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A Generalization of Characterizations for the Hybrid Definability

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1 Introduction

Invariance under (semantical) operations tells us the expressive power of the logic under consideration from the semantical viewpoint, e.g., in Birkhoff Variety Theorem for universal algebra and in the series of works for capturing the expressivity of first-order logic, due to Kochen [9], Keisler [8] and Shelah [14] (or, [4, Corollary 6.1.16]), in first-order model theory. This invariance approach have been applied also to the modal (propositional) languages [7]. Goldblatt-Thomason theorem [3, Theorem 3.19] states that: For any elementary class F of frames, F is modally definable in the unimodal propositional language (roughly, expressible by modal formulas) iff F is closed under disjoint unions, generated subframes and bounded morphic images, and $\overline{\mathsf{F}}$, the complement of F, is closed under ultrafilter extensions.

Goldblatt-Thomason theorem teaches us the limitations of modal expressivity of first-order properties. For example, we cannot express irreflexivity of the accessibility relation by any modal formulas. This is because irreflexivity is not preserved under bounded morphic images [3, Example 3.15]. In order to overcome such a lack of expressivity, various extensions with additional modal operators have been proposed, e.g., the difference operator D (e.g., [5]) and the global modality E (e.g., [6]), etc.. The author and SATO Kentaro [12] have adopted the modal-model-theoretic approach taken in [2] and proved the general version of Goldblatt-Thomason theorem for almost all of the extended modal languages with *modal operators* (see Table 1).

There are, however, other extended modal languages with a new kind of propositional variables, called *nominals*. Such extensions are called *hybrid logics* [1]. In his PhD thesis [15], ten Cate introduced the two notion of definability: ***hybrid definability*** (roughly, expressivity by arbitrary formulas of a hybrid language) and

	Necessity	Possibility	Who studied?	FO-formula
R	$\Box p$	$\Diamond p$	Kripke	$x\mathbf{R}y$
$\neq$	$\bar{\mathrm{D}}p$	$\mathrm{D}p$	de Rijke [5]	$\neg x \approx y$
$(R\cap\neq)$	$\blacksquare p$	$\blacklozenge p$	Sano [11]	$x\mathbf{R}y \wedge \neg x \approx y$
R^{-1}	$\mathrm{H}p$	$\mathrm{P}p$	Prior [10]	$y\mathbf{R}x$
W^2	$\mathrm{A}p$	$\mathrm{E}p$	Goranko	$x\mathbf{R}y \vee \neg x\mathbf{R}y$
$W \setminus R$	$\mathrm{O}\neg p$	$\neg\mathrm{O}p$	Humberstone	$\neg x\mathbf{R}y$

Table 1: Additional Modalities

pure definability (roughly, expressivity by pure formulas, i.e., formulas that do not contain the ordinary propositional variables but may contain nominals). He gave Goldblatt-Thomason-style characterizations for these two definability of three hybrid languages: $\mathcal{H}$ (modal logic extended with the nominals alone), $\mathcal{H}(@)$ ($\mathcal{H}$ extended with the satisfaction operators $@_i$), $\mathcal{H}(\mathrm{E})$ ($\mathcal{H}$ extended with the global modality E). However, he did not consider the general extended languages as we did for modal languages [12]. Thus, we adopt the approach taken in [12] and try to generalize ten Cate's characterization to the general extended hybrid language with any additional operators. In this paper, we will report current progress of this project (see Table 2 in the final section).

Let us explain the contents briefly. Section 2 defines the basic notions of hybrid logics including the notion of *frames*, *models*, *modal satisfaction relation*, and *validity*, and then introduces the relation between models called *bisimulations*. In Section 3, we briefly mention basic frame constructions preserving the validity on frames and introduce some properties (the notion of *absolute*, *trivialize*) for them. In Section 4, we define another frame construction called *ultrafilter morphic images*, which is a typical frame construction of hybrid logics, and prove the Goldblatt-Thomason-style Characterizations for the hybrid definability of the general extended languages with and without the satisfaction operators $@_i$. Finally, Section 5 introduces a new frame construction called *images of bisimulation system*, which preserves the validity of pure formulas, and then gives the characterizations for pure definability similarly to Section 4.

2 Preliminaries

2.1 Syntax and Semantical Notions

The hybrid language $\mathcal{H}(\mathsf{Mod})$ (or, $\mathcal{H}$ simply) consists of (i) *Boolean connectives*: $\wedge$, $\neg$, (ii) an arbitrary set Mod of *modal operators*: $\Box \in \mathsf{Mod}$, (iii) *proposition* *letters*: Prop $= \{p, q, r, \dots\}$, (iv) *nominal variables*: Nom $= \{i, j, k, \dots\}$. *The hybrid language* $\mathcal{H}$(Mod, @) (or $\mathcal{H}(@)$ simply) consists of the vocabulary of $\mathcal{H}$ and the *satisfaction operators* $@_i$ ($i \in$ Nom). The *formulas* of, e.g., $\mathcal{H}(@)$, are defined as:

$$\varphi ::= p \mid i \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid @_i\varphi.$$

We denote formulas by φ, ψ, θ, etc. and sets of formulas by Γ, Δ, etc.. We define the Boolean connectives $\rightarrow$, $\vee$, etc. and the modal connective $\Diamond$ as usual abbreviations (e.g., $\Diamond\varphi := \neg\Box\neg\varphi$). φ is called *pure* if it contains no proposition letters.

A (multimodal) frame $\mathfrak{F} = \langle W, \{R_\Box\}_{\Box\in \mathsf{Mod}} \rangle$ is a pair consisting of a nonempty set W and a family of binary relation $R_\Box$ on W ($\Box \in$ Mod). *A (multimodal) model* $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ is a pair consisting of a frame $\mathfrak{F} = \langle W, \{R_\Box\}_{\Box\in \mathsf{Mod}} \rangle$ and a valuation V : Prop $\cup$ Nom $\rightarrow \mathcal{P}(W)$ satisfying $|V(i)| = 1$ for any $i \in$ Nom. We denote the unique element of $V(i)$ by i^V. $|\mathfrak{M}|$ (or $|\mathfrak{F}|$) means the domain of a model $\mathfrak{M}$ (or, a frame $\mathfrak{F}$, respectively). For any binary relation R on W, $R[w]$ denotes $\{x \in W \mid wRx\}$. Then, the satisfaction relation $\Vdash$, e.g., for $\mathcal{H}(@)$, is defined inductively as:

$$\begin{aligned}
\mathfrak{M}, x \Vdash p &\iff x \in V(p).\\
\mathfrak{M}, x \Vdash i &\iff x = i^V.\\
\mathfrak{M}, x \Vdash \neg\varphi &\iff \mathfrak{M}, x \nVdash \varphi.\\
\mathfrak{M}, x \Vdash \varphi \wedge \psi &\iff \mathfrak{M}, x \Vdash \varphi \text{ and } \mathfrak{M}, x \Vdash \psi.\\
\mathfrak{M}, x \Vdash \Box\varphi &\iff (\forall y \in W)\,[xR_\Box y \Longrightarrow \mathfrak{M}, y \Vdash \varphi]\\
&\quad (\iff R_\Box[x] \subset \{y \in W \mid \mathfrak{M}, y \Vdash \varphi\})\\
\mathfrak{M}, x \Vdash @_i\varphi &\iff \mathfrak{M}, i^V \Vdash \varphi.
\end{aligned}$$

$\mathfrak{M}, w$ and $\mathfrak{N}, v$ are *modally equivalent* (written $\mathfrak{M}, w \leftrightsquigarrow \mathfrak{N}, v$) if $[\mathfrak{M}, w \Vdash \varphi \iff \mathfrak{N}, v \Vdash \varphi]$ for any formula φ. $\mathfrak{M}, w$ and $\mathfrak{N}, v$ are *purely equivalent* (written $\mathfrak{M}, w \leftrightsquigarrow_p \mathfrak{N}, v$) if $\mathfrak{M}, w$ and $\mathfrak{N}, v$ are modally equivalent with respect to the pure formulas.

Our main interest is in *unimodal* frames and models. We can deal with them in the framework of first-order languages. The *first-order (unimodal) frame language* $\mathcal{L}^f$ is the first-order language that has the identity symbol $\approx$ together with the binary predicate symbol $\mathbf{R}$. We denote $\mathcal{L}^m$ as the *first-order (unimodal) model language* which is the expanded language of $\mathcal{L}^f$ with the unary predicates $\mathbf{P}_p$ ($p \in$ Prop) and the constant symbol $\mathbf{c}_i$ ($i \in$ Nom). We write $\alpha(x)$ or $\beta(v_1, v_2)$ to denote a formula α with at most one free variable x or two distinct free variables v_1, v_2, respectively.

Note that a *unimodal* model $\mathfrak{M} = \langle W, R, V \rangle$ can be seen as the $\mathcal{L}^m$-structure defined as follows: $|\mathfrak{M}| = W$, $\mathbf{R}^{\mathfrak{M}} = R$, $\mathbf{P}_p^{\mathfrak{M}} = V(p)$ ($p \in$ Prop) and $\mathbf{c}_i^{\mathfrak{M}} = i^V$ ($i \in$ Nom). A *unimodal* frame $\mathfrak{F}$ can be seen as the $\mathcal{L}^f$-structure defined similarly. $\mathfrak{M} \models \alpha[\vec{a}]$ where $\vec{a} = \langle a_1, \dots, a_n \rangle$ is a n-tuple from $|\mathfrak{M}|$, means the usual satisfaction relation (for details, see, e.g., [4, Ch.1]). Notice that $\models$ is different from the *modal* satisfaction relation symbol $\Vdash$. In this paper, we use some notions from first-order model theory, e.g., submodel, elementary embedding, ω-saturatedness. The reader unfamiliar with them can refer to, e.g., [4].

