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A Note on the Stability Spectrum of Generic Structures

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Abstract

「超安定であるが ω 安定でないような **K**-generic 構造が存在するか」という問題がある (Baldwin [1]). このノートでは，クラス **K** が部分グラフに関して閉じているときは，そのような **K**-generic 構造が存在しないことを示した．なお，このノートは [4] の内容を整理・改良したものである．

Let L be a countable relational language and $\mathbf{K}$ a class of finite L-structures closed under subgraphs. Let $\overline{\mathbf{K}}$ be a class of L-structures such that any finite substructure belongs to $\mathbf{K}$.

Definition 1 Let $ABC \in \overline{\mathbf{K}}$. Then B and C are said to be *free* over A (in symbol, $B \perp_A C$), if it satisfies the following:
(i) $B \cap C \subset A$;
(ii) $R^{ABC} = R^{AB} \cup R^{AC}$ for any $R \in L$.

Remark 2 Let $ABCD \in \overline{\mathbf{K}}$. Then
(i) If $A \perp_B C$ and $A \perp_{BC} D$, then $A \perp_B CD$.
(ii) If $BC \perp_A D$, then $B \perp_{CA} D$.
(iii) If $BC \perp_A D$, then $B \perp_A C$ if and only if $B \perp_D C$.

Definition 3 $\delta : \mathbf{K} \to \mathbf{R}^{\geq 0}$ is said to be a *predimension*, if
(i) if $A \cong B \in \mathbf{K}$, then $\delta(A) = \delta(B)$;
(ii) $\delta(\emptyset) = 0$;
(iii) for all $AB \in \mathbf{K}, \delta(A/B) \leq \delta(A/A \cap B)$;
(iv) there is no infinite chain $A_1 \subset A_2 \subset ...$ of $A_i \in \mathbf{K}$ with $\delta(A_i) > \delta(A_{i+1})$ for

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$i \in \omega$;
(v) for any $AB \in \mathbf{K}$, $A \perp_{A \cap B} B$ if and only if $\delta(A/B) = \delta(A/A \cap B)$;
(vi) for any $ABCD \in \mathbf{K}$ with $B \cap ACD = \emptyset$, $\delta(B/AC) - \delta(B/A) \leq \delta(B/DAC) - \delta(B/DA)$,
where $\delta(X/Y)$ means $\delta(XY) - \delta(Y)$.

Definition 4 (i) For $A \subset B \in \overline{\mathbf{K}}$, we define $A \leq B$, if $\delta(X/A') \geq 0$ for any finite $X \subset B - A$ and $A' \subset A$. For $A \subset B \in \overline{\mathbf{K}}$, define $\mathrm{cl}_B(A) = \bigcap\{A' : A \subset A' \leq B\}$. By the definition of a predimension, there exists such a $\mathrm{cl}_B(A)$, and moreover if A is finite, then so is $\mathrm{cl}_B(A)$.
(ii) Fix $M \in \overline{\mathbf{K}}$. For finite $A \subset M$, define $d_M(A) = \delta(\mathrm{cl}_M(A))$. For finite $B \subset M$, $d_M(A/B) = d_M(AB) - d_M(B)$. For infinite B, $d_M(A/B) = \inf\{d_M(A/B') : B' \subset B \text{ finite}\}$. For (possibly) infinite $A, B, C \subset M$, $d_M(B/C) = d_M(B/A)$ means $d_M(B'/C) = d_M(B'/A)$ for any finite $B' \subset B$.
(iii) A countable L-structure M is said to be $(\mathbf{K}, \leq)$-generic, if $A \in \mathbf{K}$ for any finite $A \subset M$; If $A \leq B \in \mathbf{K}$, then there is $B' \cong_A B$ with $B' \leq M$.

Let $\mathcal{M}$ be a big model. The following facts can be found in [2], [5] and [6].

