A Note on the Stability Spectrum of Generic Structures

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Abstract


Let $L$ be a countable relational language and $K$ a class of finite $L$-structures closed under subgraphs. Let $\overline{K}$ be a class of $L$-structures such that any finite substructure belongs to $K$.

Definition 1
Let $ABC \in \overline{K}$. Then $B$ and $C$ are said to be free over $A$ (in symbol, $B \perp_A C$), if it satisfies the following:
(i) $B \cap C \subseteq A$;
(ii) $R^{ABC} = R^{AB} \cup R^{AC}$ for any $R \in L$.

Remark 2
Let $ABCD \in \overline{K}$. Then
(i) If $A \perp BC$ and $A \perp_B CD$, then $A \perp BCD$.
(ii) If $BC \perp_AD$, then $B \perp_C AD$.
(iii) If $BC \perp_AD$, then $B \perp_A C$ if and only if $B \perp_D C$.

Definition 3
$\delta: K \rightarrow \mathbb{R}^{\geq 0}$ is said to be a predimension, if
(i) if $A \cong B \in K$, then $\delta(A) = \delta(B)$;
(ii) $\delta(\emptyset) = 0$;
(iii) for all $AB \in K$, $\delta(A/B) \leq \delta(A/A \cap B)$;
(iv) there is no infinite chain $A_1 \subset A_2 \subset \ldots$ of $A_i \in K$ with $\delta(A_i) > \delta(A_{i+1})$ for

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\[ i \in \omega; \]

(v) for any \( AB \in K \), \( A \perp_{A \cap B} B \) if and only if \( \delta(A/B) = \delta(A/A \cap B) \);

(vi) for any \( ABCD \in K \) with \( B \cap ACD = \emptyset \), \( \delta(B/AC) - \delta(B/A) \leq \delta(B/DA) - \delta(B/DA) \),

where \( \delta(X/Y) \) means \( \delta(XY) - \delta(Y) \).

**Definition 4**

(i) For \( A \subset B \in K \), we define \( A \leq B \), if \( \delta(X/A') \geq 0 \) for any finite \( X \subset B - A \) and \( A' \subset A \). For \( A \subset B \in K \), define \( cl_B(A) = \bigcap \{ A' : A \subset A' \leq B \} \). By the definition of a predimension, there exists such a \( cl_B(A) \), and moreover if \( A \) is finite, then so is \( cl_B(A) \).

(ii) Fix \( M \in K \). For finite \( A \subset M \), define \( d_M(A) = \delta(cl_M(A)) \). For finite \( B \subset M \), \( d_M(A/B) = d_M(AB) - d_M(B) \). For infinite \( B \subset M \), \( d_M(A/B) = \inf \{ d_M(A/B') : B' \subset B \text{ finite} \} \). For (possibly) infinite \( A, B, C \subset M \), \( d_M(B/C) = d_M(B/A) \) means \( d_M(B'/C) = d_M(B'/A) \) for any finite \( B' \subset B \).

(iii) A countable \( L \)-structure \( M \) is said to be \((K, \leq)-\text{generic} \), if \( A \in K \) for any finite \( A \subset M \); If \( A \leq B \in K \), then there is \( B' \cong_{A} B \) with \( B' \leq M \).

Let \( M \) be a big model. The following facts can be found in [2], [5] and [6].

**Fact 5**

Let \( B, C \leq M \) and \( A = B \cap C \). Then the following are equivalent:

(i) \( d(B/C) = d(B/A) \);

(ii) \( B \perp_{A} C \) and \( BC \leq M \).

