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A Fast Algorithm for Computing Jones Polynomials of Montesinos Links

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Abstract

We give a fast algorithm for computing Jones polynomials of Montesinos links. Given the Tait graph with n edges of a Montesinos diagram, the algorithm runs with $\mathcal{O}(n)$ additions and multiplications in polynomials of degree $\mathcal{O}(n)$, namely in $\mathcal{O}(n^2 \log n)$ time.

1 Introduction

Knot theory is a subfield of topology. A knot is a simple (non-self-intersecting) closed curve embedded in $\mathbb{R}^3$. More generally, one may study links. A link is a finite collection of disjointly embedded knots. Works on knot theory have led to many important advances in other areas of topology, biology, chemistry and physics [1]. For classifying and characterizing links, various invariants have been defined and well studied in knot theory. The Jones polynomial [4] is a powerful invariant. L. H. Kauffman [5] gave a combinatorial method for calculating the Jones polynomial by means of the Kauffman bracket polynomial. We denote the number of the edges of the Tait graph of a link diagram $\widetilde{L}$ by $c(\widetilde{L})$. It takes $\mathcal{O}\left(2^{\mathcal{O}(c(\widetilde{L}))}\right)$ additions and multiplications in polynomials of degree $\mathcal{O}(c(\widetilde{L}))$ to compute a Jones polynomial by Kauffman's method. Actually, F. Jaeger, D. L. Vertigan and D. J. A. Welsh showed that computing the Jones polynomial is generally #**P**–hard [3, 11]. It is expected to require exponential time in the worst case.

Recently, it has been recognized that it is important to compute Jones polynomials for links with reasonable restrictions. J. A. Makowsky [6] showed that Jones polynomials are computed from a Tait graph in polynomial time if the treewidth of the Tait graph is bounded by a constant. J. Mighton [7] showed that Jones polynomials are computed from the Tait graph of a link diagram $\widetilde{L}$ with $\mathcal{O}(c(\widetilde{L})^4)$ operations in polynomials of degree $\mathcal{O}(c(\widetilde{L}))$ if the treewidth of the Tait graph is at most two. M. Hara, S. Tani and M. Yamamoto [2] showed that Jones polynomials of arborescent links are computed from the Tait graph of a link diagram $\widetilde{L}$ with $\mathcal{O}(c(\widetilde{L})^3)$ operations in polynomials of degree $\mathcal{O}(c(\widetilde{L}))$. T. Utsumi and K. Imai [10] showed that Jones polynomials of pretzel links are computed from the Tait graph of a link diagram $\widetilde{L}$ in $\mathcal{O}(c(\widetilde{L})^2)$ time. M. Murakami, M. Hara, M. Yamamoto and S. Tani [9] showed that Jones polynomials of 2–bridge links and closed 3–braid links are computed from the Tait graph of a link diagram $\widetilde{L}$ with $\mathcal{O}(c(\widetilde{L}))$ operations in polynomials of degree $\mathcal{O}(c(\widetilde{L}))$.

In this paper, we propose a fast algorithm for computing Jones polynomials of Montesinos links. Montesinos links consist of rational tangles, were introduced by J. M. Montesinos [8]. Montesinos links are a generalization of pretzel links in the sense that every Montesinos link all of whose rational tangles are integer integer tangles is a pretzel link. Montesinos links are a generalization of 2–bridge (or rational) links in the sense that every Montesinos link consisting of two rational tangles is a 2–bridge link. Montesinos diagrams, which are defined below, are standard link diagrams of Montesinos links. Our algorithm computes Jones polynomials of

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Montesinos links from the Tait graph of a Montesinos diagram $\widetilde{L}$ with $\mathcal{O}(c(\widetilde{L}))$ additions and multiplications in polynomials of degree $\mathcal{O}(c(\widetilde{L}))$, namely in $\mathcal{O}(c(\widetilde{L})^2 \log c(\widetilde{L}))$ time. Although treewidths of the Tait graphs of Montesinos diagrams are two, our algorithm is faster than Mighton's algorithm. Every rational tangle is represented by an integer sequence. Moreover, every Montesinos diagram is represented by a sequence of integer sequences.