Fact 5 Let $B, C \leq \mathcal{M}$ and $A = B \cap C$. Then the following are equivalent:
(i) $d(B/C) = d(B/A)$;
(ii) $B \perp_A C$ and $BC \leq \mathcal{M}$.

Proof (i)$\Rightarrow$(ii). First we show that $BC \leq \mathcal{M}$. If not, then there are $\bar{b} \in B, \bar{c} \in C, \bar{e} \in \mathrm{cl}(\bar{b}\bar{c}) - BC$ with $\delta(\bar{e}/\bar{b}\bar{c}) = -\gamma < 0$. Take $\bar{a} \leq A$ with $d(\bar{b}/\bar{a}) - d(\bar{b}/A) < \gamma/2$ and $d(\bar{c}/\bar{a}) - d(\bar{c}/A) < \gamma/2$. Let $\bar{b}' = \mathrm{cl}(\bar{b}\bar{a})$ and $\bar{c}' = \mathrm{cl}(\bar{c}\bar{a})$. Then $d(\bar{b}'\bar{c}'/\bar{a}) = d(\bar{b}\bar{c}/\bar{a}) \geq d(\bar{b}\bar{c}/A) = d(\bar{b}/A\bar{c}) + d(\bar{c}/A) = d(\bar{b}/A) + d(\bar{c}/A) > d(\bar{b}/\bar{a}) + d(\bar{c}/\bar{a}) - \gamma = \delta(\bar{b}'/\bar{a}) + \delta(\bar{c}'/\bar{a}) - \gamma \geq \delta(\bar{b}'\bar{c}'/\bar{a}) - \gamma$. On the other hand, we have $d(\bar{b}'\bar{c}'/\bar{a}) \leq \delta(\bar{e}\bar{b}'\bar{c}'/\bar{a}) \leq \delta(\bar{b}'\bar{c}'/\bar{a}) + \delta(\bar{e}/\bar{b}'\bar{c}') = \delta(\bar{b}'\bar{c}'/\bar{a}) - \gamma$. A contradiction. Next we show that $B \perp_A C$. If not, then there are $\bar{b} \in B, \bar{c} \in C$ with $\delta(\bar{b}/\bar{c}) < \delta(\bar{b}/\bar{a})$ where $\bar{a} = \bar{b} \cap \bar{c}$. Let $\gamma = \delta(\bar{b}/\bar{a}) - \delta(\bar{b}/\bar{c})$. Take $\bar{a}' \leq A$ with $\bar{a} \subset \bar{a}'$ and $d(\bar{b}/\bar{a}') - d(\bar{b}/A) < \gamma$. Let $\bar{b}' = \mathrm{cl}(\bar{a}'\bar{b})$ and $\bar{c}' = \mathrm{cl}(\bar{a}'\bar{c})$. By remark, we have $\delta(\bar{b}'/\bar{a}') - \delta(\bar{b}'/\bar{c}\bar{a}') \geq \delta(\bar{b}/\bar{a}) - \delta(\bar{b}/\bar{c}) = \gamma$. Then $\delta(\bar{b}'/\bar{c}\bar{a}') \geq d(\bar{b}'/\bar{c}A) = d(\bar{b}'/A) = d(\bar{b}/A) > d(\bar{b}/\bar{a}') - \gamma = \delta(\bar{b}'/\bar{a}') - \gamma \geq \delta(\bar{b}'/\bar{c}\bar{a}')$. A contradiction.
(ii)$\Rightarrow$(i). If not, then there are $\bar{b} \in B, \bar{c} \in C$ with $d(\bar{b}/\bar{c}) < d(\bar{b}/A)$. By (ii), we can take $\bar{b}', \bar{c}'$ such that $\bar{b} \subset \bar{b}' \leq B, \bar{c} \subset \bar{c}' \leq C, \bar{b}' \perp_{\bar{a}'} \bar{c}'$ and $\bar{b}'\bar{c}' \leq \mathcal{M}$ where $\bar{a}' = \bar{b}' \cap \bar{c}'$. Then $d(\bar{b}/\bar{c}) = \delta(\bar{b}'/\bar{c}') = \delta(\bar{b}'/\bar{a}') \geq d(b/\bar{a}') \geq d(b/A)$. A contradiction.