For any family $\vec{\beta} = \{\beta_\Box \mid \Box \in \mathsf{Mod}\}$ of formulas of $\mathcal{L}^f$, $\mathcal{H}([\vec{\beta}])$ (or $\mathcal{H}([\vec{\beta}], @)$) is the hybrid language $\mathcal{H}$ (or $\mathcal{H}(@)$, respectively) where the accessibility relation $R_\Box$ for $\Box \in \mathsf{Mod}$ is *defined* by the formula $\beta_\Box$. Thus, in $\mathcal{H}([\vec{\beta}])$, we usually denote $\Box(\in \mathsf{Mod})$ by $[\beta_\Box]$. From now on, we will use the following notational convention: E.g., we denote $\overline{\mathrm{D}}$ by $[\neg x \approx y]$ (or $[\neq]$ simply) and $\blacksquare$ by $[x\mathbf{R}y \wedge \neg x \approx y]$ (or $[R \cap \neq]$ simply) (see Table 1), etc.. A frame $\mathfrak{F}$ is called an $\mathcal{H}([\vec{\beta}])$-*frame* if, for any $\Box \in \mathsf{Mod}$, $\beta_\Box$ defines $R_\Box$. A model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ is called an $\mathcal{H}([\vec{\beta}])$-*model* if $\mathfrak{F}$ is an $\mathcal{H}([\vec{\beta}])$-frame. Observe that an $\mathcal{H}([\vec{\beta}])$-frame $\langle W, \{R_\Box\}_{\Box \in \mathsf{Mod}} \rangle$ (or -model) is determined by the unimodal frame $\langle W, R \rangle$ (or model, respectively). Therefore, we often regard $\langle W, R \rangle$ as $\mathcal{H}([\vec{\beta}])$-frame $\langle W, \{R_\Box\}_{\Box \in \mathsf{Mod}} \rangle$. Multimodal frames and models are only for the technical purposes, not of our original interest.

A formula φ is *valid in a model* $\mathfrak{M}$ (written $\mathfrak{M} \Vdash \varphi$) if $\mathfrak{M}, w \Vdash \varphi$ for any w in $\mathfrak{M}$. φ is *valid in a frame* $\mathfrak{F}$ (written $\mathfrak{F} \Vdash \varphi$) if $\langle \mathfrak{F}, V \rangle \Vdash \varphi$ for any valuation V : $\mathsf{Prop} \cup \mathsf{Nom} \to \mathcal{P}(|\mathfrak{F}|)$. φ is *satisfiable in a model* $\mathfrak{M}$ *(or a frame* $\mathfrak{F}$*)* if $\mathfrak{M} \nVdash \neg\varphi$ (or $\mathfrak{F} \nVdash \neg\varphi$, respectively). φ is *valid in a class* F *of frames* (written $\mathsf{F} \Vdash \varphi$) if it is valid in every $\mathfrak{F} \in \mathsf{F}$. For a set of formulas, these notions are defined similarly. Γ is *satisfiable* in F if $\langle \mathfrak{F}, V \rangle, w \Vdash \Gamma$ for some $\mathfrak{F} \in \mathsf{F}$, some V and some $w \in |\mathfrak{F}|$. Γ is *finitely satisfiable* in F if, for any $\Gamma' \subset_{\mathrm{fin}} \Gamma$ Γ' (or $\bigwedge \Gamma'$, the conjunction of all elements of Γ') is satisfiable in F. A set Γ of formulas *defines a class* F *of frames* if, for all frames $\mathfrak{F}$, $\mathfrak{F} \Vdash \Gamma \iff \mathfrak{F} \in \mathsf{F}$. A class F of frames is $\mathcal{L}$-*definable* (or, *purely* $\mathcal{L}$-*definable*) if there is some set of formulas (or, *pure* formulas, respectively) of $\mathcal{L}$ that defines F, where $\mathcal{L}$ is a hybrid language. For any class F of frames, we define $\mathrm{Th}(\mathsf{F}) = \{\varphi \mid \mathsf{F} \Vdash \varphi\}$ and $\mathrm{PTh}(\mathsf{F}) = \{\varphi\text{: pure} \mid \mathsf{F} \Vdash \varphi\}$.

Standard Translation ST_x: $\mathcal{H}(@) \to \mathcal{L}^m$ is defined as:

$$\begin{aligned}
ST_x(p) &:= \mathbf{P}_p x \\
ST_x(i) &:= x \approx \mathbf{c}_i \\
ST_x(\neg\varphi) &:= \neg ST_x(\varphi) \\
ST_x(\varphi \wedge \psi) &:= ST_x(\varphi) \wedge ST_x(\psi) \\
ST_x(\Box\varphi) &:= \forall y(x\mathbf{R}y \to ST_y(\varphi)) \quad (y\text{: a fresh variable}) \\
ST_x(@_i\varphi) &:= \exists y(y \approx \mathbf{c}_i \& ST_y(\varphi) \quad (y\text{: a fresh variable})
\end{aligned}$$

For $\mathcal{H}([\vec{\beta}])$ (or $\mathcal{H}([\vec{\beta}], @)$), we define $ST_x(\Box\varphi) := \forall y\,(\beta_\Box(x,y) \to ST_y(\varphi))$. We can easily prove that $\mathfrak{M}, w \Vdash \varphi \iff \mathfrak{M} \models ST_x(\varphi)[w]$. If we extend $\mathcal{L}^m$ to the *second-order* language, then, we have: $\mathfrak{F} \Vdash \varphi \iff \mathfrak{F} \models (\forall \vec{\mathbf{P}})(\forall \vec{\mathbf{c}})(\forall x) ST_x(\varphi)$. Thus, φ is a pure formula, then φ defines the elementary property of frames, i.e., $(\forall \vec{\mathbf{c}})(\forall x) ST_x(\varphi)$. For example, $i \to \neg\Diamond i$ defines the irreflexivity of $R_\Box$.

2.2 Bisimulations

Definition 1. A *bisimulation* between frames $\mathfrak{F} = \langle W, \{R_\Box\}_{\Box\in\mathsf{Mod}}\rangle$ and $\mathfrak{G} = \langle W', \{S_\Box\}_{\Box\in\mathsf{Mod}}\rangle$ is a binary relation $Z \subset W \times W'$ satisfying the following conditions (written $Z : \mathfrak{F} \underline{\leftrightarrow} \mathfrak{G}$): For any $\Box \in \mathsf{Mod}$,

(Zig) wZv and $wR_\Box w' \Longrightarrow$ for some $v' \in W'$ [$w'Zv'$ and $vS_\Box v'$].

(Zag) wZv and $vS_\Box v' \Longrightarrow$ for some $w' \in W$ [$w'Zv'$ and $wR_\Box w'$].

An $\mathcal{H}$*-bisimulation* between models $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ and $\mathfrak{N} = \langle \mathfrak{G}, V'\rangle$ is a bisimulation Z between $\mathfrak{F}$ and $\mathfrak{G}$ satisfying the following conditions:

(Atom) If wZv, then [$w \in V(a) \Longleftrightarrow v \in V'(a)$] for all $a \in \mathsf{Prop} \cup \mathsf{Nom}$.

An $\mathcal{H}(@)$*-bisimulation* is an $\mathcal{H}$-bisimulation Z satisfying in addition:

(Nom) for any $i \in \mathsf{Nom}$, $i^V Z i^{V'}$.

Let $\mathcal{L}$ be either $\mathcal{H}$ or $\mathcal{H}(@)$. $\mathfrak{M}, w$ and $\mathfrak{N}, v$ are $\mathcal{L}$*-bisimilar* (written $\mathfrak{M}, w \underline{\leftrightarrow}_{\mathcal{L}} \mathfrak{N}, v$) if there is an $\mathcal{L}$-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ such that wZv.

Proposition 2. Let $\mathfrak{M}, \mathfrak{N}$ be models and $w \in |\mathfrak{M}|$, $v \in |\mathfrak{N}|$. Let $\mathcal{L}$ be either $\mathcal{H}([\vec{\beta}])$ or $\mathcal{H}([\vec{\beta}], @)$. $\mathfrak{M}, w \underline{\leftrightarrow}_{\mathcal{L}} \mathfrak{N}, v$ implies $\mathfrak{M}, w \leftrightsquigarrow \mathfrak{N}, v$.

Proof. By Induction on φ [15, Theorem 4.1.2]. For the modal connectives, see, e.g., [3, Theorem 2.20]. QED

The converse of Proposition 2 does not hold in general. The following fact [15, Theorem 4.1.2], however, holds.