Given a Montesinos diagram, our algorithm analyses the structure of the Montesinos diagram, constructs a sequence of integer sequences of the Montesinos diagram and computes the Kauffman bracket polynomial of the Montesinos diagram. We show the following:

(i) A sequence of integer sequences of a Montesinos diagram $\widetilde{L}$ can be constructed from the Tait graph of $\widetilde{L}$ in $\mathcal{O}(c(\widetilde{L}))$ time.

(ii) The Kauffman bracket polynomial of a Montesinos diagrams $\widetilde{L}$ is able to be computed from a sequence of integer sequences of $\widetilde{L}$ with $\mathcal{O}(c(\widetilde{L}))$ additions and multiplications in polynomials of degree $\mathcal{O}(c(\widetilde{L}))$.

Our algorithm computes the Kauffman bracket polynomial of a Montesinos diagram from a sequence of integer sequences of the Montesinos diagram by a way similar to ones for 2–bride links and closed 3–braid links [9]. On the other hand, we investigate a linear time algorithm to construct a sequence of integer sequences for a Montesinos diagram, which is different from algorithms to compute integer sequences for 2–bridge diagrams and closed 3–braid diagrams.

This paper is organized in the following way. Section 2 contains some basic notations and definitions of knot theory. In Section 3, we provide an algorithm for computing sequences of integer sequences of Montesinos diagrams. Section 4 deals with an algorithm for computing the Kauffman bracket polynomial of a given Montesinos diagram from a sequence of integer sequences of the Montesinos diagram.

2 Preliminaries

In this section, we give some basic notations and definitions of knot theory. For details, see C. C. Adams [1].

A *link* of n components is n simple closed curves in $\mathbb{R}^3$ that are mutually disjoint. A link of one component is a *knot*. An image of a link by an orthogonal projection from $\mathbb{R}^3$ to a plane is *regular* if it contains only finitely many multiple points, all multiple points are double points and these are traverse points. A regular image of a link is called a *link diagram* if the overcrossing/undercrossing information is marked at every double point in the image (see Fig. 1). Furthermore, the double points are called *crossings*. For any link diagram $\widetilde{L}$, we denote the number of the crossings of $\widetilde{L}$ by $c(\widetilde{L})$. A *trivial* link diagram is a link diagram without a crossing. A link is *trivial* if the link has a trivial link diagram. A link is *oriented* if each of its components is given an orientation.

A continuous bijection f from $\mathbb{R}^3$ to $\mathbb{R}^3$ is called *homeomorphism* if f has a continuous inverse mapping. Let I be the closed interval $[0,1]$. A Link L is *equivalent** or *ambient isotopic* to a Link L' if there exists a homeomorphism h_t from $\mathbb{R}^3$ to $\mathbb{R}^3$ for any $t \in I$ satisfying the following:

(i) h_0 is the identity.

(ii) $h_1(L) = L'$.

(iii) For any $x \in \mathbb{R}^3$, the mapping f_x from I to $\mathbb{R}^3$ satisfying $f_x(t) = h_t(x)$ is continuous.

*Intuitively, a link L is equivalent to a link L' if L can be continuously deformed to L' without ever having any one of the loops intersect itself or any of the other loops in the process. Therefore, we can regard a knot as a "rubber band", and, deform it topologically.