**Proof**

(i) \( \Rightarrow \) (ii). First we show that \( BC \leq M \). If not, then there are \( \overline{b} \in B, \overline{c} \in C, \overline{e} \in cl(\overline{b}c) - BC \) with \( \delta(\overline{e}/\overline{bc}) = -\gamma < 0 \). Take \( \overline{a} \leq A \) with \( \delta(\overline{b}/\overline{a}) - \delta(\overline{c}/\overline{a}) < \gamma/2 \) and \( \delta(\overline{c}/\overline{a}) - \delta(\overline{e}/\overline{a}) < \gamma/2 \). Let \( \overline{b}' = cl(\overline{b}\overline{a}) \) and \( \overline{c}' = cl(\overline{c}\overline{a}) \). Then \( \delta(\overline{b}'\overline{c}'/\overline{a}) = \delta(\overline{b}\overline{c}/\overline{a}) \geq d(\overline{b}'\overline{a}/\overline{a}) = d(\overline{b}'\overline{a}) + d(\overline{e}'/\overline{a}) \geq d(\overline{b}/\overline{a}) - d(\overline{e}/\overline{a}) + \delta(\overline{e}'/\overline{a}) \geq d(\overline{b}/\overline{a}) - \gamma \). On the other hand, we have \( d(\overline{b}'\overline{c}'/\overline{a}) \leq \delta(\overline{b}'\overline{c}'/\overline{a}) \leq \delta(\overline{b}'\overline{c}'/\overline{a}) + \delta(\overline{e}'\overline{c}'/\overline{a}) = \delta(\overline{b}'\overline{c}'/\overline{a}) - \gamma \). A contradiction.

(ii) \( \Rightarrow \) (i). If not, then there are \( \overline{b}, \overline{c} \in C \) with \( \delta(\overline{b}/\overline{a}) < \delta(\overline{b}/\overline{a}) \) where \( \overline{a} = \overline{b} \cap \overline{c} \). Let \( \gamma = \delta(\overline{b}/\overline{a}) - \delta(\overline{b}/\overline{c}) \). Take \( \overline{a}' \leq A \) with \( \overline{a} \subset \overline{a}' \) and \( d(\overline{b}/\overline{a}') - d(\overline{b}/\overline{a}) < \gamma \). Let \( \overline{b}' = cl(\overline{a}'\overline{b}) \) and \( \overline{c}' = cl(\overline{a}'\overline{c}) \). By remark, we have \( \delta(\overline{b}'/\overline{a}') - \delta(\overline{b}'/\overline{a}) \geq \delta(\overline{b}/\overline{a}) - \delta(\overline{b}/\overline{a}) = \gamma \). Then \( \delta(\overline{b}'/\overline{a}') \geq d(\overline{b}/\overline{a}') - \gamma \geq \delta(\overline{b}/\overline{a}') - \gamma \geq \delta(\overline{b}/\overline{a}') - \gamma \). A contradiction.

**Fact 6**

Let \( B, C \leq M \) and \( A = B \cap C \) be algebraically closed. Then the following are equivalent:

(i) \( tp(B/C) \) does not fork over \( A \);

(ii) \( B \perp_{A} C \) and \( BC \leq M \).
Proof (i) $\Rightarrow$ (ii). Suppose that $B \downarrow_A C$. Take a sufficiently saturated model $N \supset A$ with $BC \downarrow_A N$. Then we have $B \downarrow_N C$ and $B \downarrow_A N$.

Claim 1: $d(B/N) = d(B/NC)$.

Proof: If $d(B/N) > d(B/NC)$, then there are $\overline{b} \in B, \overline{c} \in NC$ with $d(\overline{b}/N) > d(\overline{b}/\overline{c})$. Take countable $A_0 \subset N$ with $\overline{b}c \downarrow_{A_0} N$. By the saturation of $N$, we can pick $\overline{c}' \in N$ with $\text{stp}(\overline{c}/A_0) = \text{stp}(\overline{c}'/A_0)$. Since $\overline{b}c \downarrow_{A_0} N$ and $\overline{b} \downarrow_N \overline{c}$, we have $\overline{b} \downarrow_{A_0} \overline{c}$ and $\overline{b} \downarrow_{A_0} \overline{c}'$. Hence $\text{tp}(\overline{b}c/A_0) = \text{tp}(\overline{b}c'/A_0)$. Then $d(\overline{b}/\overline{c}) = d(\overline{b}/\overline{c}') \geq d(\overline{b}/N)$. A contradiction.

Claim 2: $d(B/A) = d(B/N)$.