Fact 3. Let $\mathfrak{M}, \mathfrak{N}$ be models and $w \in |\mathfrak{M}|$, $v \in |\mathfrak{N}|$. Let $\mathcal{L}$ be either $\mathcal{H}([\vec{\beta}])$ or $\mathcal{H}([\vec{\beta}], @)$. If $\mathfrak{M}$ and $\mathfrak{N}$ are ω-saturated as $\mathcal{L}^m$-models (-structures) and $\mathfrak{M}, w \leftrightsquigarrow \mathfrak{N}, v$, then $\mathfrak{M}, w \underline{\leftrightarrow}_{\mathcal{L}} \mathfrak{N}, v$.

3 Basic Frame Constructions

Definition 4. A frame $\mathfrak{G} = \langle W', \{R'_\Box\}_{\Box\in\mathsf{Mod}}\rangle$ is a *generated subframe* of $\mathfrak{F} = \langle W, \{R_\Box\}_{\Box\in\mathsf{Mod}}\rangle$ (written $\mathfrak{G} \rightarrowtail \mathfrak{F}$) if $W' \subset W$, $R'_\Box = R_\Box \cap (W')^2$, and $R_\Box[w] \subset W'$ ($w \in W'$) for any $\Box \in \mathsf{Mod}$. Let $\mathfrak{F}$ be a frame and $X \subset |\mathfrak{F}|$. The *subframe generated by* X (written $\mathfrak{F}_X$) is the smallest generated subframe of $\mathfrak{F}$ whose domain contains X. A *point generated* subframe of $\mathfrak{F}$ by $x \in |\mathfrak{F}|$ (written: $\mathfrak{F}_x$) is $\mathfrak{F}_{\{x\}}$, where x is called the *root* of the frame.

A model $\mathfrak{N} = \langle \mathfrak{G}, V'\rangle$ is a *generated model* of $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ if $\mathfrak{G}$ is a generated subframe of $\mathfrak{F}$, $i^V \in |\mathfrak{G}|$ for any $i \in \mathsf{Nom}$, and $V'(a) = V(a) \cap |\mathfrak{G}|$ for any $a \in \mathsf{Prop} \cup \mathsf{Nom}$. *Submodel generated by* X and *point generated submodel* are defined similarly to the case of frames.

Definition 5. $\vec{\beta} = \{\beta_\Box \mid \Box \in \mathsf{Mod}\}$ is *absolute for generated subframes* if, for any $\mathcal{H}([\vec{\beta}])$-frame $\mathfrak{F} = \langle W, R \rangle$, any generated subframe $\mathfrak{F}' = \langle W', \{R'_\Box\}_{\Box \in \mathsf{Mod}} \rangle$ of $\mathfrak{F}$, and any $\vec{w}$ from W', the following holds: $\langle W, R \rangle \models \beta_\Box[\vec{w}] \iff \langle W', R' \rangle \models \beta_\Box[\vec{w}]$ where $R' = R \cap (W')^2$.

For an $\mathcal{H}([\vec{\beta}])$-frame, multimodally generated subframes might differ from unimodally generated subframes since in the multimodal case the closure under all associated relations is required.

For example, any combination of the modal operators Table 1 is absolute for generated subframes since they are all quantifier free. (As for the example that $\vec{\beta}$ is not absolute for generated subframes, see [12]).

Proposition 6. Suppose that $\vec{\beta}$ is absolute for generated subframes and $\mathfrak{F} \rightarrowtail \mathfrak{G}$ for $\mathcal{H}([\vec{\beta}])$-frames $\mathfrak{F}$ and $\mathfrak{G}$. Then, $\mathfrak{G} \Vdash \varphi \Longrightarrow \mathfrak{F} \Vdash \varphi$ for any φ of $\mathcal{H}([\vec{\beta}])$ (or, $\mathcal{H}([\vec{\beta}], @)$).

Proof. By Proposition 2. For details, see [15, Theorem 4.2.1]. QED

For any binary relation R on W, R^* is the reflexive and transitive closure of R.

Definition 7. $\mathcal{H}$ *trivializes generated subframes* if $(\bigcup_{\Box \in \mathsf{Mod}} R_\Box)^* = W^2$ for every frame $\mathfrak{F} = \langle W, \{R_\Box\}_{\Box \in \mathsf{Mod}} \rangle$.

Proposition 8. $\mathcal{H}$ trivializes generated subframes $\iff$ $\mathfrak{F} \rightarrowtail \mathfrak{G}$ implies $\mathfrak{F} = \mathfrak{G}$ for every frame $\mathfrak{G}$ and $\mathfrak{F}$.

Definition 9. A frame $\mathfrak{G}$ is a *hybrid amalgamation* of a family $\{\mathfrak{F}_j \mid j \in J\}$ of frames if, for any $x \in |\mathfrak{G}|$, there exists $j \in J$ such that (up to isomorphism) $\mathfrak{G}_x \rightarrowtail \mathfrak{F}_j$ and $|\mathfrak{G}_x| \neq |\mathfrak{F}_j|$.

Definition 10. $\mathcal{H}$ *trivializes hybrid amalgamations* if, for any frame $\mathfrak{G}$ and any family $\{\mathfrak{F}_j \mid j \in J\}$ of frames, $\mathfrak{G}$ is *not* a hybrid amalgamation of $\{\mathfrak{F}_j \mid j \in J\}$.

Proposition 11. (1) $\mathcal{H}$ trivializes generated subframes $\iff$ (2) $\mathcal{H}$ trivializes hybrid amalgamations.

Proof. [(1) $\Longrightarrow$ (2)] We prove the contraposition. Assume that $\mathfrak{G}$ is a hybrid amalgamation of $\{\mathfrak{F}_j \mid j \in J\}$. Take $x \in |\mathfrak{G}|$. Then, there exists $j \in J$ such that $\mathfrak{G}_x \rightarrowtail \mathfrak{F}_j$ and $|\mathfrak{G}_x| \neq |\mathfrak{F}_j|$, which implies the negation of (1).
[(2) $\Longrightarrow$ (1)] For the contraposition, assume that $\mathcal{H}$ does not trivialize generated subframes. Then, $\mathfrak{F} \rightarrowtail \mathfrak{G}$ and $\mathfrak{F} \neq \mathfrak{G}$, for some $\mathfrak{F}$ and $\mathfrak{G}$. Suppose that $x \in |\mathfrak{F}|$. $\mathfrak{F}_x \rightarrowtail \mathfrak{F} \rightarrowtail \mathfrak{G}$ holds, which implies $|\mathfrak{F}_x| \neq |\mathfrak{G}|$. Thus, $\mathfrak{F}$ is a hybrid amalgamation of $\{\mathfrak{G}\}$. QED

Let $\mathsf{K}_{\mathsf{all}}$ be the class of all frames.

Proposition 12. (1) $\{\, \Box_1 \cdots \Box_m \neg i \mid m \in \omega \,\&\, \Box_1, \ldots, \Box_m \in \mathsf{Mod} \,\}$ is satisfiable in $\mathsf{K}_{\mathrm{all}}$ $\Longleftrightarrow$ (2) $\mathcal{H}$ does not trivialize hybrid amalgamations.

Proof. [(1) $\Longrightarrow$ (2)] Suppose that $\langle \mathfrak{G}, V \rangle, x \Vdash \Box_1 \cdots \Box_m \neg i$ for any $m \in \omega$ and any $\Box_k \in \mathsf{Mod}$ $(1 \le k \le m)$. Then, $i^V \notin |\mathfrak{G}_x|$ holds. Thus, $|\mathfrak{G}_x| \subsetneq |\mathfrak{G}|$. We conclude that $\mathfrak{G}_x$ is a hybrid amalgamation of $\{\, \mathfrak{G} \,\}$.
[(2) $\Longrightarrow$ (1)] Assume that $\mathfrak{G}$ is a hybrid amalgamation of $\{\, \mathfrak{F}_j \mid j \in J \,\} \subset \mathsf{K}_{\mathrm{all}}$. Choose $x \in |\mathfrak{G}| \neq \emptyset$. Take $\mathfrak{F}_j$ such that $\mathfrak{G}_x \rightarrowtail \mathfrak{F}_j$ and $|\mathfrak{G}_x| \neq |\mathfrak{F}_j|$. We prove that all $\Box_1 \cdots \Box_m \neg i$ $(m \in \omega,\ \Box_k \in \mathsf{Mod})$ are simultaneously satisfiable at $x \in |\mathfrak{F}_j|$. For some $* \in |\mathfrak{F}_j| \setminus |\mathfrak{G}_x|$, consider a valuation V such that $i^V = *$. Then, $\mathfrak{G}_x \rightarrowtail \mathfrak{F}_j$ and $|\mathfrak{G}_x| \neq |\mathfrak{F}_j|$. Thus, for any $m \in \omega$ and any $\Box_k \in \mathsf{Mod}$ $(1 \le k \le m)$, $\langle \mathfrak{F}_j, V \rangle, x \Vdash \Box_1 \cdots \Box_m \neg i$. QED

Corollary 13. The following are equivalent:
(1) $\mathcal{H}$ does not trivialize generated subframes,
(2) $\mathcal{H}$ does not trivialize hybrid amalgamations,
(3) $\{\, \Box_1 \cdots \Box_m \neg i \mid m \in \omega \,\&\, \Box_1, \ldots, \Box_m \in \mathsf{Mod} \,\}$ is satisfiable in $\mathsf{K}_{\mathrm{all}}$.