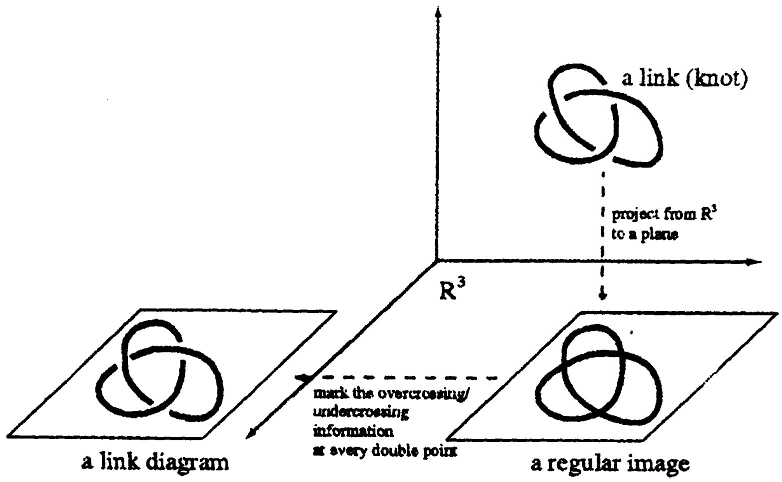


Fig. 1: A link, a regular image and a link diagram.

Definition 2.1 The *Kauffman bracket polynomial* is a function from link diagrams to the Laurent polynomial ring $\mathbb{Z}[A^{\pm 1}]$ with integer coefficients in an indeterminate A. It maps a link diagram $\widetilde{L}$ to $\langle \widetilde{L} \rangle \in \mathbb{Z}[A^{\pm 1}]$ and is characterized by

(i) $\langle \bigcirc \rangle = 1$,

(ii) $\langle \widetilde{L} \sqcup \bigcirc \rangle = (-A^{-2} - A^2)\langle \widetilde{L} \rangle$ and

(iii) $\langle \times \rangle = A\langle\,)(\,\rangle + A^{-1}\langle \asymp \rangle$.

Here, $\bigcirc$ is the trivial knot diagram and $\widetilde{L} \sqcup \bigcirc$ is the disjoint sum of $\widetilde{L}$ and $\bigcirc$. In (iii), the formula refers to three link diagrams that are exactly the same except near a point where they differ in the way indicated.

Note that for any link diagram $\widetilde{L}$, the degree of $\langle \widetilde{L} \rangle$ is $\mathcal{O}(c(\widetilde{L}))$ and the coefficients of $\langle \widetilde{L} \rangle$ are $\mathcal{O}(2^{c(\widetilde{L})})$. The *writhe* $w(\widetilde{L})$ of an oriented link diagram $\widetilde{L}$ is the sum of the signs of the crossings of $\widetilde{L}$, where each crossing has sign $+1$ or -1 as defined (by convention) in Fig. 2. The *Jones polynomial* $V(L)$ of an oriented link L is defined by

$$V(L) = (-A)^{-3w(\widetilde{L})}\langle \widetilde{L} \rangle \Big|_{t^{1/2}=A^{-2}},$$

where $\widetilde{L}$ is an oriented link diagram of L. It is known that $V(L)$ is well-defined and $V(L) \in \mathbb{Z}[t^{\pm 1/2}]$.

A *tangle* is a portion of a link diagram from which there emerge just four arcs pointing in the compass directions NW, NE, SW, SE (see Fig. 3). The tangle consisting of two vertical strings without a crossing is called 0*–tangle*. The 0–tangle twisted k times is called k*–tangle* and is denoted by I_k. They are called *integer tangles* (see Fig. 4).

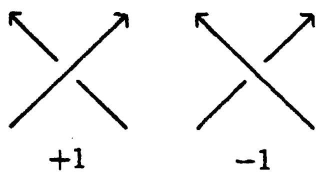


Fig. 2: Signs of crossings.

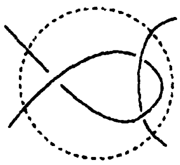


Fig. 3: A tangle.

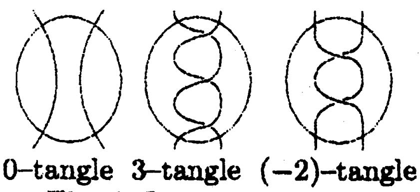


Fig. 4: Integer tangles.

