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A Note on the Stability Spectrum of Generic Structures

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Abstract

Let $L$ be a countable relational language and $K$ a class of finite $L$-structures closed under subgraphs. Let $\overline{K}$ be a class of $L$-structures such that any finite substructure belongs to $K$.

Definition 1 Let $ABC \in \overline{K}$. Then $B$ and $C$ are said to be free over $A$ (in symbol, $B \perp_A C$), if it satisfies the following:
(i) $B \cap C \subset A$;
(ii) $R^{ABC} = R^{AB} \cup R^{AC}$ for any $R \in L$.

Remark 2 Let $ABCD \in \overline{K}$. Then
(i) If $A \perp BC$ and $A \perp BCD$, then $A \perp BCD$.
(ii) If $BC \perp A D$, then $B \perp CAD$.
(iii) If $BC \perp A D$, then $B \perp A C$ if and only if $B \perp D C$.

Definition 3 $\delta : K \to R^{\geq 0}$ is said to be a predimension, if
(i) if $A \cong B \in K$, then $\delta(A) = \delta(B)$;
(ii) $\delta(\emptyset) = 0$;
(iii) for all $AB \in K$, $\delta(A/B) \leq \delta(A/A \cap B)$;
(iv) there is no infinite chain $A_1 \subset A_2 \subset \ldots$ of $A_i \in K$ with $\delta(A_i) > \delta(A_{i+1})$ for

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\( \forall i \in \omega; \)
\( (v) \) for any \( AB \in K \), \( A \perp_{A \cap B} B \) if and only if \( \delta(A/B) = \delta(A/A \cap B) \);
\( (vi) \) for any \( ABCD \in K \) with \( B \cap ACD = \emptyset \), \( \delta(B/AC) - \delta(B/A) \leq \delta(B/DA) - \delta(B/DA) \),
where \( \delta(X/Y) \) means \( \delta(XY) - \delta(Y) \).

**Definition 4**  
(i) For \( A \subset B \in \overline{K} \), we define \( A \leq B \), if \( \delta(X/A') \geq 0 \) for any finite \( X \subset B - A \) and \( A' \subset A \). For \( A \subset B \in \overline{K} \), define \( \text{cl}_B(A) = \cap \{A' : A \subset A' \leq B \} \). By the definition of a predimension, there exists such a \( \text{cl}_B(A) \), and moreover if \( A \) is finite, then so is \( \text{cl}_B(A) \).
(ii) Fix \( M \in \overline{K} \). For finite \( A \subset M \), define \( d_M(A) = \delta(\text{cl}_M(A)) \). For finite \( B \subset M \), \( d_M(A/B) = d_M(AB) - d_M(B) \). For infinite \( B \), \( d_M(A/B) = \inf \{d_M(A/B') : B' \subset B \text{ finite} \} \). For (possibly) infinite \( A, B, C \subset M \), \( d_M(B/C) = \inf \{d_M(B'/A) : B' \subset B \text{ finite} \} \).
(iii) A countable \( L \)-structure \( M \) is said to be \( (K, \leq) \)-generic, if \( A \in K \) for any finite \( A \subset M \); if \( A \leq B \in K \), then there is \( B' \cong_A B \) with \( B' \leq M \).

Let \( \mathcal{M} \) be a big model. The following facts can be found in [2], [5] and [6].

**Fact 5**  
Let \( B, C \leq \mathcal{M} \) and \( A = B \cap C \). Then the following are equivalent:
(i) \( d(B/C) = d(B/A) \);
(ii) \( B \perp_A C \) and \( BC \leq \mathcal{M} \).