Proposition 14. Assume that $\vec{\beta}$ is absolute for generated subframes and $\mathcal{H}([\vec{\beta}])$ does not trivialize generated subframes. Suppose that $\mathcal{H}([\vec{\beta}])$-frame $\mathfrak{G}$ is a hybrid amalgamation of a family $\{\, \mathfrak{F}_j \mid j \in J \,\}$ of $\mathcal{H}([\vec{\beta}])$-frames. If $\mathfrak{F}_j \Vdash \varphi$ for any $j \in J$, then $\mathfrak{G} \Vdash \varphi$.

Proof. Suppose for contraposition that $\langle \mathfrak{G}, V \rangle, v \nVdash \varphi$. Take the point-generated subframe $\mathfrak{G}_v$ of $\mathfrak{G}$. By the assumption, there exists $j \in J$ such that $\mathfrak{G}_v \rightarrowtail \mathfrak{F}_j$ and $|\mathfrak{G}_v| \neq |\mathfrak{F}_j|$. Fix $x \in |\mathfrak{F}_j| \setminus |\mathfrak{G}_v|$. Define a valuation V' on $\mathfrak{F}_j$ as follows: $V'(p) = V(p) \cap |\mathfrak{G}_v|$ $(p \in \mathsf{Prop})$ and $V'(i) = V(i)$ (if $i^V \in |\mathfrak{G}_v| \neq \emptyset$); x (o.w.) $(i \in \mathsf{Nom})$. Consider the identity relation of $|\mathfrak{G}_v|$ as a bisimulation between $\mathfrak{G}$ and $\mathfrak{F}_j$, i.e., $Z = |\mathfrak{G}_v| \times |\mathfrak{G}_v| \subset |\mathfrak{G}| \times |\mathfrak{F}_j|$. Then, Z is a bisimulation between $\mathfrak{G}$ and $\mathfrak{F}_j$. Furthermore, Z is an $\mathcal{H}$-bisimulation (i.e., $\mathcal{H}([\vec{\beta}])$- bisimulation) between $\langle \mathfrak{G}, V \rangle$ and $\langle \mathfrak{F}_j, V' \rangle$ with vZv. Since $\langle \mathfrak{G}, V \rangle, v \nVdash \varphi$, we have $\mathfrak{F}_j \nVdash \varphi$ by Proposition 2. QED

Definition 15. The *disjoint union* $\biguplus_{j \in J} \mathfrak{F}_j$ of a family $\{\, \mathfrak{F}_j \mid j \in J \,\}$, where $\mathfrak{F}_j = \langle W_j, \{\, (R_j)_\Box \,\}_{\Box \in \mathsf{Mod}} \rangle$, of pairwise disjoint frames is the pair (consisting) of $\bigcup_{j \in J} W_j$ and $\{\, \bigcup_{j \in J} (R_j)_\Box \,\}_{\Box \in \mathsf{Mod}}$.

Definition 16. $\vec{\beta}$ is *absolute for disjoint unions* if, for any family $\{\, \mathfrak{F}_j \mid j \in J \,\}$ of $\mathcal{H}([\vec{\beta}])$-frames and for any $\Box \in \mathsf{Mod}$,

$$\{\, \langle a, b \rangle \mid \langle W, R \rangle \models \beta_\Box[a, b] \,\} = \bigcup_{j \in J} \{\, \langle a, b \rangle \mid \langle W_j, R_j \rangle \models \beta_\Box[a, b] \,\},$$

where $\mathfrak{F}_j = \langle W_j, R_j \rangle$, $W = \bigcup_{j \in J} W_j$ and $R = \bigcup_{j \in J} R_j$.

As a matter of convention, write $R_{\beta=} = ((\bigcup_{\Box\in\mathrm{Mod}} R_\Box) \cup =)$ for any $\mathcal{H}([\vec{\beta}])$-frame $\mathfrak{F} = \langle W, R \rangle$. If $(R_{\beta=})^* \subset R^*$ for any $\mathcal{H}([\vec{\beta}])$-frame $\langle W, R \rangle$, then $\vec{\beta}$ is absolute for disjoint unions. For example, $\{ x\mathbf{R}y, x\mathbf{R}y \wedge \neg x \approx y \}$ (corresponding to $\{ [R], [R\cap \neq] \}$) is absolute for disjoint unions but $\{ x\mathbf{R}y, \neg x \approx y \}$ (corresponding to $\{ [R], [\neq] \}$) is not.

Proposition 17. Suppose that $\vec{\beta}$ is absolute for generated subframes and disjoint unions, then $\mathcal{H}([\vec{\beta}])$ does not trivialize generated subframes.

Proof. Assume that $\vec{\beta}$ is absolute for disjoint unions. Take any frame $\mathfrak{F}$. Then, $\mathfrak{F}$ is a hybrid amalgamation of $\{ \mathfrak{F}_w \uplus \mathfrak{F}_w \mid w \in |\mathfrak{F}| \}$. Thus, $\mathcal{H}([\vec{\beta}])$ does not trivialize hybrid amalgamations. By Corollary 13, we get the conclusion. QED

By Propositions 14 and 17, we have the following:

Corollary 18. Assume that $\vec{\beta}$ is absolute for generated subframes and disjoint unions. Suppose that an $\mathcal{H}([\vec{\beta}])$-frame $\mathfrak{G}$ is a hybrid amalgamation of a family $\{ \mathfrak{F}_j \mid j \in J \}$ of $\mathcal{H}([\vec{\beta}])$-frames. If $\mathfrak{F}_j \Vdash \varphi$ for any $j \in J$, then $\mathfrak{G} \Vdash \varphi$.

4 Goldblatt-Thomason-style Characterizations for the Hybrid Definability

4.1 Ultrafilter Morphic Images

Definition 19. A mapping $f : |\mathfrak{F}| \to |\mathfrak{G}|$ is a *bounded morphism* from a frame $\mathfrak{F} = \langle W, \{ R_\Box \}_{\Box\in\mathrm{Mod}} \rangle$ to a frame $\mathfrak{G} = \langle W', \{ S_\Box \}_{\Box\in\mathrm{Mod}} \rangle$ if, for any $\Box \in \mathrm{Mod}$, f satisfies the following:

(Forth) $wR_\Box w' \Longrightarrow f(w)S_\Box f(w')$.

(Back) $f(w)S_\Box v \Longrightarrow$ for some $w' \in |\mathfrak{F}|$, $[wR_\Box w'$ and $f(w') = v]$.

Note that f is a bounded morphism from $\mathfrak{F}$ to $\mathfrak{G}$ iff $Z = \{ \langle x, f(x) \rangle \mid x \in |\mathfrak{F}| \}$ is a bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$.

Definition 20. Given a binary relation R on a set W, we define a unary operation l_R on $\mathcal{P}(W)$: $l_R(X) := \{ w \in W \mid R[w] \subset X \}$.

The *ultrafilter extension* $\mathrm{ue}\,\mathfrak{F}$ of $\mathfrak{F} = \langle W, \{ R_\Box \}_{\Box\in\mathrm{Mod}} \rangle$ is the frame $\langle W^{\mathrm{ue}}, \{ R_\Box^{\mathrm{ue}} \}_{\Box\in\mathrm{Mod}} \rangle$, where W^{ue} is the set of (principal and non-principal) ultrafilters over W, $uR_\Box^{\mathrm{ue}}u'$ if, for any $X \subset W$, $l_{R_\Box}(X) \in u$ implies $X \in u'$.

Note that the ultrafilter extension of $\mathcal{H}([\vec{\beta}])$-frame $\mathfrak{F}$ is not necessarily an $\mathcal{H}([\vec{\beta}])$-frame.

Definition 21. Let $\mathfrak{F}$ and $\mathfrak{G}$ be frames. $\mathfrak{G}$ is an *ultrafilter morphic image* of $\mathfrak{F}$ if there is a surjective bounded morphism $f : \mathfrak{F} \to \mathrm{ue}\,\mathfrak{G}$ such that $|f^{-1}[\{u\}]| = 1$ for all *principal* ultrafilters $u \in |\mathrm{ue}\,\mathfrak{G}|$.

We can apply this notion to the $\mathcal{H}([\vec{\beta}])$-frames. If $\mathfrak{F}$ and $\mathfrak{G}$ are $\mathcal{H}([\vec{\beta}])$-frames and $\mathfrak{G}$ is an ultrafilter morphic image of $\mathfrak{F}$, then, the notion of ultrafilter morphic images links two $\mathcal{H}([\vec{\beta}])$-frames $\mathfrak{F}$ and $\mathfrak{G}$ *via (multimodal) frame* $\mathrm{ue}\,\mathfrak{G}$.

Proposition 22. Let $\mathfrak{F}$ and $\mathfrak{G}$ be $\mathcal{H}([\vec{\beta}])$-frames. Suppose that $\mathfrak{G}$ is an ultrafilter morphic image of $\mathfrak{F}$. Then, for any φ of $\mathcal{H}([\vec{\beta}])$ (or, $\mathcal{H}([\vec{\beta}], @)$), $\mathfrak{F} \Vdash \varphi$ implies $\mathfrak{G} \Vdash \varphi$.

