lemma 2.** (i) Let \( f \) be a tensor product function of the first kind with the form (2.2). Then

\[ \int_{R^n} f(x_1, \cdots, x_n)x_1^{t_1} \cdots x_n^{t_n} dx_1 \cdots dx_n = b_{t_1 \cdots t_n} \]

for \( t_1, \cdots, t_n \geq 0 \).

(ii) Let \( f \) be a tensor product function of the second kind with the form (2.3). Then

\[ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \cdots, x_n)x_{k_1}^{t_1} \cdots x_{k_q}^{t_q} dx_{k_1} \cdots dx_{k_q} \]

\[ = \lambda_{k_1, \cdots, k_q; t_1, \cdots, t_q} \left(\{x_{k_1}, \cdots, x_{k_q}\}^c\right) \]
for \(1 \leq q \leq n, \{k_1, \ldots, k_q\} \in M_q\) and \(t_1, \ldots, t_q \geq 0\).

§3. Complementary spaces of the Lizorkin spaces

For \(\{i_1, \ldots, i_p\} \in M_p, s_1, \ldots, s_p \geq 0\) and \(\varphi \in \mathcal{S}(\mathbb{R}^n)\), we define

\[
\mu_{i_1, \ldots, i_p; s_1, \ldots, s_p}(\varphi)(\{x_{i_1}, \ldots, x_{i_p}\}^c) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \varphi(x_1, \ldots, x_n)x_{i_1}^{s_1} \cdots x_{i_p}^{s_p} dx_{i_1} \cdots dx_{i_p}.
\]

Moreover, for \(a \in \mathcal{A}\) and \(\{i_1, \ldots, i_p\} \in M_p\) we set

\[
S_{i_1, \ldots, i_p}^a \in C_{i_1, \ldots, i_p}^a \quad \text{and} \quad S_a(\mathbb{R}^n) = \bigcap_{p=1}^n \bigcap_{\{i_1, \ldots, i_p\} \in M_p} S_{i_1, \ldots, i_p}^a(\mathbb{R}^n).
\]

If \(\varphi \in \Phi_1(\mathbb{R}^n)\), then \(\mu_{1, \ldots, n; s_1, \ldots, s_n}(\varphi) = 0\) for \(s_1, \ldots, s_n \geq 0\). Hence \(\Phi_1(\mathbb{R}^n) \subset S_{1, \ldots, n}^a(\mathbb{R}^n)\) for any \(a \in \mathcal{A}\). If \(\varphi \in \Phi_2(\mathbb{R}^n)\), then \(\mu_{i_1, \ldots, i_p; s_1, \ldots, s_p}(\varphi) = 0\) for \(1 \leq p \leq n, \{i_1, \ldots, i_p\} \in M_p\) and \(s_1, \ldots, s_p \geq 0\). Hence \(\Phi_2(\mathbb{R}^n) \subset S_a(\mathbb{R}^n)\) for any \(a \in \mathcal{A}\). Moreover, By Lemma 2 (i), (ii) and the definitions of \(T_{1}^a, T_{2}^a\) we see that \(T_{1}^a(\mathbb{R}^n) \subset S_{1, \ldots, n}^a(\mathbb{R}^n)\) and \(T_{2}^a(\mathbb{R}^n) \subset S_a(\mathbb{R}^n)\). We introduce some operators. For \(\varphi \in S_{1, \ldots, n}(\mathbb{R}^n)\), we define

\[
T_{1, \ldots, n}^a \varphi(x) = \sum_{s_1, \ldots, s_n=0}^\infty \mu_{1, \ldots, n; s_1, \ldots, s_n}(\varphi)\theta_{s_1}^a(x_1) \cdots \theta_{s_n}^a(x_n)
\]

and

\[
U_{1, \ldots, n}^a \varphi = \varphi - T_{1, \ldots, n}^a \varphi.
\]