Fact 6 Let $B, C \leq \mathcal{M}$ and $A = B \cap C$ be algebraically closed. Then the following are equivalent:
(i) $\mathrm{tp}(B/C)$ does not fork over A;
(ii) $B \perp_A C$ and $BC \leq \mathcal{M}$.

Proof (i) $\Rightarrow$ (ii). Suppose that $B \downarrow_A C$. Take a sufficiently saturated model $N \supset A$ with $BC \downarrow_A N$. Then we have $B \downarrow_N C$ and $B \downarrow_A N$.
Claim 1: $d(B/N) = d(B/NC)$.
Proof: If $d(B/N) > d(B/NC)$, then there are $\bar{b} \in B, \bar{c} \in NC$ with $d(\bar{b}/N) > d(\bar{b}/\bar{c})$. Take countable $A_0 \subset N$ with $\bar{b}\bar{c} \downarrow_{A_0} N$. By the saturation of N, we can pick $\bar{c}' \in N$ with $\mathrm{stp}(\bar{c}/A_0) = \mathrm{stp}(\bar{c}'/A_0)$. Since $\bar{b}\bar{c} \downarrow_{A_0} N$ and $\bar{b} \downarrow_N \bar{c}$, we have $\bar{b} \downarrow_{A_0} \bar{c}$ and $\bar{b} \downarrow_{A_0} \bar{c}'$. Hence $\mathrm{tp}(b\bar{c}/A_0) = \mathrm{tp}(b\bar{c}'/A_0)$. Then $d(b/\bar{c}) = d(b/\bar{c}') \geq d(b/N)$. A contradiction.
Claim 2: $d(B/A) = d(B/N)$.
Proof: Let $B^* = \mathrm{acl}(B)$. We can take A_1 with $d(B^*/N) = d(B^*/A_1)$ where $A \subset A_1 \subset N$ and $|A_1| = |B| + \aleph_0$. A_1 acl??? By the saturation of N there is $A_2 \subset N$ with $\mathrm{tp}(A_2/A) = \mathrm{tp}(A_1/A)$ and $A_1 \downarrow_A A_2$. Note that $A_1 \downarrow_{B^*} A_2$ by $B \downarrow_A N$. Let $B_1^* = \mathrm{cl}(A_1B^*)$ and $B_2^* = \mathrm{cl}(A_2B^*)$. Then $B_1^* \cap B_2^* = B^*$. By fact 6, we have $B_1^*N, B_2^*N \leq \mathcal{M}$ since $d(B^*/N) = d(B^*/A_1) = d(B^*/A_2)$. Hence $B^*N = B_1^*N \cap B_2^*N \leq \mathcal{M}$. On the other hand, we have $B^* \perp_A N$. (Proof: Suppose that $B^* \not\perp_A N$. Note that $B^* \perp_{A_1} N$ and $B^* \perp_{A_2} N$ since $d(B^*/N) = d(B^*/A_1) = d(B^*/A_2)$. So we have $B^* \not\perp_A A_1$ and $B^* \not\perp_A A_2$. Since $A_1 \downarrow_A A_2$, we have $A_1 \cap A_2 = A$. A contradiction.) Hence $d(B/N) = d(B/A)$.

By claim 1,2, we have $d(B/A) = d(B/NC)$, and hence $d(B/A) = d(B/C)$.
(ii) $\Rightarrow$ (i). Take B' such that $\mathrm{tp}(B'/C)$ does not fork over A and $\mathrm{tp}(B/A) = \mathrm{tp}(B'/A)$. By (i) $\Rightarrow$ (ii), we have $B' \perp_A C$ and $B'C \leq \mathcal{M}$. So we have $\mathrm{tp}(BC/A) = \mathrm{tp}(B'C/A)$, and hence $\mathrm{tp}(B/C)$ does not fork over A.