Given any link diagram $\widetilde{L}$, we can color the faces black and white in such a way that no two faces with a common edge are the same color. We color the unique unbounded face white. Such a coloring is called the *Tait coloring* of $\widetilde{L}$. As shown in Fig. 5, we can get an edge-labeled planar graph G of $\widetilde{L}$. Its vertices are the black faces of the Tait coloring and two vertices are joined by a labeled edge if they share a crossing. The label of the edge is $+1$ or -1 according to the (conventional) rule shown in Fig. 6. We may call the label the *sign*. We call G the *Tait graph* of $\widetilde{L}$. Note that the number of the edges of G is $c(\widetilde{L})$. A Tait graph G is *isomorphic* to a Tait graph G' if there exists a bijection f from the vertex set of G to the vertex set of G' satisfying the following:

(i) For any pair of vertices u and v of G, the number of the edges in G that joins u and v and are labeled "+1" is equal to the number of the edges in G' that joins $f(u)$ and $f(v)$ and are labeled "+1".

(ii) For any pair of vertices u and v of G, the number of the edges in G that joins u and v and are labeled "−1" is equal to the number of the edges in G' that joins $f(u)$ and $f(v)$ and are labeled "−1".

Such a function f is called an *isomorphism* from G to G'.

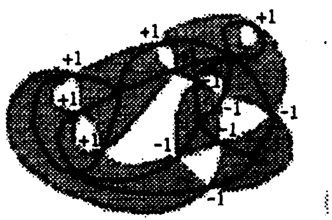
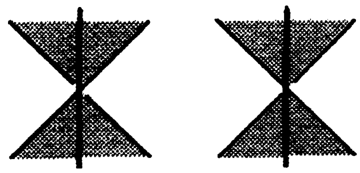


Fig. 5: A Tait coloring and a Tait graph.



+1 −1

Fig. 6: Signs of edges.

Let $a_{11}, \ldots, a_{1m_1}, \ldots, a_{l1}, \ldots, a_{lm_l}$ and a be integers. We denote the link diagram consisting of integer tangles $I_{a_{11}}, \ldots, I_{a_{1m_1}}, \ldots, I_{a_{l1}}, \ldots, I_{a_{lm_l}}$ and I_a as shown in Fig. 7 by $\widetilde{M}(a_{11}, \ldots, a_{1m_1}|\cdots|a_{l1}, \ldots, a_{lm_l}||a)$ and its Tait graph by $G_M(a_{11}, \ldots, a_{1m_1}|\cdots|a_{l1}, \ldots, a_{lm_l}||a)$. We call $\widetilde{M}(a_{11}, \ldots, a_{1m_1}|\cdots|a_{l1}, \ldots, a_{lm_l}||a)$ a *Montesinos diagram* if $l \geq 3$, $m_i \geq 3$, m_i is an odd number and $a_{ij} \neq 0$ for $i = 1, \ldots, l$ and $j = 1, \ldots, m_i$ (see Fig. 8). A link is called a *Montesinos link* if there exists a Montesinos diagram representing the link. A Montesinos link that consists of two rational tangles is a 2–bridge link. A Montesinos link all of whose rational tangles are integer tangles is a pretzel link. For convenience, $\widetilde{M}(a_{11}, \ldots, a_{1m_1}|\cdots|a_{l1}, \ldots, a_{lm_l})$ denotes $\widetilde{M}(a_{11}, \ldots, a_{1m_1}|\cdots|a_{l1}, \ldots, a_{lm_l}||0)$. For any non-zero integer n, we set $\text{sign}(n) = n/|n|$.

