Title: On quasi-minimal \( \omega \)-stable groups
(Model theoretic aspects of the notion of independence and dimension)

Author(s): MAESONO, Hisatomo

Citation: 数理解析研究所講究録 (2007), 1555: 70-72

Issue Date: 2007-05

URL: http://hdl.handle.net/2433/80985

Type: Departmental Bulletin Paper

Textversion: publisher

Kyoto University
On quasi-minimal $\omega$-stable groups

前園 久智 (Hisatomo MAESONO)
早稲田大学メディアネットワークセンター
(Media Network Center, Waseda University)

Abstract
Itai and Wakai investigated some group as an example of quasi-minimal structures [1]. We try to characterize such groups more.

1 Quasi-minimal structures and groups

We recall the definition of quasi-minimality. The notion of quasi-minimality is a generalization of that of strong minimality.

Definition 1 An uncountable structure $M$ is called quasi-minimal if every definable subset of $M$ with parameters is at most countable or co-countable.

Itai, Tsuboi and Wakai investigated quasi-minimal structures [2]. After that Itai and Wakai showed an example of such structures [1]. They characterized the group $(Q^\omega, +, \sigma, 0)$ where $Q$ is the set of rational numbers and $\sigma$ is the shift function.

Definition 2 A function $\sigma$ is a shift function if $\sigma : Q^\omega \rightarrow Q^\omega$ and for $\bar{x} = (x_0, x_1, x_2, \cdots) \in Q^\omega$, $\sigma(\bar{x}) = (x_1, x_2, x_3, \cdots) \in Q^\omega$.

They showed that the theory $\text{Th}(Q^\omega, +, \sigma, 0)$ is $\omega$-stable and has the elimination of quantifiers. Thus I tried to characterize structural properties of quasi-minimal $\omega$-stable groups.

2 Quasi-minimal $\omega$-stable groups

$(Q^\omega, +)$ is a divisible abelian group. And it is known that its theory is strongly minimal. So I wondered whether quasi-minimal groups are abelian. By using known Facts about stable groups, it is shown that quasi-minimal nonabelian groups have the strict order property substantially.
Definition 3 A formula \( \varphi(x, y) \) has the strict order property if there are \( a_i \) (\( i < \omega \)) such that for any \( i, j < \omega \), \( \models \exists x [\neg \varphi(x, a_i) \land \varphi(x, a_j)] \iff i < j \). A theory \( T \) has the strict order property if some formula \( \varphi(x, y) \) has the strict order property.

Proposition 4 Let \( G \) be a quasi-minimal group. And let \( Z \) be the center of \( G \). If \( G/Z \) is not abelian, then \( Th(G) \) has the strict order property.

Proof. Suppose that \( G/Z \) is nonabelian. As \( Z \) is definable subgroup of \( G \), \(|Z|\) is countable. For \( a \in G - Z \), let \( C_a = \{g \in G | a^g = g^{-1}ag = a\} \). Since \( C_a \) is definable subgroup of \( G \), \(|C_a|\) is countable. Thus the orbit of \( a \), denoted by \( O(a) \), is uncountable set. As orbits are definable equivalence classes, \( G \) has only one infinite orbit. In the following, let \( G \) be \( G/Z \) for convenience of notation. Hence now \( G \) has only one nontrivial orbit. So there is \( a \in G \) with \( a \neq a^{-1} \). As \( a^{-1} \in O(a) \), there is \( b \in G \) such that \( a^b = a^{-1} \). Let \( C_G(b) = \{g \in G | g^b = g\} \). Since \( a^{b^2} = a \) and \( a^b \neq a \), \( C_G(b^2) \supsetneq C_G(b) \). As \( b \in O(a) \), \( b^2 \neq 1 \) and there is \( c \in G \) such that \( b^c = b^2 \). Then we get \( C_G(b) < C_G(b^c) < C_G(b^{2c}) < \cdots \cdots \) .

Thus we can see that quasi-minimal simple (in stability theoretic meaning) groups are abelian essentially. However, strongly minimal groups and \( \omega \)-stable abelian groups were characterized completely.

Theorem 5 (Reineke [3]) Let \( G \) be a group. Then the followings are equivalent:
(1) \( G \) is strongly minimal.
(2) \( G \) is minimal.
(3) \( G \) is abelian and has the form \( G = \oplus_{\alpha} Q \oplus \oplus_{\gamma} Z_{p^\infty}^{\beta_{p}} \) where \( \alpha \geq 0 \), \( \beta_{p} \) is finite, or the form \( G = \oplus_{\gamma} Z_{p} \) where \( \gamma \) is infinite.

Theorem 6 (Macintyre [4]) Let \( G \) be an abelian group. Then \( Th(G) \) is totally transcendental if and only if \( G \) is of the form \( D \oplus H \) where \( D \) is divisible and \( H \) is of bounded order.

And by the following facts about infinite abelian groups, we can see that \( \omega \)-stable abelian groups are direct sums of strongly minimal groups. (These facts are well known, see e.g. [5]. In them, groups means abelian groups.)

Fact 7 Let \( G \) be a group. Then \( G \) has the maximal divisible direct summand.

Fact 8 Let \( G \) be a divisible group. Then \( G \) has the form \( G = \oplus_{\alpha} Q \oplus \oplus_{\gamma} Z_{p^\infty}^{\beta_{p}} \).
Fact 9  Let $G$ be a group of bounded order. Then $G$ is a direct sum of cyclic groups.

But we can easily check that $\omega$-stable abelian groups $G = D \oplus H$ in which $H$ has infinitely many summands are not quasi-minimal. Then

Conclusion

Quasi-minimal $\omega$-stable pure groups (i.e. groups reduced to the group language) are strongly minimal substantially.

Thus we should put the next problem last.

Problem

Find quasi-minimal non-$\omega$-stable groups.

References

[1] M.Itai and K.Wakai, $\omega$-saturated quasi-minimal models of $Th(Q^\omega, +, \sigma, 0)$, Math. Log. Quart, vol. 51 (2005) pp. 258-262