For each $A \leq B \in \mathbf{K}$, B is said to be *minimal*, if $C = A$ or B for any C with $A \leq C \leq B$.

Lemma 7 Let $A \leq B \in \mathbf{K}$ with $B \leq \mathcal{M}$. Let B be minimal over A. If $\mathrm{tp}(B/A)$ is algebraic, then $B \perp_A C$ for any $C \leq \mathcal{M}$ with $B \cap C = A$.

Proof Suppose that $\delta(B/C) < \delta(B/A)$ for some $C \leq BC \in \mathbf{K}$ with $B \cap C = A$.
Claim: There is a set $\{B_i\}_{i<\omega}$ of copies of B over A with the following conditions:
(i) $C \leq CB_j \leq CB_0B_1 \cdots B_i \in \mathbf{K}$ for each $j \leq i < \omega$;
(ii) $B_i \cap B_j = A$ for each $j < i < \omega$;
(iii) B_i, C are free over A for each $i < \omega$.
Proof: Suppose that $\{B_i\}_{i \leq n}$ has been defined. By our assumption, we have $C \leq CB \in \mathbf{K}$, and by (i) we have $C \leq CB_0B_1...B_n \in \mathbf{K}$. By amalgamation, we can take a copy B^* of B over C such that $CB_0...B_n, CB^* \leq CB_0...B_nB^* \in \mathbf{K}$. By (iii) and $\delta(B^*/C) < \delta(B^*/A)$, we have $B_i \neq B^*$ for all $i \leq n$. Since B is minimal over A, we have $B^* \cap B_i = A$. Since $\mathbf{K}$ is closed under L-subgraphs, there is $B_{n+1} \cong_{AB_0B_1...B_n} B^*$ such that $CB_0B_1...B_nB_{n+1} \in \mathbf{K}$ and B_{n+1}, C are free over A. So (ii) and (iii) hold. It is not difficult to check that $CB_j \leq CB_0B_1 \cdots B_{n+1} \in \mathbf{K}$ for each $j \leq n+1$. So (i) holds. (End of Proof of Claim)

By claim, we have $AB_j \leq AB_0...B_i \in \mathbf{K}$ for each $j \leq i < \omega$. We can assume that $AB_0...B_i \leq \mathcal{M}$. Thus we have $\mathrm{tp}(B_j/A) = \mathrm{tp}(B/A)$ for each $j \leq i$. By (ii) of claim, B_j's are pairwise distinct. Hence $\mathrm{tp}(B/A)$ is not algebraic.

Lemma 8 Let $A \leq B \in \mathbf{K}$ with $B \leq \mathcal{M}$. Let B be minimal over A. If $\mathrm{tp}(B/A)$ is algebraic, then $BC \leq \mathcal{M}$ for any $C \leq \mathcal{M}$ with $B \cap C = A$.