Remark 2.2

$$\widetilde{M}(a_{11}, \ldots, a_{1m_1}|\cdots|a_{l1}, \ldots, a_{lm_l}||a) = \begin{cases} \widetilde{M}(a_{11}, \ldots, a_{1m_1}|\cdots|a_{l1}, \ldots, a_{lm_l}) & \textit{if } a = 0 \\ \widetilde{M}(a_{11}, \ldots, a_{1m_1}|\cdots|a_{l1}, \ldots, a_{lm_l}|\underbrace{-\text{sign}(a)|\cdots|-\text{sign}(a)}_{|a|}) & \textit{if } a \neq 0 \end{cases}$$

For any Tait graph $G = G_M(a_{11}, \ldots, a_{1m_1}|\cdots|a_{l1}, \ldots, a_{lm_l}||a)$ of a Montesinos diagram, we call the path of the Tait graph corresponding to I_a the *special path* of G and a the *special path number* of G if $a \neq 0$. We call the contraction by the edges of the special path of the Tait graph G of a Montesinos diagram the *special contraction* of G (see Fig. 9).

Let $G = (V, E, s)$ be a Tait graph, where V is the vertex set of G, E is the edge set of G and s is the edge-labeling function from E to $\{-1, +1\}$. For any vertex $v \in V$, $\deg_G(v)$ denotes the degree of v in G and $N_G(v)$ denotes the set of the neighbors of v in G. For any subset V' of V, $G[V']$ denotes the subgraph of G induced by V'. We define a function edge_sum_G from $V \times V$ to $\mathbf{Z}$ such that for any pair of vertices u and v of G, $\text{edge\_sum}_G(u, v)$ is the sum of the signs of the edges of G that join u and v. For a set S, we denote the size of S by $|S|$.

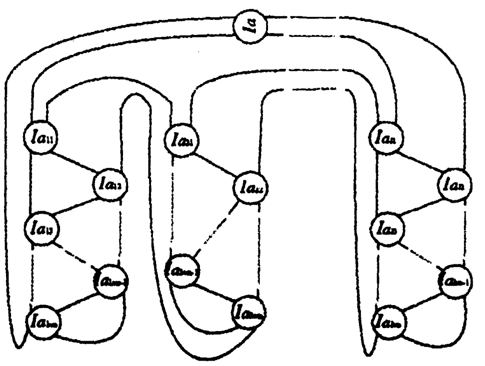
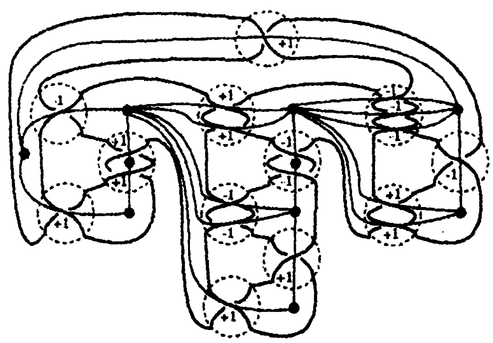
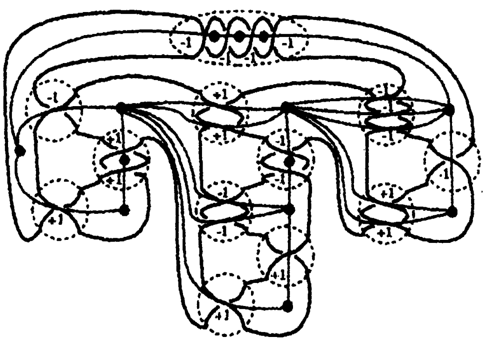


Fig. 7: $\widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l}||a)$.