**Proof**  
(i)\( \Rightarrow \) (ii). First we show that \( BC \leq \mathcal{M} \). If not, then there are \( \bar{b} \in B, \bar{c} \in C, \bar{e} \in \text{cl} (\bar{b} \bar{c}) - BC \) with \( \delta(\bar{e}/\bar{b} \bar{c}) = -\gamma < 0 \). Take \( \bar{a} \subset A \) with \( d(\bar{b}/\bar{a}) - d(\bar{b}/A) < \gamma/2 \) and \( d(\bar{c}/\bar{a}) - d(\bar{c}/A) < \gamma/2 \). Let \( \bar{b}' = \text{cl}(\bar{b} \bar{a}) \) and \( \bar{c}' = \text{cl}(\bar{c} \bar{a}) \). Then \( d(\bar{b}' \bar{c}'/\bar{a}) = d(\bar{b} \bar{c}/\bar{a}) \geq d(\bar{b} \bar{c}/A) = d(\bar{b}' \bar{a}/\bar{c}) + d(\bar{c}/A) = d(\bar{b}/\bar{a}) + d(\bar{c}/\bar{a}) \geq \gamma \). On the other hand, we have \( d(\bar{b}' \bar{c}'/\bar{a}) \leq \delta(\bar{b}' \bar{c}'/\bar{a}) \leq \delta(\bar{b}'/\bar{a}') + \delta(\bar{e}/\bar{b}' \bar{c}') = \delta(\bar{b}' \bar{c}'/\bar{a}) - \gamma \). A contradiction. Next we show that \( B \perp_A C \). If not, then there are \( \bar{b} \in B, \bar{c} \in C \) with \( \delta(\bar{b}/\bar{c}) < \delta(\bar{b}/\bar{a}) \) where \( \bar{a} = \bar{b} \cap \bar{c} \). Let \( \gamma = \delta(\bar{b}/\bar{a}) - \delta(\bar{b}/\bar{c}) \). Take \( \bar{a}' \subset A \) with \( \bar{a} \subset \bar{a}' \) and \( d(\bar{b}/\bar{a}') - d(\bar{b}/A) < \gamma \). Let \( \bar{b}' = \text{cl}(\bar{a}' \bar{b}) \) and \( \bar{c}' = \text{cl}(\bar{a}' \bar{c}) \). By remark, we have \( \delta(\bar{b}' \bar{a}') - \delta(\bar{b}' \bar{a}) \geq \delta(\bar{b}/\bar{a}) - \delta(\bar{b}/\bar{c}) = \gamma \). Then \( \delta(\bar{b}' \bar{a}) \geq d(\bar{b}'/\bar{c} A) = d(\bar{b}/A) + d(\bar{a}/A) - \gamma \). A contradiction.
(ii)\( \Rightarrow \) (i). If not, then there are \( \bar{b} \in B, \bar{c} \in C \) with \( d(\bar{b}/\bar{c}) < d(\bar{b}/A) \). By (ii), we can take \( \bar{b}', \bar{c}' \) such that \( \bar{b} \subset \bar{b}' \leq B, \bar{c} \subset \bar{c}' \leq C, \bar{b} \perp_{\bar{a}'} \bar{c}' \) and \( \bar{b}' \subset M \) where \( \bar{a}' = \bar{b} \cap \bar{c}' \). Then \( d(\bar{b}/\bar{c}) = \delta(\bar{b}'/\bar{c}') = \delta(\bar{b}'/\bar{a}') \geq d(\bar{b}/\bar{a}') \geq d(\bar{b}/A) \). A contradiction.

**Fact 6**  
Let \( B, C \leq \mathcal{M} \) and \( A = B \cap C \) be algebraically closed. Then the following are equivalent:
(i) \( \text{tp}(B/C) \) does not fork over \( A \);
(ii) \( B \perp_A C \) and \( BC \leq \mathcal{M} \).
Proof (i) ⇒ (ii). Suppose that $B \downarrow_A C$. Take a sufficiently saturated model $N \supset A$ with $BC \downarrow_A N$. Then we have $B \downarrow_N C$ and $B \downarrow_A N$.

Claim 1: $d(B/N) = d(B/NC)$.

Proof: If $d(B/N) > d(B/NC)$, then there are $\overline{b} \in B, \overline{c} \in NC$ with $d(\overline{b}/N) > d(\overline{b}/\overline{c})$. Take countable $A_0 \subset N$ with $\overline{b} \overline{c} \downarrow_{A_0} N$. By the saturation of $N$, we can pick $\overline{c}' \in N$ with $\text{stp}(\overline{c}/A_0) = \text{stp}(\overline{c}'/A_0)$. Since $\overline{b} \overline{c} \downarrow_{A_0} N$ and $\overline{b} \downarrow_N \overline{c}$, we have $\overline{b} \downarrow_{A_0} \overline{c}'$ and $\overline{b} \downarrow_{A_0} \overline{c}'$. Hence $\text{tp}(\overline{b}\overline{c}/A_0) = \text{tp}(\overline{b}\overline{c}'/A_0)$. Then $d(\overline{b}/\overline{c}) = d(\overline{b}/\overline{c}') \geq d(\overline{b}/N)$. A contradiction.

Claim 2: $d(B/A) = d(B/N)$.