Proof: Let $B^* = \text{acl}(B)$. We can take $A_1$ with $d(B^*/N) = d(B^*/A_1)$ where $A \subset A_1 \subset N$ and $|A_1| = |B| + \aleph_0$. $A_1$ acl??? By the saturation of $N$ there is $A_2 \subset N$ with $\text{tp}(A_2/A) = \text{tp}(A_1/A)$ and $A_1 \downarrow_A A_2$. Note that $A_1 \downarrow_{B^*} A_2$ by $B \downarrow_A N$. Let $B^*_1 = \text{cl}(A_1B^*)$ and $B^*_2 = \text{cl}(A_2B^*)$. Then $B^*_1 \cap B^*_2 = B^*$. By fact 6, we have $B^*_1N, B^*_2N \leq \mathcal{M}$ since $d(B^*/N) = d(B^*/A_1) = d(B^*/A_2)$. Hence $B^*N = B^*_1N \cap B^*_2N \leq \mathcal{M}$. On the other hand, we have $B^* \perp_{A} N$. (Proof: Suppose that $B^* \perp_{A} N$. Note that $B^* \perp_{A_1} N$ and $B^* \perp_{A_2} N$ since $d(B^*/N) = d(B^*/A_1) = d(B^*/A_2)$. So we have $B^* \perp_{A_1} A_1$ and $B^* \perp_{A_2} A_2$. Since $A_1 \downarrow_A A_2$, we have $A_1 \cap A_2 = A$. A contradiction.) Hence $d(B/N) = d(B/A)$.

By claim 1,2, we have $d(B/A) = d(B/NC)$, and hence $d(B/A) = d(B/C)$.

(ii) $\Rightarrow$ (i). Take $B'$ such that $\text{tp}(B'/C)$ does not fork over $A$ and $\text{tp}(B'/A) = \text{tp}(B'/A)$. By (i) $\Rightarrow$ (ii), we have $B' \perp_{A} C$ and $B'C \leq \mathcal{M}$. So we have $\text{tp}(BC/A) = \text{tp}(BC/A)$, and hence $\text{tp}(B/C)$ does not fork over $A$.

For each $A \leq B \in \mathcal{K}$, $B$ is said to be minimal, if $C = A$ or $B$ for any $C$ with $A \leq C \leq B$.

Lemma 7 Let $A \leq B \in \mathcal{K}$ with $B \leq \mathcal{M}$. Let $B$ be minimal over $A$. If $\text{tp}(B/A)$ is algebraic, then $B \perp_{A} C$ for any $C \leq \mathcal{M}$ with $B \cap C = A$.

Proof Suppose that $\delta(B/C) < \delta(B/A)$ for some $C \leq BC \in \mathcal{K}$ with $B\cap C = A$.

Claim: There is a set $\{B_i\}_{i<\omega}$ of copies of $B$ over $A$ with the following conditions:

(i) $C \leq CB_j \leq CB_0B_1 \cdots B_i \in \mathcal{K}$ for each $j \leq i < \omega$;

(ii) $B_i \cap B_j = A$ for each $j < i < \omega$;

(iii) $B_i, C$ are free over $A$ for each $i < \omega$.

Proof: Suppose that $\{B_i\}_{i \leq n}$ has been defined. By our assumption, we have $C \leq CB \in \mathcal{K}$, and by (i) we have $C \leq CB_0B_1 \cdots B_n \in \mathcal{K}$. By amalgamation, we can take a copy $B^*$ of $B$ over $C$ such that $CB_0 \cdots B_n, CB^* \leq CB_0 \cdots B_n B^* \in \mathcal{K}$. By (iii) and $\delta(B^*/C) < \delta(B^*/A)$, we have $B_i \neq B^*$ for all $i \leq n$. Since $B$ is minimal over $A$, we have $B^* \cap B_i = A$. Since $\mathcal{K}$ is closed under $L$-subgraphs, there is $B_{n+1} \cong_{AB_n} B_1 \cdots B_n B^*$ such that $CB_0B_1 \cdots B_n B_{n+1} \in \mathcal{K}$ and $B_{n+1}, C$ are free over $A$. So (ii) and (iii) hold. It is not difficult to check that $CB_j \leq CB_0B_1 \cdots B_{n+1} \in \mathcal{K}$ for each $j \leq n + 1$. So (i) holds. (End of Proof of Claim)
By claim, we have $AB_j \leq AB_0...B_i \in K$ for each $j \leq i < \omega$. We can assume that $AB_0...B_i \leq M$. Thus we have $\text{tp}(B_j/A) = \text{tp}(B/A)$ for each $j \leq i$. By (ii) of claim, $B_j$'s are pairwise distinct. Hence $\text{tp}(B/A)$ is not algebraic.