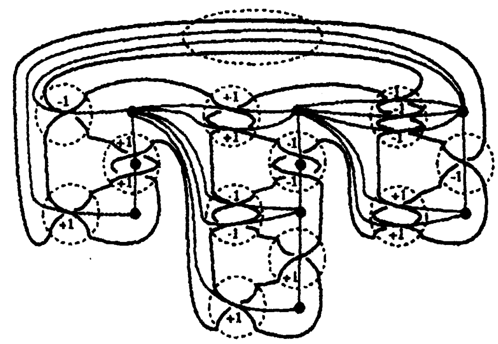


$\widetilde{M}(1,2,-1|-2,-2,2,1,-1|3,-1,-2|1)$ and
$G_M(1,2,-1|-2,-2,2,1,-1|3,-1,-2|1)$

Fig. 8: A montesinos diagram and its Tait graph.



$G_M(1,2,-1|-2,-2,2,1,-1|3,-1,-2||-4)$.



The special contraction of
$G_M(1,2,-1|-2,-2,2,1,-1|3,-1,-2||-4)$.

Fig. 9: The Tait graph of a Montesinos diagram and its special contraction.

3 Constructing sequences of integer sequences

In this section, we show that a sequence of integer sequences of a Montesinos diagram $\widetilde{L}$ can be constructed from the Tait graph of $\widetilde{L}$ in $\mathcal{O}(c(\widetilde{L}))$ time.

Lemma 3.1 *Let $G=(V,E,s)$ be the Tait graph of a Montesinos diagram, $P=(V',E',s')$ a path of G and v_1 and v_2 the endvertices of the path P ($|N_G(v_1)|\le|N_G(v_2)|$). P is the special path of G if and only if P satisfying the following:*

(i) $|V'|\ge 2$ *and* $G[V']=P$.

(ii) $|N_G(v_1)|=3$.

(iii) $N_G(v_1)\cap N_G(v_2)\subset V'$.

(iv) *There exists no path P' which is not P and every vertex of P' except for v_2 has at most three neighbors.*

Theorem 3.2 *Let $G=(V,E,s)$ be the Tait graph of a Montesinos diagram. We can determine whether there exists the special path of G in $\mathcal{O}(|E|)$ time. If there exists the special path of G, then we can construct the graph obtained by the special contraction from G and obtain the special path number of G in $\mathcal{O}(|E|)$ time.*

Given the graph G by the special contraction from the Tait graph of a Montesinos diagram, Procedure seq_montesinos constructs a sequence of integer sequences $(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l})$ such that $G_M(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l})$ is isomorphic to G.

Procedure seq_montesinos

Input: The graph $G = (V, E)$ by the special contraction from the Tait graph of a Montesinos diagram.

Output: A sequence of integer sequences $(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l})$ such that $G_M(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l})$ is isomorphic to G.

