A Note on the Stability Spectrum of Generic Structures

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Abstract

「超安定であるがω安定でないような K-generic 構造が存在するか」という問題がある (Baldwin [1])。このノートでは、クラス K が部分グラフに関して閉じているときは、そのような K-generic 構造が存在しないことを示した。なお、このノートは [4] の内容を整理・改良したものである。

Let $L$ be a countable relational language and $K$ a class of finite $L$-structures closed under subgraphs. Let $\overline{K}$ be a class of $L$-structures such that any finite substructure belongs to $K$.

**Definition 1** Let $ABC \in \overline{K}$. Then $B$ and $C$ are said to be free over $A$ (in symbol, $B \perp_A C$), if it satisfies the following:
(i) $B \cap C \subset A$;
(ii) $R^{ABC} = R^{AB} \cup R^{AC}$ for any $R \in L$.

**Remark 2** Let $ABCD \in \overline{K}$. Then
(i) If $A \perp BCD$ and $A \perp BCD$, then $A \perp BCD$.
(ii) If $BC \perp AD$, then $B \perp CAD$.
(iii) If $BC \perp AD$, then $B \perp AC$ if and only if $B \perp D$.

**Definition 3** $\delta : K \to \mathbb{R}^{\geq 0}$ is said to be a predimension, if
(i) if $A \cong B \in K$, then $\delta(A) = \delta(B)$;
(ii) $\delta(\emptyset) = 0$;
(iii) for all $AB \in K, \delta(A/B) \leq \delta(A/A \cap B)$;
(iv) there is no infinite chain $A_1 \subset A_2 \subset \ldots$ of $A_i \in K$ with $\delta(A_i) > \delta(A_{i+1})$ for

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\[ i \in \omega; \]

(v) for any \( AB \in K \), \( A \perp_{AN} B \) if and only if \( \delta(A/B) = \delta(A/A \cap B) \);

(vi) for any \( ABCD \in K \) with \( B \cap ACD = \emptyset \), \( \delta(B/AC) - \delta(B/A) \leq \delta(B/DAC) - \delta(B/DA) \),

where \( \delta(X/Y) \) means \( \delta(XY) - \delta(Y) \).

**Definition 4** (i) For \( A \subset B \in \overline{K} \), we define \( A \leq B \), if \( \delta(X/A') \geq 0 \) for any finite \( X \subset B - A \) and \( A' \subset A \). For \( A \subset B \in \overline{K} \), define \( cl_B(A) = \cap \{ A' : A \subset A' \leq B \} \). By the definition of a predimension, there exists such a \( cl_B(A) \), and moreover if \( A \) is finite, then so is \( cl_B(A) \).

(ii) Fix \( M \in \overline{K} \). For finite \( A \subset M \), define \( d_M(A) = \delta(cl_M(A)) \). For finite \( B \subset M \), \( d_M(A/B) = d_M(AB) - d_M(B) \). For infinite \( B \subset M \), \( d_M(A/B) = \inf \{ d_M(A/B') : B' \subset B \text{ finite} \} \). For (possibly) infinite \( A, B, C \subset M, d_M(B/C) = d_M(B/A) \) means \( d_M(B'/C) = d_M(B'/A) \) for any finite \( B' \subset B \).

(iii) A countable \( L \)-structure \( M \) is said to be \( (K, \leq) \)-generic, if \( A \subset K \) for any finite \( A \subset M \); If \( A \subset B \subset K \), then there is \( B' \cong_A B \) with \( B' \leq M \).

Let \( M \) be a big model. The following facts can be found in [2], [5] and [6].

**Fact 5** Let \( B, C \leq M \) and \( A = B \cap C \). Then the following are equivalent:

(i) \( d(B/C) = d(B/A) \);

(ii) \( B \perp_A C \) and \( BC \leq M \).

**Proof** (i)\( \Rightarrow \) (ii). First we show that \( BC \leq M \). If not, then there are \( \bar{b} \in B, \bar{c} \in C, \bar{e} \in cl(\bar{b}\bar{c}) - BC \) with \( \delta(\bar{e}/\bar{b}\bar{c}) = -\gamma < 0 \). Take \( \bar{a} \subset A \) with \( d(\bar{b}/\bar{a}) - d(\bar{b}/A) < \gamma/2 \) and \( d(\bar{c}/\bar{a}) - d(\bar{c}/A) < \gamma/2 \). Let \( \bar{b}' = cl(\bar{b}\bar{a}) \) and \( \bar{c}' = cl(\bar{c}\bar{a}) \).