Proof. Similar to [15, Proposition 4.2.6]. QED

Proposition 23. Assume that $\vec{\beta}$ is absolute for disjoint unions. Let $\mathfrak{F}_i$ and $\mathfrak{G}_i$ ($i \in \{1, 2\}$) be $\mathcal{H}([\vec{\beta}])$-frames. If $\mathfrak{F}_i$ is an ultrafilter morphic image of $\mathfrak{G}_i$ for any $i \in \{1, 2\}$, then $\mathfrak{F}_1 \uplus \mathfrak{F}_2$ is an ultrafilter morphic image of $\mathfrak{G}_1 \uplus \mathfrak{G}_2$.

Proof. Similar to [15, Lemma 4.2.13]. QED

Proposition 24. Assume that $\vec{\beta}$ is absolute for disjoint unions. Let $\mathfrak{F}_i$ and $\mathfrak{G}_i$ ($i \in \{1, 2\}$) be $\mathcal{H}([\vec{\beta}])$-frames. If $\mathfrak{F}_i$ and $\mathfrak{G}_i$ are elementarily equivalent for any $i \in \{1, 2\}$, then $\mathfrak{F}_1 \uplus \mathfrak{F}_2$ and $\mathfrak{G}_1 \uplus \mathfrak{G}_2$ are elementarily equivalent.

Proof. Similar to [15, Lemma 4.2.14]. QED

4.2 Characterizations for the Hybrid Definability

Theorem 25. Suppose that $\vec{\beta}$ is absolute for generated subframes and disjoint unions. Then, for any elementary family F of $\mathcal{H}([\vec{\beta}])$-frames, F is $\mathcal{H}([\vec{\beta}])$-definable $\iff$ F is closed under (i) ultrafilter morphic images, (ii) generated subframes, and (iii) hybrid amalgamations.

Proof. Suppose that $\vec{\beta}$ is absolute for generated subframes and disjoint unions. The left-to-right-direction is clear from Propositions 6, 22 and Corollary 18. We will prove that the right-to-left-direction. It suffices to prove that, for any $\mathcal{H}([\vec{\beta}])$-frame $\mathfrak{F}$, $\mathfrak{F} \Vdash \mathrm{Th}(\mathsf{F}) \Longrightarrow \mathfrak{F} \in \mathsf{F}$. Suppose that $\mathfrak{F} \Vdash \mathrm{Th}(\mathsf{F})$.

We divide the proof into two cases: (Case 1) and (Case 2).

(Case 1) Let us assume that $\mathfrak{F}$ is point generated by w. We can suppose that $\mathsf{Prop} = \{p_X \mid X \subset |\mathfrak{F}|\}$ and $\mathsf{Nom} = \{i_x \mid x \in |\mathfrak{F}|\}$ [1]. Let $\mathfrak{M} = \langle \mathfrak{F}, V_0 \rangle$, where V_0 is

[1] $\mathfrak{F} \Vdash \mathrm{Th}(\mathsf{F}) \overset{\mathrm{def}}{\iff} \mathfrak{F} \Vdash \{\varphi \text{ of } \mathcal{H}([\vec{\beta}]) \mid \mathsf{F} \Vdash \varphi\} \iff \mathfrak{F} \Vdash \{\varphi \text{ of } \mathcal{H}'([\vec{\beta}]) \mid \mathsf{F} \Vdash \varphi\}$, where $\mathcal{H}([\vec{\beta}])$ and $\mathcal{H}'([\vec{\beta}])$ are alike except that the set of propositional and nominal variables of $\mathcal{H}'$ is $\{p_X \mid X \subset |\mathfrak{F}|\} \cup \{i_x \mid x \in |\mathfrak{F}|\}$.

the natural valuation with $V_0(p_X) = X$ and $V_0(i_x) = \{x\}$. Let Δ be the set consisting of the following, for all $X, Y \subset W$ and $x \in |\mathfrak{F}|$,

$$p_{W\setminus X} \leftrightarrow \neg p_X;\ p_{X\cap Y} \leftrightarrow p_X \wedge p_Y;\ i_x \leftrightarrow p_{\{x\}};\ p_{l_{R_\Box}(X)} \leftrightarrow \Box p_X \quad (\Box \in \mathsf{Mod}).$$

Let $\Delta_{\mathfrak{F}}$ be the following set:

$$\{i_w\} \cup \{\Box_1 \cdots \Box_m \varphi \mid \varphi \in \Delta \text{ and } m \in \omega \text{ and } \Box_1, \ldots, \Box_m \in \mathsf{Mod}\}.$$

It is easy to see that $\Delta_{\mathfrak{F}}$ is satisfiable on $\mathfrak{F}$ at w under the natural valuation V_0. Then, we can prove the following (for the proof, see, e.g., [15, p.59, Claim 1]): $\Delta_{\mathfrak{F}}$ is satisfiable in $\mathfrak{G}$ for some $\mathfrak{G} \in \mathsf{F}$. [2]

By this, we may infer that $\langle \mathfrak{G}, V\rangle, v \Vdash \Delta_{\mathfrak{F}}$ for some valuation V and some v in $\mathfrak{G}$ for some $\mathfrak{G} \in \mathsf{F}$. It follows that all nominals in the set $\{i_x \mid x \in |\mathfrak{F}|\}$ denote points in $\langle \mathfrak{G}, V\rangle$ that are reachable from v [3]. Thus, we can think of V as a valuation for the frame $\mathfrak{G}_v$. In this way, we can consider the point-generated submodel $\langle \mathfrak{G}_v, V\rangle$ of $\langle \mathfrak{G}, V\rangle$, which implies $\mathfrak{G}_v \in \mathsf{F}$ by (ii). Then, we can prove that $\langle \mathfrak{G}_v, V\rangle \Vdash \Delta$ and $\langle \mathfrak{G}_v, V\rangle, v \Vdash p_X$ for all $X \subset |\mathfrak{F}|$ with $w \in X$ ($\because$ $\langle \mathfrak{G}_v, V\rangle, v \Vdash i_w$ and $i_w \leftrightarrow p_{\{w\}}, p_{\{w\}} \leftrightarrow p_X \wedge p_{\{w\}} \in \Delta$).

Let $\langle \mathfrak{G}_v^*, V^*\rangle$ be an ω-saturated elementary extension of $\langle \mathfrak{G}_v, V\rangle$, which implies $\mathfrak{G}_v^* \in \mathsf{F}$. It follows that $\langle \mathfrak{G}_v^*, V^*\rangle \Vdash \Delta$ and $\langle \mathfrak{G}_v^*, V^*\rangle, v^* \Vdash p_X$ for all $X \subset |\mathfrak{F}_w|$ with $w \in X$ where v^* is the element corresponding to v, since the satisfaction relation is elementary.

Claim 1. $\mathfrak{F}$ is an ultrafilter morphic image of $\mathfrak{G}_v^*$.

(Proof of Claim) For any s in $\mathfrak{G}_v^*$, $\{X \subset |\mathfrak{F}| \mid \langle \mathfrak{G}_v^*, V^*\rangle, s \Vdash p_X\}$ is an ultrafilter. This defines the mapping f from $|\mathfrak{G}_v^*|$ to $|\mathrm{ue}\,\mathfrak{F}|$. We can prove that f is a surjective bounded morphism and satisfies the condition of ultrafilter morphic image (For the detailed proof of these, see [15, Claim 2 in the proof of Theorem 4.3.4]). ◄

Thus, we can conclude that $\mathfrak{F} \in \mathsf{F}$ from $\mathfrak{G}_v^* \in \mathsf{F}$ by (i).

(Case 2) Assume that $\mathfrak{F}$ is *not* point generated. Here, we need the assumption of absoluteness of disjoint unions and the closure condition (iii). Take any point-generated subframe $\mathfrak{F}_w$ of $\mathfrak{F}$. In what follows, we will show that $\mathfrak{F}_w \uplus \mathfrak{F}_w \in \mathsf{F}$. It then follows by (iii) that $\mathfrak{F} \in \mathsf{F}$.

As in (Case 1), we also suppose that $\mathsf{Prop} = \{p_X \mid X \subset |\mathfrak{F}|\}$ and $\mathsf{Nom} = \{i_x \mid x \in |\mathfrak{F}|\} \cup \{i_\emptyset\}$, where $i_\emptyset$ *is a distinct nominal from* $\{i_x \mid x \in |\mathfrak{F}|\}$.

[2] ($\because$) We prove that $\Delta_{\mathfrak{F}}$ is finitely satisfiable in F. Let $\Delta' \subset_{\text{fin}} \Delta_{\mathfrak{F}}$. Since $\mathfrak{F} \Vdash \mathrm{Th}(\mathsf{F})$ and $\langle \mathfrak{F}, V_0\rangle, w \Vdash \bigwedge \Delta'$, Δ' is satisfiable in F. By the elementariness of F, $\Delta_{\mathfrak{F}}$ is satisfiable in F. Note that F is closed under ultraproducts by elementariness.

[3] This is because $\{\Box_1 \cdots \Box_m(i_x \to \Diamond i_y) \mid$ for any $m \in \omega$ and $\Box_i \in \mathsf{Mod}\}$ expresses the information of $xR_\Box y$. Note that $i_x \to \Diamond i_y$ is equivalent to $i_x \vee \Diamond i_y \leftrightarrow \Diamond i_y$, which is equivalent to $p_{\{x\}\cup(W\setminus l_{R_\Box}(W\setminus\{y\}))} \leftrightarrow p_{W\setminus l_{R_\Box}(W\setminus\{y\})} \in \Delta$.