Proof Suppose by way of contradiction that $BC \not\leq \mathcal{M}$ for some $C \leq \mathcal{M}$ with $B \cap C = A$. Then there is finite $X \subset \mathcal{M} - BC$ such that $\delta(X/BC) < 0$.
Claim 1: There is a set $\{B_i\}_{i<\omega}$ of copies of B with the following conditions:
(i) $B_i \cong_{CB_0...B_{i-1}} B$ for each $i < \omega$;
(ii) $CB_0...B_i, CB_0...B_{i-1}BX \leq CB_0...B_iBX \in \mathbf{K}$ for each $i < \omega$;
(iii) $XB \cap B_i = B_j \cap B_i = A$ for each $j < i < \omega$.
Proof: Suppose that $\{B_i\}_{i\leq n}$ has been defined. By (ii), $CB_0...B_n \leq CB_0...B_nBX \in \mathbf{K}$, and so we have $CB_0...B_n \leq CB_0...B_nB \in \mathbf{K}$. By amalgamation, we can take a copy B_{n+1} of B over $CB_0...B_n$ such that $CB_0...B_nBX, CB_0...B_nB_{n+1} \leq CB_0...B_nB_{n+1}BX \in \mathbf{K}$. Hence (i) and (ii) hold. On the other hand, $B_{n+1} \cap B_i = A$ for each $i \leq n$, since $B_{n+1} \cong_{CB_0...B_n} B$. So, to see that (iii) holds, it is enough to show that $B_{n+1} \cap XB = A$. Let $B' = B_{n+1} \cap XB$. First, suppose that $B' = B_{n+1}$. Then we have $B_{n+1} \subset BX$, and so $CB_{n+1} \not\leq CBX$, since $\delta(XB/CB_{n+1}) = \delta(XB/C) - \delta(B_{n+1}/C) = \delta(XB/C) - \delta(B/C) = \delta(X/BC) < 0$. This contradicts our choice of B_{n+1}. Hence we have $B' \neq B_{n+1}$. We have to see that $B' = A$. This can be shown as follows: By our choice of B_{n+1}, we have $CB_0...B_nBX \leq CB_0...B_nB_{n+1}BX$, and so $B' \leq B_{n+1}$. Since B is minimal and $B' \neq B_{n+1}$, we have $B' = A$. (End of Proof of Claim 1)
Claim 2: $B, B_j \leq B_0...B_iB (\in \mathbf{K})$ for $j \leq i < \omega$
Proof: We prove by induction on i. By (ii) of claim 1, $B_0...B_iB \leq B_0...B_{i+1}B$. By induction hypothesis, we have $B, B_j \leq B_0...B_iB$ for $j \leq i$. Hence $B, B_j \leq B_0...B_{i+1}B$ for $j \leq i$. So, it is enough to show that $B_{i+1} \leq B_0...B_{i+1}B$. By induction hypothesis again, we have $B \leq B_0...B_iB$. From (i) of claim 1, it follows that $B_{i+1} \leq B_0...B_{i+1}$. By (ii) of claim 1, $B_0...B_{i+1} \leq B_0...B_{i+1}B$. Hence we have $B_{i+1} \leq B_0...B_{i+1}B$. (End of Proof of Claim 2)

We show that $\mathrm{tp}(B/A)$ is non-algebraic. By claim 2, we can assume that $B, B_j \leq BB_0...B_i \leq \mathcal{M}$ for each i, j with $j \leq i < \omega$. So we have $\mathrm{tp}(B_j/A) = \mathrm{tp}(B/A)$ for each $j < \omega$. By (iii) of claim 1, B_j's are pairwise distinct. Hence $\mathrm{tp}(B/A)$ is not algebraic.

Proposition 9 Let $A \leq B \leq \mathcal{M}$ and $A = \mathrm{acl}(A) \cap B$. Then $\mathrm{acl}(A) \perp_A B$ and $\mathrm{acl}(A) \cup B \leq \mathcal{M}$.

Proof We can assume that A, B are finite. We wil show that $A^* \perp_A B$ and $A^*B \leq \mathcal{M}$ for any finite $A^* \leq \mathrm{acl}(A)$ with $A \subset A^*$. Take $A = A_0 \leq A_1 \leq ... \leq A_n = A^*$ with A_{i+1} minimal over A_i for each $i < n$. Then it is enough to show that $A_i \perp_{A_0} B$ and $A_i B \leq \mathcal{M}$ for each $i \leq n$. (Proof: We prove by induction on i. Clearly $A_i \leq A_{i+1}, A_{i+1} \cap A_i B = A_i$ and $\mathrm{tp}(A_{i+1}/A_i)$ is algebraic. By inducition hypothesis, $A_i B \leq \mathcal{M}$. So we have $A_{i+1} \perp_{A_i} B$ and $A_{i+1} B \leq \mathcal{M}$ by lemma. By induction hypothesis, $A_i \perp_{A_0} B$, and hence $A_{i+1} \perp_{A_0} B$.)