Then \( d(\bar{b}'\bar{c}'/\bar{a}) = \delta(\bar{b}\bar{c}/\bar{a}) \geq \delta(\bar{b}\bar{c}/A) = d(\bar{b}/A\bar{c}) + d(\bar{c}/A) = d(\bar{b}/A) + d(\bar{c}/A) > d(\bar{b}/\bar{a}) - d(\bar{c}/\bar{a}) - \gamma = \delta(\overline{b}/\overline{a}) + \delta(\overline{c}/\overline{a}) - \gamma \geq \delta(\overline{b}'/\overline{a}'\overline{c}') - \gamma \). On the other hand, we have \( d(\bar{b}'\bar{c}'/\bar{a}) \leq \delta(\overline{b}/\overline{a}') \overline{c}'/\overline{a}' \leq \delta(\overline{b}'/\overline{a}'\overline{c}') = \delta(\overline{b}'\overline{c}'/\overline{a}') - \gamma \). A contradiction. Next we show that \( B \perp_A C \). If not, then there are \( \bar{b} \in B, \bar{c} \in C \) with \( \delta(\bar{b}'/\bar{a}') \leq \delta(\overline{b}'\overline{c}'/\overline{a}') \) where \( \bar{a} = \overline{b}\cap\overline{c} \). Let \( \gamma = \delta(\overline{b}/\overline{a}) - \delta(\overline{b}/\overline{c}) \). Take \( \bar{a}' \subset A \) with \( \bar{a} \subset \bar{a}' \) and \( d(\bar{b}/\bar{a}) - d(\bar{b}/A) < \gamma \). Let \( \bar{b}' = cl(\bar{a}'\bar{b}) \) and \( \bar{c}' = cl(\bar{a}'\bar{c}) \). By remark, we have \( \delta(\overline{b}'/\overline{a}') \leq \delta(\overline{b}/\overline{a}'\overline{c}') \geq \delta(\overline{b}/\overline{a}) - \delta(\overline{b}/\overline{c}) = \gamma \). Then \( \delta(\overline{b}'/\overline{a}') \geq \delta(\overline{b}'/\overline{c}'\overline{a}') = \delta(\overline{b}/\overline{a}') - \gamma \). A contradiction.

(ii)\( \Rightarrow \) (i). If not, then there are \( \bar{b} \in B, \bar{c} \in C \) with \( d(\bar{b}/\overline{b}') < d(\bar{b}/A) \). By (ii), we can take \( \bar{b}', \bar{c}' \) such that \( \bar{b} \subset \bar{b}' \subset B, \bar{c} \subset \bar{c}' \subset C, \bar{b}' \perp_{\bar{a}'} \bar{c}' \) and \( \bar{b}' \leq M \) where \( \bar{a}' = \overline{b}\cap\overline{c}' \). Then \( d(\bar{b}/\overline{c}') \leq \delta(\overline{b}/\overline{c}') = \delta(\overline{b}/\overline{a}') \geq \delta(\overline{b}/\overline{a}) \). A contradiction.

**Fact 6** Let \( B, C \leq M \) and \( A = B \cap C \) be algebraically closed. Then the following are equivalent:

(i) \( tp(B/C) \) does not fork over \( A \);

(ii) \( B \perp_A C \) and \( BC \leq M \).
Proof (i) ⇒ (ii). Suppose that \( B \downarrow_A C \). Take a sufficiently saturated model \( N \supset A \) with \( BC \downarrow_A N \). Then we have \( B \downarrow_N C \) and \( B \downarrow_A N \).

Claim 1: \( d(B/N) = d(B/NC) \). Proof: If \( d(B/N) > d(B/NC) \), then there are \( \bar{b} \in B, \bar{c} \in NC \) with \( d(\bar{b}/N) > d(\bar{b}/\bar{c}) \). Take countable \( A_0 \subset N \) with \( \bar{b} \bar{c} \downarrow_{A_0} N \). By the saturation of \( N \), we can pick \( \bar{c}' \in N \) with \( \text{stp}(\bar{c}/A_0) = \text{stp}(\bar{c}'/A_0) \). Since \( \bar{b} \bar{c} \downarrow_{A_0} N \) and \( \bar{b} \downarrow_N \bar{c} \), we have \( \bar{b} \downarrow_{A_0} \bar{c} \) and \( \bar{b} \downarrow_{A_0} \bar{c}' \). Hence \( \text{tp}(\bar{b}c/A_0) = \text{tp}(\bar{b}c'/A_0) \). Then \( d(\bar{b}/\bar{c}) = d(\bar{b}/\bar{c}') \geq d(\bar{b}/N) \). A contradiction.