Further, for \(\varphi \in S^a(\mathbb{R}^n)\) we define

\[
T_j^a \varphi(x) = \sum_{s=0}^\infty \mu_{j; s}(\varphi)(\{x_j\}^c)\theta_{s}^a(x_j), \quad j = 1, \ldots, n
\]
and
\[ U_j^a \varphi = \varphi - T_j^a \varphi. \]
Moreover
\[ U^a \varphi = U_1^a \cdots U_n^a \varphi. \]
We see that
\[ U^a \varphi = \varphi - \sum_{p=1}^{n} (-1)^{p+1} \sum_{\{i_1, \cdots, i_p\} \in M_p} T_{i_1, \cdots, i_p}^a \varphi \]
where
\[ T_{i_1, \cdots, i_p}^a \varphi(x) = \sum_{s_1, \cdots, s_p=0}^{\infty} \mu_{i_1, \cdots, i_p; s_1, \cdots, s_p} (\varphi)(\{x_{i_1}, \cdots, x_{i_p}\}^c) \theta_{s_1}^a(x_{i_1}) \cdots \theta_{s_p}^a(x_{i_p}). \]
We put
\[ T^a = \sum_{p=1}^{n} (-1)^{p+1} \sum_{\{i_1, \cdots, i_p\} \in M_p} T_{i_1, \cdots, i_p}^a. \]
We establish properties of these operators which are necessary for decompositions of \( S_{1, \cdots, n}^a(R^n) \) and \( S^a(R^n) \). About ranges of these operators we have

**Lemma 3.** (i) If \( \varphi \in S_{1, \cdots, n}^a(R^n) \), then \( T_{1, \cdots, n}^a \varphi, U_{1, \cdots, n}^a \varphi \in S_{1, \cdots, n}^a(R^n) \). (ii) If \( \varphi \in S^a(R^n) \), then \( T^a \varphi, U^a \varphi \in S^a(R^n) \).

**Lemma 4.** (i) \( \varphi \in S_{1, \cdots, n}^a(R^n) \), then \( U_{1, \cdots, n}^a \varphi \in \Phi_1(R^n) \). (ii) If \( \varphi \in S^a(R^n) \), then \( U^a \varphi \in \Phi_2(R^n) \).

**Lemma 5.** (i) \( \varphi \in S_{1, \cdots, n}^a(R^n) \), then \( T_{1, \cdots, n}^a \varphi \in T_{1}^a(R^n) \). (ii) If \( \varphi \in S^a(R^n) \), then \( T^a \varphi \in T_2^a(R^n) \).

These operators become the identity operators on each proper subspace. In fact we have

**Lemma 6.** (i) \( \varphi \in \Phi_1(R^n) \), then \( U_{1, \cdots, n}^a \varphi = \varphi \).
(ii) If $\varphi \in \Phi_2(R^n)$, then $U^a \varphi = \varphi$

**Lemma 7.** $\varphi \in T_1^a(R^n)$, then $T_1^a \varphi = \varphi$.

(ii) If $\varphi \in T_2^a(R^n)$, then $T^a \varphi = \varphi$.

Now we give decompositions of $S_{1,\ldots,n}(R^n)$ and $S^a(R^n)$.

**Theorem 8.** (i) $S_{1,\ldots,n}(R^n) = \Phi_1(R^n) \oplus T_1^a(R^n)$.

(ii) $S^a(R^n) = \Phi_2(R^n) \oplus T_2^a(R^n)$.

In order to give a decomposition of $S(R^n)$, we need a relation between $S(R^n)$ and $S^a(R^n)$ (or $S_{1,\ldots,n}(R^n)$). We have

**Lemma 9.** $S(R^n) = \bigcup_{a \in A} S^a(R^n)$, $S(R^n) = \bigcup_{a \in A} S_{1,\ldots,n}(R^n)$.

Taking Lemma 9 into account we put

$$T_1(R^n) = \bigcup_{a \in A} T_1^a(R^n), \quad T_2(R^n) = \bigcup_{a \in A} T_2^a(R^n).$$

Then we have

**Theorem 10.** (i) $S(R^n) = \Phi_1(R^n) \oplus T_1(R^n)$.

(ii) $S(R^n) = \Phi_2(R^n) \oplus T_2(R^n)$.

**References**


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