Theorem 10 Let $B, C \leq \mathcal{M}$ and $A = B \cap C$. Then the following are equivalent:
(i) $\mathrm{tp}(B/C)$ does not fork over A;
(ii) $B \perp_A C$ and $BC \cup \mathrm{acl}(A) \leq \mathcal{M}$.

Proof By proposition 9, $B \cup \mathrm{acl}(A), C \cup \mathrm{acl}(A) \leq \mathcal{M}$. So, by fact 7, (i) is equivalent to $B \perp_{\mathrm{acl}(A)} C$ and $BC \cup \mathrm{acl}(A) \leq \mathcal{M}$. Therefore, proving that (i) and (ii) are equivalent, it is enough to show that $B \perp_{\mathrm{acl}(A)} C$ if and only if $B \perp_A C$. We can assume that A, B, C is finite. Take any finite $A^* \leq \mathrm{acl}(A)$ with $BC \cap \mathrm{acl}(A) \subset A^*$. Then we will show that $B \perp_{A^*} C$ if and only if $B \perp_A C$. Let $B' = B \cap A^*, C' = C \cap A^*$.
($\Rightarrow$) Since $\mathrm{tp}(A^*/B'C')$ is algebraic, we have $A^* \perp_{B'C'} BC$. So, from $B \perp_{A^*} C$ it follows that $B \perp_{B'C'} C$. On the other hand, since $\mathrm{tp}(B'/C')$ and $\mathrm{tp}(C'/A)$ are algebraic, we have $B' \perp_{C'} C$ and $B \perp_A C'$. Hence we have $B \perp_A C$.
($\Leftarrow$) By $B \perp_A C$, we have $B \perp_{B'C'} C$. On the other hand, since $\mathrm{tp}(A^*/B'C')$ is algebraic, we have $A^* \perp_{B'C'} BC$. Hence $B \perp_{A^*} C$.

Corollary 11 Let L be a countable relational language and **K** a class of finte L-structures that is derived from a predimension δ. Then there is no **K**-generic structure that is superstable but not ω-stable.

Proof Suppose that a theory T of a **K**-generic structure is superstable. Take any countable model N of T.
Claim: For any $p \in S(N)$ there is finite $A \subset N$ such that p does not fork over A and $p|A$ is stationary.
Proof: Take a realization $\bar{b}$ of p. By superstability, there is finite $X \subset N$ such that p does not fork over X. Let $B = \mathrm{cl}(X\bar{b})$ and $A = B \cap N$. Clearly p does not fork over A. We show that $\mathrm{tp}(\bar{b}/A)$ is stationary. Take any $\bar{b}'$ such that $\mathrm{tp}(\bar{b}'/A) = \mathrm{tp}(\bar{b}/A)$ and $\mathrm{tp}(\bar{b}'/N)$ does not fork over A. Let $B' = \mathrm{cl}(\bar{b}'A)$. Then $\mathrm{tp}(B/N)$ and $\mathrm{tp}(B'/N)$ do not fork over A. Since $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{b}'/A)$, we have $B \cong_A B'$. Note that $B \cap N = B' \cap N = A$. By theorem, $B \perp_A N, B' \perp_A N$ and $BN, B'N \leq \mathcal{M}$. In particular, $BN \cong B'N$. It follows that $\mathrm{tp}(BN) = \mathrm{tp}(B'N)$ and hence $\mathrm{tp}(b/N) = \mathrm{tp}(b'/N)$. (End of Proof of Claim)

By claim, we have $|S(N)| \leq \aleph_0 \cdot |S(T)| = \aleph_0$. Hence T is ω-stable.

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