Claim 2: \( d(B/A) = d(B/N) \). Proof: Let \( B^* = \text{acl}(B) \). We can take \( A_1 \) with \( d(B^*/N) = d(B^*/A_1) \) where \( A \subset A_1 \subset N \) and \( |A_1| = |B| + \aleph_0 \). \( A_1 \) acl??? By the saturation of \( N \) there is \( A_2 \subset N \) with \( \text{tp}(A_2/A) = \text{tp}(A_1/A) \) and \( A_1 \downarrow_A A_2 \). Note that \( A_1 \downarrow_{B^*} A_2 \) by \( B \perp_A N \). Let \( B_1^* = \text{acl}(A_1B^*) \) and \( B_2^* = \text{acl}(A_2B^*) \). Then \( B_1^* \cap B_2^* = B^* \). By fact 6, we have \( B_1^*N, B_2^*N \leq M \) since \( d(B^*/N) = d(B^*/A_1) = d(B^*/A_2) \). Hence \( B^*N = B_1^*N \cap B_2^*N \leq M \). On the other hand, we have \( B^* \perp_A N \). (Proof: Suppose that \( B^* \not\perp_A N \). Note that \( B^* \perp_{A_1} N \) and \( B^* \perp_{A_2} N \) since \( d(B^*/N) = d(B^*/A_1) = d(B^*/A_2) \). So we have \( B^* \not\perp_{A_1} A_1 \) and \( B^* \not\perp_{A_2} A_2 \). Since \( A_1 \downarrow_A A_2 \), we have \( A_1 \cap A_2 = A \). A contradiction.) Hence \( d(B/N) = d(B/A) \).

By claim 1,2, we have \( d(B/A) = d(B/NC) \), and hence \( d(B/A) = d(B/C) \).

(ii) ⇒ (i). Take \( B' \) such that \( \text{tp}(B'/C) \) does not fork over \( A \) and \( \text{tp}(B/A) = \text{tp}(B'/A) \). By (i) ⇒ (ii), we have \( B' \perp_A C \) and \( B'C \leq M \). So we have \( \text{tp}(BC/A) = \text{tp}(B'C/A) \), and hence \( \text{tp}(B/C) \) does not fork over \( A \).

For each \( A \leq B \in K \), \( B \) is said to be minimal, if \( C = A \) or \( B \) for any \( C \) with \( A \leq C \leq B \).

Lemma 7 Let \( A \leq B \in K \) with \( B \leq M \). Let \( B \) be minimal over \( A \). If \( \text{tp}(B/A) \) is algebraic, then \( B \perp_A C \) for any \( C \leq M \) with \( B \cap C = A \).

Proof Suppose that \( \delta(B/C) < \delta(B/A) \) for some \( C \leq BC \in K \) with \( B \cap C = A \). Claim: There is a set \( \{B_i\}_{i<\omega} \) of copies of \( B \) over \( A \) with the following conditions:

(i) \( C \leq CB_j \leq CB_0B_1 \cdots B_i \in K \) for each \( j \leq i < \omega \);
(ii) \( B_i \cap B_j = A \) for each \( j < i < \omega \);
(iii) \( B_i \cap C \) are free over \( A \) for each \( i < \omega \).

Proof: Suppose that \( \{B_i\}_{i \leq n} \) has been defined. By our assumption, we have \( C \leq CB \in K \), and by (i) we have \( C \leq CB_0B_1 \cdots B_n \in K \). By amalgamation, we can take a copy \( B^* \) of \( B \) over \( C \) such that \( CB_0 \cdots B_n, CB^* \leq CB_0 \cdots B_n B^* \in K \). By (iii) and \( \delta(B^*/C) < \delta(B^*/A) \), we have \( B_i \neq B^* \) for all \( i \leq n \). Since \( B \) is minimal over \( A \), we have \( B^* \cap B_i = A \). Since \( K \) is closed under \( L \)-subgraphs, there is \( B_{n+1} \cong_{A B_0} B_1 \cdots B_n B^* \) such that \( CB_0B_1 \cdots B_n B_{n+1} \in K \) and \( B_{n+1}, C \) are free over \( A \). So (ii) and (iii) hold. It is not difficult to check that \( CB_j \leq CB_0B_1 \cdots B_{n+1} \in K \) for each \( j \leq n + 1 \). So (i) holds. (End of Proof of Claim)
By claim, we have $AB_j \leq AB_0 \ldots B_i \in K$ for each $j \leq i < \omega$. We can assume that $AB_0 \ldots B_i \leq M$. Thus we have $tp(B_j/A) = tp(B/A)$ for each $j \leq i$. By (ii) of claim, $B_j$'s are pairwise distinct. Hence $tp(B/A)$ is not algebraic.