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Compute all of the values in {edge_sum_G(u,v) : u,v ∈ V are adjacent};
Construct N_G(v) at most four vertices for all v ∈ V;
Set V' = {v ∈ V : |N_G(v)| ≥ 4} and l = |V'|;
Construct N'_G(v) = {u ∈ N_G(v) : |N_G(u)| = 2} for all v ∈ V';
Index a vertex of V' v_01;
Set v_drc as a vertex of N'_G(v_01);
Initialize v_prv as v_drc and i as 1;
repeat
  Initialize v_crr as the vertex of N_G(v_prv) − {v_{i-1 1}} and flr as 0;
  repeat
    if there exist multiple edges connecting v_prv and v_crr, v_{i-1 1} ∉ N_G(v_prv) and |N_G(v_prv)| = 3
    or v_{i-1 1} ∉ N_G(v_prv) and the two edges incident to v_prv have different signs then
      begin v_prv := v ∈ N'_G(v_01) − {v_drc}; Initialize i as 1; flr := 1; end;
    else if |N_G(v_crr)| ≤ 3 then
      begin v_tmp := v_crr; v_crr := v ∈ N_G(v_crr) − {v_i1, v_prv}; v_prv := v_tmp; end;
  until |N_G(v_crr)| ≥ 4 or flr = 1;
  if flr = 0 then
    begin Index v_i1 v_crr and v_i2 v_prv; v_prv := v ∈ N'_G(v_i1) − {v_prv}; Increment i; end;
until i = l + 1;
for i := 1 to l do begin
  Initialize v_crr as v_i1 and m_i as 1;
  repeat
    { m_i is an odd number }
    Set a_{im_i} = −edge_sum_G(v_{i-1 1}, v_crr);
    Increment m_i;
    { m_i is an even number }
    Initialize a_{im_i} as 0;
    repeat
      v_tmp := v_crr;
      if v_crr = v_i1 then v_crr := v_i2 else v_crr := v ∈ N_G(v_crr) − {v_{i-1 1}, v_prv};
      v_prv := v_tmp;
      a_{im_i} := a_{im_i} + edge_sum_G(v_prv, v_crr);
    until v_{i-1 1} ∈ N_G(v_crr) ;
    Increment m_i;
  until |N_G(v_crr)| = 2;
  Set a_{im_i} = −edge_sum_G(v_{i-1 1}, v_crr);
end;
```

Theorem 3.3 Procedure seq_montesinos *constructs a sequence of integer sequences* $(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l})$ *such that* $G_M(a_{11}, \ldots, a_{1m_1} | \cdots | a_{l1}, \ldots, a_{lm_l})$ *is isomorphic to* G *in* $\mathcal{O}(|E|)$ *time.*

Theorems 3.2, 3.3 Remark 2.2 imply the following.

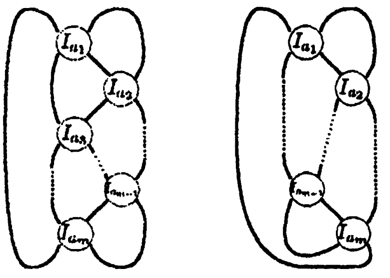
Corollary 3.4 *Given the Tait graph* G *of a Montesinos diagram* $\widetilde{L}$, *one can construct a sequence* *of integer sequences* $(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l})$ *such that* $G_M(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l})$ *is isomorphic to* G *in* $\mathcal{O}(c(\widetilde{L}))$ *time.*

4 Computing Kauffman bracket polynomials

In this section, we show that the Kauffman bracket polynomial of a Montesinos diagram $\widetilde{L}$ is able to be computed from a sequence of integer sequences of $\widetilde{L}$ with $\mathcal{O}(c(\widetilde{L}))$ additions and multiplications in polynomials of degree $\mathcal{O}(c(\widetilde{L}))$.

We denote the link diagram consisting of integer tangles $I_{a_1},\ldots,I_{a_m}$ as shown in Fig. 10 by $\widetilde{R}(a_1,\ldots,a_m)$ and the link diagram consisting of integer tangles $I_{a_{ij}}$ as shown in Fig. 11 by $\widetilde{N}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l})$. For an integer n, we set

$$Q_n = \frac{1-(-A^4)^n}{1-(-A^4)} = \begin{cases} 1+(-A^4)+\cdots+(-A^4)^{n-1} & \text{if } n>0, \\ 0 & \text{if } n=0, \\ -(-A^4)^{-1}-(-A^4)^{-2}-\cdots-(-A^4)^{n} & \text{if } n<0. \end{cases}$$



m is odd $\qquad$ m is even

Fig. 10: $\widetilde{R}(a_1,\ldots,a_m)$.

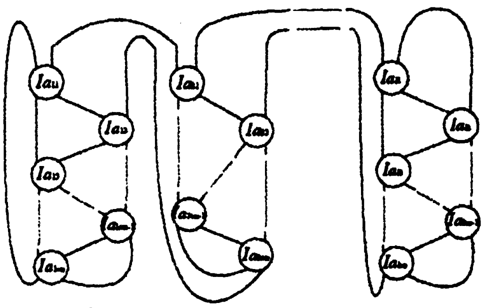