Let $\mathfrak{M} = \langle \mathfrak{F}_w, V_0 \rangle$, where V_0 is a natural valuation with $V_0(p_X) = X$, $i_x^{V_0} = x$ and $i_\emptyset^{V_0} = * \in |\mathfrak{F}| \setminus |\mathfrak{F}_w|$. We define Δ in the same way as in (Case 1). Let $\Delta_{\mathfrak{F}_w}$ be the following set:

$$\{ i_w \} \cup \{ \Box_1 \cdots \Box_m \varphi \mid \varphi \in \Delta \text{ and } m \in \omega \text{ and } \Box_1, \ldots, \Box_m \in \mathsf{Mod} \}$$
$$\cup \{ \Box_1 \cdots \Box_m \neg i_\emptyset \mid m \in \omega \text{ and } \Box_1, \ldots, \Box_m \in \mathsf{Mod} \}.$$

Easily, $\Delta_{\mathfrak{F}_w}$ is satisfiable on $\mathfrak{F}_w$ at w under a natural valuation. Then, in the same way as (Case 1), we can prove that $\langle \mathfrak{G}, V \rangle, v \Vdash \Delta_{\mathfrak{F}_w}$ for some valuation V and some v in $\mathfrak{G}$ for some $\mathfrak{G} \in \mathsf{F}$.

Let $\mathfrak{G}_v$ be the subframe of $\mathfrak{G}$ generated by v. By the construction of $\Delta_{\mathfrak{F}_w}$, $\mathfrak{G}_v$ is a proper generated subframe of $\mathfrak{G}$. Thus, by (ii) and (iii), $\mathfrak{G}_v \uplus \mathfrak{G}_v \in \mathsf{F}$. It follows from $\langle \mathfrak{G}, V \rangle, v \Vdash \Delta_{\mathfrak{F}_w}$ that all nominals in the set $\{ i_x \mid x \in |\mathfrak{F}| \}$ *(except $i_\emptyset$)* denote points in $\langle \mathfrak{G}, V \rangle$ that are reachable from v. Thus, we can think of V as a valuation for the frame $\mathfrak{G}_v$ by removing $i_\emptyset$ from our vocabulary. In this way, we obtain the point-generated submodel $\langle \mathfrak{G}_v, V \rangle$ of $\langle \mathfrak{G}, V \rangle$. Then we can prove that $\langle \mathfrak{G}_v, V \rangle \Vdash \Delta$ and $\langle \mathfrak{G}_v, V \rangle, v \Vdash p_X$ for all $X \subset |\mathfrak{F}|$ with $w \in X$.

In the same way as in (Case 1), we can take an ω-saturated elementary extension $\langle \mathfrak{G}_v^*, V^* \rangle$ of $\langle \mathfrak{G}_v, V \rangle$ and prove that $\mathfrak{F}_w$ is an ultrafilter morphic image of $\mathfrak{G}_v^*$.

Thus, we can conclude that $\mathfrak{F}_w \uplus \mathfrak{F}_w$ is an ultrafilter morphic image of $\mathfrak{G}_v^* \uplus \mathfrak{G}_v^*$ by Proposition 23. By Proposition 24, $\mathfrak{G}_v^* \uplus \mathfrak{G}_v^*$ is elementarily equivalent to $\mathfrak{G}_v \uplus \mathfrak{G}_v$, which implies $\mathfrak{G}_v^* \uplus \mathfrak{G}_v^* \in \mathsf{F}$. It follows that $\mathfrak{F}_w \uplus \mathfrak{F}_w \in \mathsf{F}$ by (i). QED

Theorem 26. Suppose that $\vec{\beta}$ is absolute for generated subframes. Then, for any elementary family F of $\mathcal{H}([\vec{\beta}])$-frames, F is $\mathcal{H}([\vec{\beta}])$-definable $\Longleftrightarrow$ F is closed under ultrafilter morphic images.

Proof. We can prove this theorem similarly to the proof of Theorem 25. It suffice to consider (Case 1), i.e., the case where $\mathfrak{F}$ is point generated, in the proof of the right-to-left-direction. QED

For $\mathcal{H}([\vec{\beta}], @)$, we can prove the following characterization.

Theorem 27. Suppose that $\vec{\beta}$ is absolute for generated subframe and $\mathcal{H}([\vec{\beta}])$ does not trivializes generated subframes. Then, for any elementary family F of $\mathcal{H}([\vec{\beta}])$-frames, F is $\mathcal{H}([\vec{\beta}], @)$-definable $\Longleftrightarrow$ F is closed under (i) ultrafilter morphic images and (ii) generated subframes.

Proof. Suppose that $\vec{\beta}$ is absolute for generated subframes.

We will prove that the right-to-left-direction. It suffices to prove that, for any $\mathcal{H}([\vec{\beta}])$-frame $\mathfrak{F}$, $\mathfrak{F} \Vdash \mathrm{Th}(\mathsf{F}) \Longrightarrow \mathfrak{F} \in \mathsf{F}$. Suppose that $\mathfrak{F} \Vdash \mathrm{Th}(\mathsf{F})$. We can suppose that $\mathsf{Prop} = \{ p_X \mid X \subset |\mathfrak{F}| \}$ and $\mathsf{Nom} = \{ i_x \mid x \in |\mathfrak{F}| \}$. Let $\mathfrak{M} = \langle \mathfrak{F}, V_0 \rangle$, where V_0 is the natural valuation with $V_0(p_X) = X$ and $i_x^{V_0} = x$. We define Δ in the same way as in (Case 1) at the proof of Theorem 25. Let $\Delta_{\mathfrak{F}}$ be the following set:

$$\{ @_{i_v}\Box_1 \cdots \Box_m\varphi \mid v \in |\mathfrak{F}| \text{ and } \varphi \in \Delta \text{ and } m \in \omega \text{ and } \Box_1, \ldots, \Box_m \in \mathsf{Mod} \}.$$

It is easy to see that $\Delta_{\mathfrak{F}}$ is satisfiable on $\mathfrak{F}$ at w under the natural valuation. Then we can prove that $\langle \mathfrak{G}, V \rangle \Vdash \Delta_{\mathfrak{F}}$ for some valuation V for some $\mathfrak{G} \in \mathsf{F}$. *Since* F *is closed under generated subframes, we may assume that* $\mathfrak{G}$ *is generated by the set of points that are named by nominals*, i.e., $\{ i_x^V \mid x \in |\mathfrak{F}| \}$. Then, we can prove that $\langle \mathfrak{G}, V \rangle \Vdash \Delta$.

Let $\langle \mathfrak{G}^*, V^* \rangle$ be an ω-saturated elementary extension of $\langle \mathfrak{G}, V \rangle$, which implies $\mathfrak{G}^* \in \mathsf{F}$. It follows that $\langle \mathfrak{G}^*, V^* \rangle \Vdash \Delta$. We can prove the following claim in the same way as in [15, Claim 2 in the proof of Theorem 4.3.4]: $\mathfrak{F}$ is an ultrafilter morphic image of $\mathfrak{G}^*$. Thus, we can conclude that $\mathfrak{F} \in \mathsf{F}$. QED

In the case where $\vec{\beta}$ is absolute for generated subframes and $\mathcal{H}([\vec{\beta}])$ trivializes generated subframes, we can prove, in the same way as Theorem 27, that for any elementary family F of $\mathcal{H}([\vec{\beta}])$-frames, F is $\mathcal{H}([\vec{\beta}], @)$-definable $\Longleftrightarrow$ F is closed under ultrafilter morphic images. Thus, in this case, $\mathcal{H}([\vec{\beta}], @)$ and $\mathcal{H}([\vec{\beta}])$ have the same expressive power with respect to any elementary family of frames.

As corollaries of theorems in this section, we can obtain several semantical characterizations of the extended hybrid languages whose characterizations were previously unknown. For example, by Theorem 25 (or 26, 27), we can get the characterization for the hybrid language whose operators are $\{ [R], [R^{-1} \cap \neq] \}$ (or, $\{ [R \cap \neq], [\neq] \}$, $\{ [R \cap \neq], [R^{-1} \cap \neq], [\neq] \}$, respectively).

Remark 28. In this remark, let us restrict our interest to the set $\vec{\beta}$ of quantifier free (QF-) formulas of $\mathcal{L}^f$. Then, $\mathcal{H}([\vec{\beta}])$ is absolute for generated subframes. We can give the general characterization for extended *modal* languages (the modal version of $\mathcal{H}([\vec{\beta}])$) without assuming the absoluteness for disjoint unions [12]. In our characterization (Theorems 25 and 26) of this paper, however, we need to class the hybrid languages $\mathcal{H}([\vec{\beta}])$ under two cases: the case where we assume the absoluteness for disjoint unions and the case where $\mathcal{H}([\vec{\beta}])$ trivializes generated subframes (see Table 2 in the final section). Thus, the range where we apply our characterizations (Theorems 25 and 26), is smaller than that of [12].