**Lemma 8** Let $A \leq B \in K$ with $B \leq M$. Let $B$ be minimal over $A$. If $tp(B/A)$ is algebraic, then $BC \leq M$ for any $C \leq M$ with $B \cap C = A$.

**Proof** Suppose by way of contradiction that $BC \not\leq M$ for some $C \leq M$ with $B \cap C = A$. Then there is finite $X \subset M - BC$ such that $\delta(X/BC) < 0$.

Claim 1: There is a set $\{B_i\}_{i < \omega}$ of copies of $B$ with the following conditions:

(i) $B_i \cong_{CB_0 \ldots B_{i-1}} B$ for each $i < \omega$;
(ii) $CB_0 \ldots B_i CB_0 \ldots B_{i-1} BX \leq CB_0 \ldots B_i BX \in K$ for each $i < \omega$;
(iii) $XB \cap B_i = B_j \cap B_i = A$ for each $j < i < \omega$.

Proof: Suppose that $\{B_i\}_{i < n}$ has been defined. By (ii), $CB_0 \ldots B_n \leq CB_0 \ldots B_n BX \in K$, and so we have $CB_0 \ldots B_n \leq CB_0 \ldots B_n B \in K$. By amalgamation, we can take a copy $B_{n+1}$ of $B$ over $CB_0 \ldots B_n$ such that $CB_0 \ldots B_n BX, CB_0 \ldots B_n B_{n+1} \leq CB_0 \ldots B_n B_{n+1} BX \in K$. Hence (i) and (ii) hold. On the other hand, $B_{n+1} \cap B_i = A$ for each $i < n$, since $B_{n+1} \cong_{CB_0 \ldots B_n} B$. So, to see that (iii) holds, it is enough to show that $B_{n+1} \cap XB = A$. Let $B' = B_{n+1} \cap XB$. First, suppose that $B' = B_{n+1}$. Then we have $B_{n+1} \leq BX$, and so $CB_{n+1} \not\leq CBX$, since $\delta(XB/CB_{n+1}) = \delta(XB/C) - \delta(B_{n+1}/C) = \delta(XB/C) - \delta(B/C) = \delta(X/BC) < 0$. This contradicts our choice of $B_{n+1}$. Hence we have $B' \neq B_{n+1}$. We have to see that $B' = A$. This can be shown as follows: By our choice of $B_{n+1}$, we have $CB_0 \ldots B_n BX \leq CB_0 \ldots B_n B_{n+1} BX$, and so $B' \leq B_{n+1}$. Since $B$ is minimal and $B' \neq B_{n+1}$, we have $B' = A$. (End of Proof of Claim 1)

Claim 2: $B, B_j \leq B_0 \ldots B_i B (\in K)$ for $j < i < \omega$

Proof: We prove by induction on $i$. By (ii) of claim 1, $B_0 \ldots B_i B \leq B_0 \ldots B_{i+1} B$. By induction hypothesis, we have $B, B_j \leq B_0 \ldots B_i B$ for $j \leq i$. Hence $B, B_j \leq B_0 \ldots B_i B_{i+1} B$ for $j < i$. So, it is enough to show that $B_{i+1} \leq B_0 \ldots B_{i+1} B$. By induction hypothesis again, we have $B \leq B_0 \ldots B_i B$. From (i) of claim 1, it follows that $B_{i+1} \leq B_0 \ldots B_{i+1} B$. By (ii) of claim 1, $B_0 \ldots B_{i+1} \leq B_0 \ldots B_{i+1} B$. Hence we have $B_{i+1} \leq B_0 \ldots B_{i+1} B$. (End of Proof of Claim 2)

We show that $tp(B/A)$ is non-algebraic. By claim 2, we can assume that $B, B_j \leq BB_0 \ldots B_i \leq M$ for each $i, j$ with $j < i < \omega$. So we have $tp(B_j/A) = tp(B/A)$ for each $j < \omega$. By (iii) of claim 1, $B_j$'s are pairwise distinct. Hence $tp(B/A)$ is not algebraic.