Proof: Let $B^* = \text{acl}(B)$. We can take $A_1$ with $d(B^*/N) = d(B^*/A_1)$ where $A \subset A_1 \subset N$ and $|A_1| = |B| + \aleph_0$. $A_1$ ac??? By the saturation of $N$ there is $A_2 \subset N$ with $\text{tp}(A_2/A) = \text{tp}(A_1/A)$ and $A_1 \downarrow_A A_2$. Note that $A_1 \downarrow_{B^*} A_2$ by $B \downarrow_A N$. Let $B_1^* = \text{cl}(A_1B^*)$ and $B_2^* = \text{cl}(A_2B^*)$. Then $B_1^* \cap B_2^* = B^*$. By fact 6, we have $B_1^* N, B_2^* N \leq M$ since $d(B^*/N) = d(B^*/A_1) = d(B^*/A_2)$. Hence $B^* N = B_1^* N \cap B_2^* N \leq M$. On the other hand, we have $B^* \perp_A N$. (Proof: Suppose that $B^* \perp_A N$. Note that $B^* \perp_{A_1} N$ and $B^* \perp_{A_2} N$ since $d(B^*/N) = d(B^*/A_1) = d(B^*/A_2)$. So we have $B^* \not\perp_A A_1$ and $B^* \not\perp_A A_2$. Since $A_1 \downarrow_A A_2$, we have $A_1 \cap A_2 = A$. A contradiction.) Hence $d(B/N) = d(B/A)$.

By claim 1,2, we have $d(B/A) = d(B/NC)$, and hence $d(B/A) = d(B/C)$.

(ii) ⇒ (i). Take $B'$ such that $\text{tp}(B'/C)$ does not fork over $A$ and $\text{tp}(B'/A) = \text{tp}(B'/A)$. By (i) ⇒ (ii), we have $B' \perp_A C$ and $B' \leq M$. So we have $\text{tp}(BC/A) = \text{tp}(B'C/A)$, and hence $\text{tp}(B/C)$ does not fork over $A$.

For each $A \leq B \in K$, $B$ is said to be minimal, if $C = A$ or $B$ for any $C$ with $A \leq C \leq B$.

Lemma 7 Let $A \leq B \in K$ with $B \leq M$. Let $B$ be minimal over $A$. If $\text{tp}(B/A)$ is algebraic, then $B \perp_A C$ for any $C \leq M$ with $B \cap C = A$.

Proof Suppose that $\delta(B/C) < \delta(B/A)$ for some $C \leq BC \in K$ with $B \cap C = A$.

Claim: There is a set $\{B_i\}_{i<\omega}$ of copies of $B$ over $A$ with the following conditions:

(i) $C \leq CB_j \leq CB_0B_1 \cdots B_i \in K$ for each $j < i < \omega$;

(ii) $B_i \cap B_j = A$ for each $j < i < \omega$;

(iii) $B_i, C$ are free over $A$ for each $i < \omega$.

Proof: Suppose that $\{B_i\}_{i \leq n}$ has been defined. By our assumption, we have $C \leq CB \in K$, and by (i) we have $C \leq CB_0B_1 \cdots B_n \in K$. By amalgamation, we can take a copy $B^*$ of $B$ over $C$ such that $CB_0 \cdots B_n, CB^* \leq CB_0 \cdots B_nB^* \in K$.

By (iii) and $\delta(B^*/C) < \delta(B^*/A)$, we have $B_i \neq B^*$ for all $i \leq n$. Since $B$ is minimal over $A$, we have $B^* \cap B_i = A$. Since $K$ is closed under $L$-subgraphs, there is $B_{n+1} \cong_{A, B_0B_1 \cdots B_n} B^*$ such that $CB_0B_1 \cdots B_nB_{n+1} \in K$ and $B_{n+1}, C$ are free over $A$. So (ii) and (iii) hold. It is not difficult to check that $CB_j \leq CB_0B_1 \cdots B_{n+1} \in K$ for each $j \leq n + 1$. So (i) holds. (End of Proof of Claim)
By claim, we have $AB_j \leq AB_0...B_i \in K$ for each $j \leq i < \omega$. We can assume that $AB_0...B_i \leq \mathcal{M}$. Thus we have $\text{tp}(B_j/A) = \text{tp}(B/A)$ for each $j \leq i$. By (ii) of claim, $B_j$'s are pairwise distinct. Hence $\text{tp}(B/A)$ is not algebraic.