**Lemma 8** Let $A \leq B \in K$ with $B \leq M$. Let $B$ be minimal over $A$. If $\text{tp}(B/A)$ is algebraic, then $BC \leq M$ for any $C \leq M$ with $B \cap C = A$.

**Proof** Suppose by way of contradiction that $BC \not\leq M$ for some $C \leq M$ with $B \cap C = A$. Then there is finite $X \subset M - BC$ such that $\delta(X/BC) < 0$.

Claim 1: There is a set $\{B_i\}_{i<\omega}$ of copies of $B$ with the following conditions:
(i) $B_i \cong_{CB_0...B_{i-1}} B$ for each $i < \omega$;
(ii) $CB_0...B_i, CB_0...B_{i-1}BX \leq CB_0...B_iBX \in K$ for each $i < \omega$;
(iii) $XB \cap B = B_j \cap B_i = A$ for each $j < i < \omega$.

Proof: Suppose that $\{B_i\}_{i<\omega}$ has been defined. By (ii), $CB_0...B_n \leq CB_0...B_nBX \in K$, and so we have $CB_0...B_n \leq CB_0...B_nB \in K$. By amalgamation, we can take a copy $B_{n+1}$ of $B$ over $CB_0...B_n$ such that $CB_0...B_nBX, CB_0...B_nB_{n+1} \leq CB_0...B_nB_{n+1}BX \in K$. Hence (i) and (ii) hold. On the other hand, $B_{n+1} \cap B_i = A$ for each $i \leq n$, since $B_{n+1} \cong_{CB_0...B_n} B$. So, to see that (iii) holds, it is enough to show that $B_{n+1} \cap XB = A$. Let $B' = B_{n+1} \cap XB$. First, suppose that $B' \neq B_{n+1}$. Then we have $B_{n+1} \subset BX$, and so $CB_{n+1} \not\leq CBX$, since $\delta(XB/BC_{n+1}) = \delta(XB/C) - \delta(B_{n+1}/C) = \delta(XB/C) - \delta(B/C) = \delta(X/BC) < 0$. This contradicts our choice of $B_{n+1}$. Hence we have $B' = B_{n+1}$. We have to see that $B' = A$. This can be shown as follows: By our choice of $B_{n+1}$, we have $CB_0...B_nBX \leq CB_0...B_nB_{n+1}BX$, and so $B' \leq B_{n+1}$. Since $B$ is minimal and $B' \neq B_{n+1}$, we have $B' = A$. (End of Proof of Claim 1)

Claim 2: $B, B_j \leq B_0...B_iB(\in K)$ for $j \leq i < \omega$

Proof: We prove by induction on $i$. By (ii) of claim 1, $B_0...B_iB \leq B_0...B_{i+1}B$. By induction hypothesis, we have $B, B_j \leq B_0...B_iB$ for $j \leq i$. Hence $B, B_j \leq B_0...B_{i+1}B$ for $j \leq i$. So, it is enough to show that $B_{i+1} \leq B_0...B_{i+1}B$. By induction hypothesis again, we have $B \leq B_0...B_iB$. From (i) of claim 1, it follows that $B_{i+1} \leq B_0...B_{i+1}B$. By (ii) of claim 1, $B_0...B_{i+1} \leq B_0...B_{i+1}B$. Hence we have $B_{i+1} \leq B_0...B_{i+1}B$. (End of Proof of Claim 2)

We show that $\text{tp}(B/A)$ is non-algebraic. By claim 2, we can assume that $B, B_j \leq BB_0...B_i \leq M$ for each $i, j$ with $j \leq i < \omega$. So we have $\text{tp}(B_j/A) = \text{tp}(B/A)$ for each $j < \omega$. By (iii) of claim 1, $B_j$'s are pairwise distinct. Hence $\text{tp}(B/A)$ is not algebraic.

**Proposition 9** Let $A \leq B \leq M$ and $A = \text{acl}(A) \cap B$. Then $\text{acl}(A) \perp A B$ and $\text{acl}(A) \cup B \leq M$.