**Proposition 9** Let $A \leq B \leq M$ and $A = acl(A) \cap B$. Then $acl(A) \perp_A B$ and $acl(A) \cup B \leq M$.

**Proof** We can assume that $A, B$ are finite. We will show that $A^* \perp_A B$ and $A^* B \leq M$ for any finite $A^* \leq acl(A)$ with $A \subset A^*$. Take $A = A_0 \leq A_1 \leq \ldots \leq A_n = A^*$ with $A_{i+1}$ minimal over $A_i$ for each $i < n$. Then it is enough to show
that $A_i \perp_{A_0} B$ and $A_i B \leq \mathcal{M}$ for each $i \leq n$. (Proof: We prove by induction on $i$. Clearly $A_i \leq A_{i+1}, A_{i+1} \cap A_i B = A_i$ and $\text{tp}(A_{i+1}/A_i)$ is algebraic. By induction hypothesis, $A_i B \leq \mathcal{M}$. So we have $A_{i+1} \perp A_i B$ and $A_{i+1} B \leq \mathcal{M}$ by lemma. By induction hypothesis, $A_i \perp_{A_0} B$, and hence $A_{i+1} \perp_{A_0} B$.)

**Theorem 10**  Let $B, C \leq \mathcal{M}$ and $A = B \cap C$. Then the following are equivalent:

(i) $\text{tp}(B/C)$ does not fork over $A$;

(ii) $B \perp A C$ and $BC \cup \text{acl}(A) \leq \mathcal{M}$.

**Proof**  By proposition 9, $B \cup \text{acl}(A), C \cup \text{acl}(A) \leq \mathcal{M}$. So, by fact 7, (i) is equivalent to $B \perp_{\text{acl}(A)} C$ and $BC \cup \text{acl}(A) \leq \mathcal{M}$. Therefore, proving that (i) and (ii) are equivalent, it is enough to show that $B \perp_{\text{acl}(A)} C$ if and only if $B \perp A C$. We can assume that $A, B, C$ is finite. Take any finite $A^* \leq \text{acl}(A)$ with $BC \cap \text{acl}(A) \subset A^*$. Then we will show that $B \perp A^* C$ if and only if $B \perp A C$. Let $B' = B \cap A^*, C' = C \cap A^*$.

$(\Rightarrow)$ Since $\text{tp}(A^*/B'C')$ is algebraic, we have $A^* \perp_{B'C'} BC$. So, from $B \perp A^* C$ it follows that $B \perp_{B'C'} C$. On the other hand, since $\text{tp}(B'/C')$ and $\text{tp}(C'/A)$ are algebraic, we have $B' \perp_{C'} C$ and $B \perp A C'$. Hence we have $B \perp A C$.

$(\Leftarrow)$ By $B \perp A C$, we have $B \perp_{B'C'} C$. On the other hand, since $\text{tp}(A^*/B'C')$ is algebraic, we have $A^* \perp_{B'C'} BC$. Hence $B \perp A^* C$.

**Corollary 11**  Let $L$ be a countable relational language and $K$ a class of finite $L$-structures that is derived from a predimension $\delta$. Then there is no $K$-generic structure that is superstable but not $\omega$-stable.

**Proof**  Suppose that a theory $T$ of a $K$-generic structure is superstable. Take any countable model $N$ of $T$.

Claim: For any $p \in S(N)$ there is finite $A \subset N$ such that $p$ does not fork over $A$ and $p|A$ is stationary.

Proof: Take a realization $\bar{b}$ of $p$. By superstability, there is finite $X \subset N$ such that $p$ does not fork over $X$. Let $B = \text{cl}(X\bar{b})$ and $A = B \cap N$. Clearly $p$ does not fork over $A$. We show that $\text{tp}(\bar{b}/A)$ is stationary. Take any $\bar{b}'$ such that $\text{tp}(\bar{b}'/A) = \text{tp}(\bar{b}/A)$ and $\text{tp}(\bar{b}'/N)$ does not fork over $A$. Let $B' = \text{cl}(\bar{b}'A)$. Then $\text{tp}(B/N)$ and $\text{tp}(B'/N)$ do not fork over $A$. Since $\text{tp}(\bar{b}/A) = \text{tp}(\bar{b}'/A)$, we have $B \cong_A B'$. Note that $B \cap N = B' \cap N = A$. By theorem, $B \perp_A N, B' \perp_A N$ and $BN, B'N \leq \mathcal{M}$. In particular, $BN \cong B'N$. It follows that $\text{tp}(BN) = \text{tp}(B'N)$ and hence $\text{tp}(b/N) = \text{tp}(\bar{b}'/N)$. (End of Proof of Claim)

By claim, we have $|S(N)| \leq \aleph_0 \cdot |S(T)| = \aleph_0$. Hence $T$ is $\omega$-stable.

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