**Lemma 8** Let $A \leq B \in K$ with $B \leq \mathcal{M}$. Let $B$ be minimal over $A$. If $\text{tp}(B/A)$ is algebraic, then $BC \leq \mathcal{M}$ for any $C \leq \mathcal{M}$ with $B \cap C = A$.

**Proof** Suppose by way of contradiction that $BC \not\leq \mathcal{M}$ for some $C \leq \mathcal{M}$ with $B \cap C = A$. Then there is finite $X \subset \mathcal{M} - BC$ such that $\delta(X/BC) < 0$.

Claim 1: There is a set $\{B_i\}_{i<\omega}$ of copies of $B$ with the following conditions:

(i) $B_i \cong_{CB_0...B_{i-1}} B$ for each $i < \omega$;

(ii) $CB_0...B_i, CB_0...B_{i-1}BX \leq CB_0...B_iBX \in K$ for each $i < \omega$;

(iii) $XB \cap B_i = B_j \cap B_i = A$ for each $j < i < \omega$.

Proof: Suppose that $\{B_i\}_{i \leq n}$ has been defined. By (ii), $CB_0...B_n \leq CB_0...B_nBX \in K$, and so we have $CB_0...B_n \leq CB_0...B_n B \in K$. By amalgamation, we can take a copy $B_{n+1}$ of $B$ over $CB_0...B_n$ such that $CB_0...B_nBX, CB_0...B_nB_{n+1} \leq CB_0...B_nB_{n+1}BX \in K$. Hence (i) and (ii) hold. On the other hand, $B_{n+1} \cap B_i = A$ for each $i \leq n$, since $B_{n+1} \cong_{CB_0...B_n} B$. So, to see that (iii) holds, it is enough to show that $B_{n+1} \cap XB = A$. Let $B' = B_{n+1} \cap XB$. First, suppose that $B' = B_{n+1}$. Then we have $B_{n+1} \subset BX$, and so $CB_{n+1} \leq CBX$, since $\delta(XB/CB_{n+1}) = \delta(XB/C) - \delta(B_{n+1}/C) = \delta(XB/C) - \delta(B/C) = \delta(X/BC) < 0$. This contradicts our choice of $B_{n+1}$. Hence we have $B' \neq B_{n+1}$. We have to see that $B' = A$. This can be shown as follows: By our choice of $B_{n+1}$, we have $CB_0...B_nBX \leq CB_0...B_nB_{n+1}BX$, and so $B' \leq B_{n+1}$. Since $B$ is minimal and $B' \neq B_{n+1}$, we have $B' = A$. (End of Proof of Claim 1)

Claim 2: $B, B_j \leq B_0...B_i B \in K$ for $j \leq i < \omega$.

Proof: We prove by induction on $i$. By (ii) of claim 1, $B_0...B_i B \leq B_0...B_{i+1} B$. By induction hypothesis, we have $B, B_j \leq B_0...B_i B$ for $j \leq i$. Hence $B, B_j \leq B_0...B_{i+1} B$ for $j \leq i$. So, it is enough to show that $B_{i+1} \leq B_0...B_{i+1} B$. By induction hypothesis again, we have $B \leq B_0...B_i B$. From (i) of claim 1, it follows that $B_{i+1} \leq B_0...B_{i+1}$. By (ii) of claim 1, $B_0...B_{i+1} \leq B_0...B_{i+1} B$. Hence we have $B_{i+1} \leq B_0...B_{i+1} B$. (End of Proof of Claim 2)

We show that $\text{tp}(B/A)$ is non-algebraic. By claim 2, we can assume that $B, B_j \leq BB_0...B_i \leq \mathcal{M}$ for each $i, j$ with $j \leq i < \omega$. So we have $\text{tp}(B_j/A) = \text{tp}(B/A)$ for each $j < \omega$. By (iii) of claim 1, $B_j$'s are pairwise distinct. Hence $\text{tp}(B/A)$ is not algebraic.

**Proposition 9** Let $A \leq B \leq \mathcal{M}$ and $A = \text{acl}(A) \cap B$. Then $\text{acl}(A) \perp A B$ and $\text{acl}(A) \cup B \leq \mathcal{M}$.