The satisfaction operators $@_i$ can change the situation. For $\mathcal{H}([\vec{\beta}])$, we can give the general characterization (Theorem 27) without assuming the absoluteness for disjoint unions. Therefore, the range of Theorem 27 is the same as that of [12]. It should be noted that SATO Kentaro [13] gives another Goldblatt-Thomason-style characterizations of hybrid (not pure) definability for (@-free) $\mathcal{H}([\vec{\beta}])$, without assuming the absoluteness for disjoint unions. In order to give characterizations, he introduces the notion of *Kripke frame with designation* and defines semantical operations for it. Thus, together with his results, we have the comparable characterizations for hybrid definability of $\mathcal{H}([\vec{\beta}])$ and $\mathcal{H}([\vec{\beta}], @)$ to [12].

5 Goldblatt-Thomason-style Characterizations for the Pure Definability

5.1 Images of Bisimulation System

Definition 29. Given a bisimulation Z between frames $\mathfrak{F}$ and $\mathfrak{G}$, and a subset X of $|\mathfrak{G}|$, Z *respects* X if the following two conditions hold for all $x \in X$:

1. $(\exists! w)$ such that wZx.
2. For all $w \in |\mathfrak{F}|$ and $v \in |\mathfrak{G}|$, if wZx and wZv, then $x = v$.

In other words, Z respects $X \subset |\mathfrak{G}|$ if $Z^{-1} \subset |\mathfrak{G}| \times |\mathfrak{F}|$ is a function on X and $Z \subset |\mathfrak{F}| \times |\mathfrak{G}|$ is a function on $Z^{-1}[X]$.

Definition 30. Given a bisimulation Z between $\mathfrak{F}$ and $\mathfrak{G}$, Z is *total* if $(\forall s \in |\mathfrak{F}|)$ $(\exists t \in |\mathfrak{G}|)\, sZt$ and $(\forall s \in |\mathfrak{G}|)$ $(\exists t \in |\mathfrak{F}|)\, tZs$. In other words, Z is total if $\mathrm{dom}\, Z = |\mathfrak{F}|$ and $\mathrm{ran}\, Z = |\mathfrak{G}|$.

Definition 31. A *bisimulation system* from a frame $\mathfrak{F}$ to a frame $\mathfrak{G}$ is a function

$$\mathcal{Z} : \{ X \mid X \subset_{\mathrm{fin}} |\mathfrak{G}| \} \to \{ Z : \mathfrak{F} \underline{\leftrightarrow} \mathfrak{G} \mid Z \text{ is total} \}$$

satisfying that $\mathcal{Z}(X)$ respects X. If there exists a bisimulation system $\mathcal{Z}$ from a frame $\mathfrak{F}$ to a frame $\mathfrak{G}$, then, $\mathfrak{G}$ is an *image of bisimulation system* from $\mathfrak{F}$.

We can also apply this notion to the $\mathcal{H}([\vec{\beta}])$-frames. Compared with the notion of ultrafilter morphic images, notice that there is no need to consider multimodal frames in order to link two $\mathcal{H}([\vec{\beta}])$-frames.

Recall that φ is pure if φ contains no proposition letters.

Proposition 32. Let $\mathfrak{F}$ and $\mathfrak{G}$ be $\mathcal{H}([\vec{\beta}])$-frames. Suppose that $\mathfrak{G}$ is an image of bisimulation system from $\mathfrak{F}$. Then, for any pure formula φ of $\mathcal{H}([\vec{\beta}])$ (or, $\mathcal{H}([\vec{\beta}], @)$), $\mathfrak{F} \Vdash \varphi$ implies $\mathfrak{G} \Vdash \varphi$.

Proof. See [15, Theorem 4.2.10]. QED

Proposition 33. Assume that $\vec{\beta}$ is absolute for disjoint unions. Let $\mathfrak{F}_i$ and $\mathfrak{G}_i$ $(i \in \{1, 2\})$ be $\mathcal{H}([\vec{\beta}])$-frames. If $\mathfrak{F}_i$ is an image of bisimulation system from $\mathfrak{G}_i$ for any $i \in \{1, 2\}$, then $\mathfrak{F}_1 \uplus \mathfrak{F}_2$ is an image of bisimulation system from $\mathfrak{G}_1 \uplus \mathfrak{G}_2$.

Proof. See [15, Lemma 4.2.15]. QED

5.2 Characterizations for the Pure Definability

Recall that pure formulas define elementary properties of frames by standard translation.

Theorem 34. Suppose that $\vec{\beta}$ is absolute for generated subframes and disjoint unions. Then, for any family F of $\mathcal{H}([\vec{\beta}])$-frames, F is purely $\mathcal{H}([\vec{\beta}])$-definable $\Longleftrightarrow$ F is elementary and F is closed under (i) images of bisimulation system, (ii) generated subframes, and (iii) hybrid amalgamations.

Proof. Suppose that $\vec{\beta}$ is absolute for generated subframes and disjoint unions. We prove the right-to-left-direction. It suffices to show that, for all frames $\mathfrak{F}$, $\mathfrak{F} \Vdash \mathrm{PTh}(\mathsf{F}) \Longrightarrow \mathfrak{F} \in \mathsf{F}$. Suppose that $\mathfrak{F} \Vdash \mathrm{PTh}(\mathsf{F})$.

We divide the proof into two cases: (Case 1) and (Case 2).

(Case 1) Assume that $\mathfrak{F}$ is point generated by w. We can suppose that $\mathsf{Nom} = \{ i_x \mid x \in |\mathfrak{F}| \}$. Let $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, where V is the natural valuation with $i_x^V = x$.

Let $\Delta_{\mathfrak{F}} = \{ \varphi : \text{pure} \mid \langle \mathfrak{F}, V \rangle, w \Vdash \varphi \}$. Clearly, $\Delta_{\mathfrak{F}}$ is satisfiable on $\mathfrak{F}$ under V. Then, we can prove that $\Delta_{\mathfrak{F}}$ is satisfiable in F, similarly to Theorem 25. Let $\langle \mathfrak{G}, U \rangle, v \Vdash \Delta_{\mathfrak{F}}$ for some $\mathfrak{G} \in \mathsf{F}$. Let $\mathfrak{G}_v$ be the subframe of $\mathfrak{G}$ generated by v, which implies $\mathfrak{G}_v \in \mathsf{F}$ by (ii). It follows from $\langle \mathfrak{G}, U \rangle, v \Vdash \Delta_{\mathfrak{F}}$ that all nominals in the set $\{ i_x \mid x \in |\mathfrak{F}| \}$ denote points in $\langle \mathfrak{G}, U \rangle$ that are reachable from v. Thus, we can think of U as a valuation for the frame $\mathfrak{G}_v$. In this way, we can consider the point-generated submodel $\langle \mathfrak{G}_v, U \rangle$ of $\langle \mathfrak{G}, U \rangle$.

We can prove the following [15, Claim 2 in the proof of Theorem 4.4.4].

Claim 2. For all pure $\mathcal{H}([\vec{\beta}])$-formulas φ, $\langle \mathfrak{F}, V \rangle \Vdash \varphi \Longleftrightarrow \langle \mathfrak{G}_v, U \rangle \Vdash \varphi$.

Let $\langle \mathfrak{F}^*, V^* \rangle$ and $\langle \mathfrak{G}_v^*, U^* \rangle$ be ω-saturated elementary extensions. By elementariness, $\mathfrak{G}_v^* \in \mathsf{F}$.

In what follows, we will construct a bisimulation system from $\mathfrak{G}_v^*$ to $\mathfrak{F}^*$. Fix $w_1, \ldots, w_n \in |\mathfrak{F}^*|$, and introduce new nominals $\vec{j} = (j_1, \ldots, j_n)$. We will write $\langle \mathfrak{F}^*, V^*, w_1, \ldots, w_n \rangle$ (or simply $\langle \mathfrak{F}^*, V^*, \vec{w} \rangle$) for the expansion of $\langle \mathfrak{F}^*, V^* \rangle$ in which $j_k^{V^*} = w_k$ $(1 \le k \le n)$. We can prove the following analogous to [15, Claim 3 in the proof of Theorem 4.4.1].

Claim 3. There exists $\vec{v} \in |\mathfrak{G}_v^*|$ such that, for any pure $\mathcal{H}([\vec{\beta}])[\vec{j}]$-formula φ, $\langle \mathfrak{F}^*, V^*, \vec{w} \rangle \Vdash \varphi \Longleftrightarrow \langle \mathfrak{G}_v^*, U^*, \vec{v} \rangle \Vdash \varphi$.

Define the binary relation Z between $|\mathfrak{G}_v^*|$ and $|\mathfrak{F}^*|$ such that $sZt \overset{\text{def}}{\Longleftrightarrow} \langle \mathfrak{F}^*, V^*, \vec{w} \rangle, s \leftrightsquigarrow_p \langle \mathfrak{G}_v^*, U^*, \vec{v} \rangle, t$ in $\mathcal{H}([\vec{\beta}])[\vec{j}]$. Then, we can prove that Z is a total bisimulation between $\mathfrak{G}_v^*$ and $\mathfrak{F}^*$ respecting $\{ w_1, \ldots, w_n \}$ (see, e.g., [15, Claim 4 in the proof of Theorem 4.4.1]). We have constructed a bisimulation system from $\mathfrak{G}_v^*$ to $\mathfrak{F}^*$. By (i), $\mathfrak{F}^* \in \mathsf{F}$ and then, by elementariness, $\mathfrak{F} \in \mathsf{F}$.

