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<td>Author(s)</td>
<td>MAESONO, Hisatomo</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2007), 1555: 70-72</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2007-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/80985">http://hdl.handle.net/2433/80985</a></td>
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<td>Right</td>
<td>Type</td>
</tr>
<tr>
<td>Textversion</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>publisher</td>
<td>Kyoto University</td>
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</tbody>
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On quasi-minimal $\omega$-stable groups

前園久智 (Hisatomo MAESONO)
早稲田大学メディアネットワークセンター
(Media Network Center, Waseda University)

Abstract

Itai and Wakai investigated some group as an example of quasi-minimal structures [1]. We try to characterize such groups more.

1 Quasi-minimal structures and groups

We recall the definition of quasi-minimality. The notion of quasi-minimality is a generalization of that of strong minimality.

Definition 1 An uncountable structure $M$ is called quasi-minimal if every definable subset of $M$ with parameters is at most countable or co-countable.

Itai, Tsuboi and Wakai investigated quasi-minimal structures [2]. After that Itai and Wakai showed an example of such structures [1]. They characterized the group $(Q^\omega, +, \sigma, 0)$ where $Q$ is the set of rational numbers and $\sigma$ is the shift function.

Definition 2 A function $\sigma$ is a shift function if $\sigma : Q^\omega \to Q^\omega$ and for $\bar{x} = (x_0, x_1, x_2, \cdots) \in Q^\omega$, $\sigma(\bar{x}) = (x_1, x_2, x_3, \cdots) \in Q^\omega$.

They showed that the theory $\text{Th}(Q^\omega, +, \sigma, 0)$ is $\omega$-stable and has the elimination of quantifiers. Thus I tried to characterize structural properties of quasi-minimal $\omega$-stable groups.

2 Quasi-minimal $\omega$-stable groups

$(Q^\omega, +)$ is a divisible abelian group. And it is known that its theory is strongly minimal. So I wondered whether quasi-minimal groups are abelian. By using known Facts about stable groups, it is shown that quasi-minimal nonabelian groups have the strict order property substantially.
Definition 3 A formula $\varphi(x, y)$ has the strict order property if there are $a_i$ ($i < \omega$) such that for any $i, j < \omega$, $\models \exists x [\neg \varphi(x, a_i) \land \varphi(x, a_j)] \iff i < j$. A theory $T$ has the strict order property if some formula $\varphi(x, y)$ has the strict order property.

Proposition 4 Let $G$ be a quasi-minimal group. And let $Z$ be the center of $G$. If $G/Z$ is not abelian, then $\text{Th}(G)$ has the strict order property.

Proof. Suppose that $G/Z$ is nonabelian. As $Z$ is definable subgroup of $G$, $|Z|$ is countable. For $a \in G - Z$, let $C_a = \{g \in G | a^g = g^{-1}ag = a\}$. Since $C_a$ is definable subgroup of $G$, $|C_a|$ is countable. Thus the orbit of $a$, denoted by $O(a)$, is uncountable set. As orbits are definable equivalence classes, $G$ has only one infinite orbit. In the following, let $G$ be $G/Z$ for convenience of notation. Hence now $G$ has only one nontrivial orbit. So there is $a \in G$ with $a \neq a^{-1}$. As $a^{-1} \in O(a)$, there is $b \in G$ such that $a^b = a^{-1}$. Let $C_G(b) = \{g \in G | g^b = g\}$. Since $a^{b^2} = a$ and $a^b \neq a$, $C_G(b^2) \supsetneq C_G(b)$. As $b \in O(a)$, $b^2 \neq 1$ and there is $c \in G$ such that $b^c = b^2$. Then we get $C_G(b) < C_G(b^c) < C_G(b^{c^2}) < \cdots \cdots$.

Thus we can see that quasi-minimal simple (in stability theoretic meaning) groups are abelian essentially. However, strongly minimal groups and $\omega$-stable abelian groups were characterized completely.

Theorem 5 (Reineke [3]) Let $G$ be a group. Then the followings are equivalent;
1. $G$ is strongly minimal.
2. $G$ is minimal.
3. $G$ is abelian and has the form $G = \oplus_{\alpha} Q \oplus \oplus_p Z_p^{\beta_p}$ where $\alpha \geq 0$, $\beta_p$ is finite, or the form $G = \oplus_{\gamma} Z_p$ where $\gamma$ is infinite.

Theorem 6 (Macintyre [4]) Let $G$ be an abelian group. Then $\text{Th}(G)$ is totally transcendental if and only if $G$ is of the form $D \oplus H$ where $D$ is divisible and $H$ is of bounded order.

And by the following facts about infinite abelian groups, we can see that $\omega$-stable abelian groups are direct sums of strongly minimal groups. (These facts are well known, see e.g. [5]. In them, groups means abelian groups.)

Fact 7 Let $G$ be a group. Then $G$ has the maximal divisible direct summand.

Fact 8 Let $G$ be a divisible group. Then $G$ has the form $G = \oplus_{\alpha} Q \oplus \oplus_p Z_p^{\beta_p}$. 
**Fact 9** Let $G$ be a group of bounded order. Then $G$ is a direct sum of cyclic groups.

But we can easily check that $\omega$-stable abelian groups $G = D \oplus H$ in which $H$ has infinitely many summands are not quasi-minimal. Then

**Conclusion**

Quasi-minimal $\omega$-stable pure groups (i.e. groups reduced to the group language) are strongly minimal substantially.

Thus we should put the next problem last.

**Problem**

*Find quasi-minimal non-\(\omega\)-stable groups.*

**References**