Fig. 11: $\widetilde{N}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l})$.

Lemma 4.1 *For any sequence of integer sequences* $(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l})$*, the following recurrence formula holds.*

$$\begin{aligned}
&\langle\widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l})\rangle \\
&= \begin{cases}
(-A^{-3})^{a_{11}} & \textit{if } l=1, m_1=1, \\
(-A^{-3})^{a_{11}}\langle\widetilde{R}(a_{12},\ldots,a_{1m_1})\rangle & \textit{if } l=1, m_1\geq 2, \\
A^{a_{l1}}\langle\widetilde{N}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l-11},\ldots,a_{l-1m_{l-1}})\rangle & \\
\quad -(-A)^{-3a_{l1}+2}Q_{a_{l1}} & \\
\quad\quad \times\langle\widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l-11},\ldots,a_{l-1m_{l-1}})\rangle & \textit{if } l\geq 2, m_l=1, \\
(-1)^{a_{l1}}A^{-3a_{l1}+a_{l2}}\langle\widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l-11},\ldots,a_{l-1m_{l-1}})\rangle & \\
\quad -(-A)^{-3a_{l2}+2}Q_{a_{l2}} & \\
\quad\quad \times\langle\widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l-11},\ldots,a_{l-1m_{l-1}}|a_{l1})\rangle & \textit{if } l\geq 2, m_l=2, \\
(-1)^{a_{lm_l-1}}A^{-3a_{lm_l-1}+a_{lm_l}} & \\
\quad \times\langle\widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l-2})\rangle & \\
\quad\quad -(-A)^{-3a_{lm_l}+2}Q_{a_{lm_l}} & \\
\quad\quad\quad \times\langle\widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l-1})\rangle & \textit{if } l\geq 2, m_l\geq 3.
\end{cases}
\end{aligned}$$

$$\langle\widetilde{N}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l})\rangle$$

$$
= \begin{cases}
\langle \widetilde{R}(a_{11},\ldots,a_{1m_1})\rangle & \textit{if } l=1,\\
(A^{a_{l1}}(-A^{-2}-A^{2})-(-A)^{-3a_{l1}+2}Q_{a_{l1}}) & \\
\quad \times \langle \widetilde{N}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l-11},\ldots,a_{l-1m_{l-1}})\rangle & \textit{if } l\geq 2, m_l=1,\\
(-1)^{a_{l1}}A^{-3a_{l1}+a_{l2}}\langle \widetilde{N}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l-11},\ldots,a_{l-1m_{l-1}})\rangle & \\
\quad -(-A)^{-3a_{l2}+2}Q_{a_{l2}} & \\
\qquad \times \langle \widetilde{N}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l-11},\ldots,a_{l-1m_{l-1}}|a_{l1})\rangle & \textit{if } l\geq 2, m_l=2,\\
(-1)^{a_{lm_l-1}}A^{-3a_{lm_l-1}+a_{lm_l}} & \\
\quad \times \langle \widetilde{N}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l-2})\rangle & \\
\qquad -(-A)^{-3a_{lm_l}+2}Q_{a_{lm_l}} & \\
\qquad\quad \times \langle \widetilde{N}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l-1})\rangle & \textit{if } l\geq 2, m_l\geq 3.
\end{cases}
$$

Theorem 4.2 Procedure bra_montesinos *computes the Kauffman bracket polynomial* $\langle \widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l})\rangle$ *with* $O(c(\widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l})))$ *additions and multiplications in polynomials of degree* $O(c(\widetilde{M}(a_{11},\ldots,a_{1m_1}|\cdots|a_{l1},\ldots,a_{lm_l})))$.

Corollary 4.3 *The Jones polynomial of a Montesinos link is computed from the Tait graph of a Montesinos diagram* $\widetilde{L}$ *with* $O(c(\widetilde{L}))$ *additions and multiplications in polynomials of degree* $O(c(\widetilde{L}))$, *namely in* $O(c(\widetilde{L})^2 \log c(\widetilde{L}))$ *time.*

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