(Case 2) Assume that $\mathfrak{F}$ is *not* point-generated. Take any point-generated subframe $\mathfrak{F}_w$ of $\mathfrak{F}$. In what follows, we will show that $\mathfrak{F}_w \uplus \mathfrak{F}_w \in \mathsf{F}$. It then follows by (iii) that $\mathfrak{F} \in \mathsf{F}$.

We can suppose that $\mathsf{Nom} = \{ i_x \mid x \in |\mathfrak{F}| \} \cup \{ i_\emptyset \}$ where $i_\emptyset$ is a distinct nominal from $\{ i_x \mid x \in |\mathfrak{F}| \}$. Let $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, where V is a natural valuation with $i_x^V = x$ and $i_\emptyset^V = * \in |\mathfrak{F}| \setminus |\mathfrak{F}_w|$. Let $\Delta_{\mathfrak{F}_w} = \{ \varphi : \text{pure} \mid \langle \mathfrak{F}, V \rangle, w \Vdash \varphi \}$. Clearly, $\Delta_{\mathfrak{F}_w}$ is satisfiable on $\mathfrak{F}_w$ under V. Then, we can prove that $\Delta_{\mathfrak{F}_w}$ is satisfiable in F, similarly to Theorem 25.

Let $\langle \mathfrak{G}, U \rangle, v \Vdash \Delta_{\mathfrak{F}_w}$ for some $\mathfrak{G} \in \mathsf{F}$. Let $\mathfrak{G}_v$ be the subframe of $\mathfrak{G}$ generated by v. By the construction, $\mathfrak{G}_v$ is a proper generated subframe of $\mathfrak{G}$. Hence, by (ii) and (iii), $\mathfrak{G}_v \uplus \mathfrak{G}_v \in \mathsf{F}$.

It follows from $\langle \mathfrak{G}, U \rangle, v \Vdash \Delta_{\mathfrak{F}_w}$ that all nominals in the set $\{ i_x \mid x \in |\mathfrak{F}_w| \}$ (except $i_\emptyset$) denote points in $\langle \mathfrak{G}, U \rangle$ that are reachable from v. Hence we can think U as a valuation for the frame $\mathfrak{G}_v$ by removing $i_\emptyset$ from our vocabulary. In this way, we can consider the point-generated submodel $\langle \mathfrak{G}_v, U \rangle$ of $\langle \mathfrak{G}, U \rangle$. We can prove the following as in (Case 1): For all pure $\mathcal{H}([\vec{\beta}])$-formulas φ, $\langle \mathfrak{F}_w, V \rangle \Vdash \varphi \iff \langle \mathfrak{G}, U \rangle \Vdash \varphi$.

Let $\langle \mathfrak{F}_w^*, V^* \rangle$ and $\langle \mathfrak{G}_v^*, U^* \rangle$ be ω-saturated elementary extensions. By elementariness, $\mathfrak{G}_v^* \in \mathsf{F}$. In the same way as (Case 1), we can construct a bisimulation system from $\mathfrak{G}_v^*$ to $\mathfrak{F}_w^*$. Thus, we can conclude that $\mathfrak{F}_w \uplus \mathfrak{F}_w$ is a image of bisimulation system from $\mathfrak{G}_v^* \uplus \mathfrak{G}_v^*$ by Proposition 33. By Proposition 24, $\mathfrak{G}_v^* \uplus \mathfrak{G}_v^*$ is elementarily equivalent to $\mathfrak{G}_v \uplus \mathfrak{G}_v$, which implies $\mathfrak{G}_v^* \uplus \mathfrak{G}_v^* \in \mathsf{F}$. It follows that $\mathfrak{F}_w \uplus \mathfrak{F}_w \in \mathsf{F}$ by (i). QED

Theorem 35. Suppose that $\vec{\beta}$ is absolute for generated subframes and $\mathcal{H}([\vec{\beta}])$ trivializes generated subframes. Then, for any family F of $\mathcal{H}([\vec{\beta}])$-frames, F is purely $\mathcal{H}([\vec{\beta}])$-definable $\iff$ F is elementary and F is closed under images of bisimulation system.

Proof. We can prove this theorem similarly to the proof of Theorem 34. It suffice to consider (Case 1), i.e., the case where $\mathfrak{F}$ is point generated, in the proof of the right-to-left-direction. QED

Theorem 36. Suppose that $\vec{\beta}$ is absolute for generated subframes. Then, for any family F of $\mathcal{H}([\vec{\beta}])$-frames, F is purely $\mathcal{H}([\vec{\beta}], @)$-definable $\iff$ F is elementary and F is closed under (i) images of bisimulation system and (ii) generated subframes.

Proof. Suppose that $\vec{\beta}$ is absolute for generated subframes. It suffices to show that for all frames $\mathfrak{F}$, $\mathfrak{F} \Vdash \mathrm{PTh}(\mathsf{F}) \Longrightarrow \mathfrak{F} \in \mathsf{F}$. Suppose that $\mathfrak{F} \Vdash \mathrm{PTh}(\mathsf{F})$. We can suppose that $\mathsf{Nom} = \{ i_x \mid x \in |\mathfrak{F}| \}$. Let $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, where V is the natural valuation with $i_x^V = x$.

Let $\Delta_{\mathfrak{F}} = \{ @_{i_x}\varphi \mid \varphi$: pure, and, $\langle \mathfrak{F}, V \rangle, x \Vdash \varphi \}$. Clearly, $\Delta_{\mathfrak{F}}$ is satisfiable on $\mathfrak{F}$ under V. Then, we can prove that $\Delta_{\mathfrak{F}}$ is satisfiable in F, similarly to Theorem 25. Let $\langle \mathfrak{G}, U \rangle \Vdash \Delta_{\mathfrak{F}}$ for some $\mathfrak{G} \in$ F. Since F is closed under generated subframes, we may assume that $\mathfrak{G}$ is generated by $\{ i_x^U \mid x \in |\mathfrak{F}| \}$ (= Nom). We can prove the following [15, Claim 2 in the proof of Theorem 4.4.1]:

Claim 4. For all pure $\mathcal{H}([\vec{\beta}], @)$-formulas φ, $\langle \mathfrak{F}, V \rangle \Vdash \varphi \Longleftrightarrow \langle \mathfrak{G}, U \rangle \Vdash \varphi$.

Let $\langle \mathfrak{F}^*, V^* \rangle$ and $\langle \mathfrak{G}^*, U^* \rangle$ be ω-saturated elementary extensions. By elementariness, $\mathfrak{G}^* \in$ F. In the same way in (Case 1) of the proof of Theorem 34, we can construct a bisimulation system from $\mathfrak{G}^*$ to $\mathfrak{F}^*$. By (i), $\mathfrak{F}^* \in$ F and hence, by elementariness, $\mathfrak{F} \in$ F. QED

In the case where $\vec{\beta}$ is absolute for generated subframes and $\mathcal{H}([\vec{\beta}])$ trivializes generated subframes, we can prove, in the same way as Theorem 36, that for any family F of $\mathcal{H}([\vec{\beta}])$-frames, F is purely $\mathcal{H}([\vec{\beta}], @)$-definable $\Longleftrightarrow$ F is closed under images of bisimulation system. Thus, in this case, $\mathcal{H}([\vec{\beta}], @)$ and $\mathcal{H}([\vec{\beta}])$ have the same expressive power with respect to an elementary family of frames.

As in the same way in Section 4, we can obtain several semantical characterizations of the extended hybrid languages whose characterizations (for pure definability) were previously unknown.

6 Conclusion

We can summarize our results as in Table 2. In the table, [Ags] (or [Adu]) means

Languages	Assumptions	Hybrid Definability	Pure Definability
$\mathcal{H}([\vec{\beta}])$ E.g. $\mathcal{H}([R], [R\cap\neq])$	[Ags], [Adu]	Theorem 25 (um), (gs), (ha)	Theorem 34 (bs), (gs), (ha), (ele)
$\mathcal{H}([\vec{\beta}])$ E.g. $\mathcal{H}([R], [W \setminus R])$	[Ags], [Tgs]	Theorem 26 (um)	Theorem 35 (bs), (ele)
$\mathcal{H}([\vec{\beta}], @)$ E.g. $\mathcal{H}([R\cap\neq], @)$	[Ags]	Theorem 27 (um), (gs)	Theorem 36 (bs), (gs), (ele)

Table 2: Summary of this paper

that $\vec{\beta} = \{ \beta_\Box \mid \Box \in$ Mod $\}$ of the generalized hybrid language $\mathcal{H}([\vec{\beta}])$ is absolute for generated subframes (or disjoint unions, respectively). [Tgs] means that $\mathcal{H}([\vec{\beta}])$ trivializes generated subframes. (gs) (or, (ha), (um), (bs)) denotes the closure condition under generated subframes (or, hybrid amalgamations, ultrafilter morphic images, images of bisimulation systems, respectively). (ele) means that a given class of frames is elementary. Theorems 27 and 36 cover almost all extensions of $\mathcal{H}(@)$ with modal operators that have been already introduced, e.g., any non-empty subsets of

$$\{\, [R], [R^{-1}], [W \setminus R], [W \setminus R^{-1}], [(R\cap =)], [(R\cap \neq)], [W^2], [\neq] \,\}.$$

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