**Proof** We can assume that $A, B$ are finite. We will show that $A^* \perp A B$ and $A^* B \leq \mathcal{M}$ for any finite $A^* \leq \text{acl}(A)$ with $A \subset A^*$. Take $A = A_0 \leq A_1 \leq ... \leq A_n = A^*$ with $A_{i+1}$ minimal over $A_i$ for each $i < n$. Then it is enough to show
that \( A_i \perp_{A_0} B \) and \( A_i B \leq \mathcal{M} \) for each \( i \leq n \). (Proof: We prove by induction on \( i \). Clearly \( A_i \leq A_{i+1} \), \( A_{i+1} \cap A_i B = A_i \) and \( \text{tp}(A_{i+1}/A_i) \) is algebraic. By induction hypothesis, \( A_i B \leq \mathcal{M} \). So we have \( A_{i+1} \perp_{A_i} B \) and \( A_{i+1} B \leq \mathcal{M} \) by lemma. By induction hypothesis, \( A_i \perp_{A_0} B \), and hence \( A_{i+1} \perp_{A_0} B \).)

**Theorem 10** Let \( B, C \leq \mathcal{M} \) and \( A = B \cap C \). Then the following are equivalent:

(i) \( \text{tp}(B/C) \) does not fork over \( A \);

(ii) \( B \perp_{A} C \) and \( BC \cup \text{acl}(A) \leq \mathcal{M} \).

**Proof** By proposition 9, \( B \cup \text{acl}(A), C \cup \text{acl}(A) \leq \mathcal{M} \). So, by fact 7, (i) is equivalent to \( B \perp_{\text{acl}(A)} C \) and \( BC \cup \text{acl}(A) \leq \mathcal{M} \). Therefore, proving that (i) and (ii) are equivalent, it is enough to show that \( B \perp_{A} C \) if and only if \( B \perp_{A} C \). We can assume that \( A, B, C \) is finite. Take any finite \( A^* \leq \text{acl}(A) \) with \( BC \cap \text{acl}(A) \subset A^* \). Then we will show that \( B \perp_{A} C \) if and only if \( B \perp_{A} C \). Let \( B' = B \cap A^*, C' = C \cap A^* \).

(\( \Rightarrow \)) Since \( \text{tp}(A^*/B'C') \) is algebraic, we have \( A^* \perp_{B'C'} BC \). So, from \( B \perp_{A} C \) it follows that \( B \perp_{B'C'} C \). On the other hand, since \( \text{tp}(B'/C') \) and \( \text{tp}(C'/A) \) are algebraic, we have \( B' \perp_{C'} C \) and \( B \perp_{A} C \). Hence we have \( B \perp_{A} C \).

(\( \Leftarrow \)) By \( B \perp_{A} C \), we have \( B \perp_{B'C'} C \). On the other hand, since \( \text{tp}(A^*/B'C') \) is algebraic, we have \( A^* \perp_{B'C'} BC \). Hence \( B \perp_{A} C \).

**Corollary 11** Let \( L \) be a countable relational language and \( K \) a class of finite \( L \)-structures that is derived from a predimension \( \delta \). Then there is no \( K \)-generic structure that is superstable but not \( \omega \)-stable.

**Proof** Suppose that a theory \( T \) of a \( K \)-generic structure is superstable. Take any countable model \( N \) of \( T \).

Claim: For any \( p \in S(N) \) there is finite \( A \subset N \) such that \( p \) does not fork over \( A \) and \( p|A \) is stationary.

Proof: Take a realization \( \bar{b} \) of \( p \). By superstability, there is finite \( X \subset N \) such that \( p \) does not fork over \( X \). Let \( B = \text{cl}(X\bar{b}) \) and \( A = B \cap N \). Clearly \( p \) does not fork over \( A \). We show that \( \text{tp}(\bar{b}/A) \) is stationary. Take any \( \bar{b}' \) such that \( \text{tp}(\bar{b}'/A) = \text{tp}(\bar{b}/A) \) and \( \text{tp}(\bar{b}'/N) \) does not fork over \( A \). Let \( B' = \text{cl}(\bar{b}'A) \). Then \( \text{tp}(B/N) \) and \( \text{tp}(B'/N) \) do not fork over \( A \). Since \( \text{tp}(\bar{b}/A) = \text{tp}(\bar{b}'/A) \), we have \( B \equiv A B' \). Note that \( B \cap N = B' \cap N = A \). By theorem, \( B \perp_{A} N, B' \perp_{A} N \) and \( BN, B'N \leq \mathcal{M} \). In particular, \( BN \cong B'N \). It follows that \( \text{tp}(BN) = \text{tp}(B'N) \) and hence \( \text{tp}(b/N) = \text{tp}(b'/N) \). (End of Proof of Claim)

By claim, we have \( |S(N)| \leq \aleph_0 \cdot |S(T)| = \aleph_0 \). Hence \( T \) is \( \omega \)-stable.

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