**Proof** We can assume that $A, B$ are finite. We will show that $A^* \perp A B$ and $A^* B \leq M$ for any finite $A^* \leq \text{acl}(A)$ with $A \subset A^*$. Take $A = A_0 \leq A_1 \leq \ldots \leq A_n = A^*$ with $A_{i+1}$ minimal over $A_i$ for each $i < n$. Then it is enough to show
that $A_i \perp_{A_0} B$ and $A_i B \leq \mathcal{M}$ for each $i \leq n$. (Proof: We prove by induction on $i$. Clearly $A_i \leq A_{i+1}, A_{i+1} \cap A_i B = A_i$ and $\text{tp}(A_{i+1}/A_i)$ is algebraic. By induction hypothesis, $A_i B \leq \mathcal{M}$. So we have $A_{i+1} \perp_{A_i} B$ and $A_{i+1} B \leq \mathcal{M}$ by lemma. By induction hypothesis, $A_i \perp_{A_0} B$, and hence $A_{i+1} \perp_{A_0} B$.)

**Theorem 10** Let $B, C \leq \mathcal{M}$ and $A = B \cap C$. Then the following are equivalent:

(i) $\text{tp}(B/C)$ does not fork over $A$;

(ii) $B \perp_{A} C$, and $BC \cup \text{acl}(A) \leq \mathcal{M}$.

**Proof** By proposition 9, $B \cup \text{cl}(A), C \cup \text{acl}(A) \leq \mathcal{M}$. So, by fact 7, (i) is equivalent to $B \perp_{\text{acl}(A)} C$ and $BC \cup \text{acl}(A) \leq \mathcal{M}$. Therefore, proving that (i) and (ii) are equivalent, it is enough to show that $B \perp_{\text{acl}(A)} C$ if and only if $B \perp_{A} C$. We can assume that $A, B, C$ is finite. Take any finite $A^* \leq \text{acl}(A)$ with $BC \cap \text{acl}(A) \subset A^*$. Then we will show that $B \perp_{A} C$ if and only if $B \perp_{A} C^*$. Let $B' = B \cap A^*, C' = C \cap A^*$.

$(\Rightarrow)$ Since $\text{tp}(A^*/B'C')$ is algebraic, we have $A^* \perp_{B'C'} BC$. So, from $B \perp_{A} C^*$ it follows that $B \perp_{B'C'} C$. On the other hand, since $\text{tp}(B'/C')$ and $\text{tp}(C'/A)$ are algebraic, we have $B' \perp_{C'} C$ and $B \perp_{A} C'$. Hence we have $B \perp_{A} C$.

$(\Leftarrow)$ By $B \perp_{A} C$, we have $B \perp_{B'C'} C$. On the other hand, since $\text{tp}(A^*/B'C')$ is algebraic, we have $A^* \perp_{B'C'} BC$. Hence $B \perp_{A} C$.

**Corollary 11** Let $L$ be a countable relational language and $K$ a class of finite $L$-structures that is derived from a predimension $\delta$. Then there is no $K$-generic structure that is superstable but not $\omega$-stable.

**Proof** Suppose that a theory $T$ of a $K$-generic structure is superstable. Take any countable model $N$ of $T$.

Claim: For any $p \in S(N)$ there is finite $A \subset N$ such that $p$ does not fork over $A$ and $p|A$ is stationary.

Proof: Take a realization $\bar{b}$ of $p$. By superstability, there is finite $X \subset N$ such that $p$ does not fork over $X$. Let $B = \text{cl}(X\bar{b})$ and $A = B \cap N$. Clearly $p$ does not fork over $A$. We show that $\text{tp}(\bar{b}/A)$ is stationary. Take any $\bar{b}'$ such that $\text{tp}(\bar{b}'/A) = \text{tp}(\bar{b}/A)$ and $\text{tp}(\bar{b}'/N)$ does not fork over $A$. Let $B' = \text{cl}(\bar{b}'A)$. Then $\text{tp}(B/N)$ and $\text{tp}(B'/N)$ do not fork over $A$. Since $\text{tp}(\bar{b}/A) = \text{tp}(\bar{b}'/A)$, we have $B \equiv_A B'$. Note that $B \cap N = B' \cap N = A$. By theorem, $B \perp_{A} N, B' \perp_{A} N$ and $BN, B'N \leq \mathcal{M}$. In particular, $BN \equiv B'N$. It follows that $\text{tp}(BN) = \text{tp}(B'N)$ and hence $\text{tp}(b/N) = \text{tp}(b'/N)$. (End of Proof of Claim)

By claim, we have $|S(N)| \leq \aleph_0 \cdot |S(T)| = \aleph_0$. Hence $T$ is $\omega$-stable.

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