<table>
<thead>
<tr>
<th>Title</th>
<th>Quantifier Elimination for Products of Ordered Abelian Groups (Model theoretic aspects of the notion of independence and dimension)</th>
</tr>
</thead>
<tbody>
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Quantifier Elimination for Products of Ordered Abelian Groups

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Introduction

Komori [1] and Weispfenning [2] gave an axiomatization for $Z \times Q$ in the language of Presburger, $\{+,-,0,1,\equiv_{n}|n \geq 1\}$, and showed quantifier elimination, where $x \equiv_{n} y$ is interpreted as $\exists z(x - y = nz)$, and $1$ as $(1,0)$.

If $H$ is an ordered abelian group with extra relations and constants, the lexicographic product $H \times Q$ can be naturally viewed as a structure for the language of $H$. Suzuki [3] generalized Komori’s result and proved that when $H$ admits quantifier elimination, so $H \times Q$ does, by adding another relation $I = [0] \times Q$ to $H \times Q$.


Adapting their argument, we show quantifier elimination for $H \times Z$ in the extended language by $I = [0] \times Z$ and the language of Presburger.

1 Preliminaries

We write $N^+$ for the natural numbers greater than $0$, and $\bar{x}$ for the sequence of variables $(x_1, x_2, \cdots, x_n)$. Let $L_{og} = \{+,-,0,<\}$, the language of the ordered abelian groups, and $L_{Pr} = L_{og} \cup \{1\} \cup \{\equiv_{n}|n \in N^+\}$, the language of Presburger Arithmetic, where $1$ is a constant symbol and $\equiv_{n}$ are binary relation symbols. The notation $nx$ stands for the term $x + \cdots + x$ (n times) for each $n \in N^+$.

Let $H$ be an ordered abelian group as an $L_{og}$-structure and have an additional interpretation of $L_{RC}$, where $L_{RC}$ consists of relation and constant symbols. For simplicity, we define $L_{og+} = L_{og} \cup L_{RC} \cup \{I\}$ and $L_{Pr+} = L_{Pr} \cup L_{RC} \cup \{I\}$, where $I$ is a unary relation symbol which does not appear in $L_{Pr} \cup L_{RC}$.

Here, we give a structure expanding the direct sum of two structures.

Definition 1.1 (Product interpretation) Suppose that $K$ is an ordered abelian group as an $L_{og}$-structure (resp., a model of Presburger arithmetic as an $L_{Pr}$-structure). We call $G$ the structure with the product interpretation of $H$ and $K$ for $L_{og+}$ (resp., $L_{Pr+}$) if $G$ is the direct product of $H$ and $K$ with the following interpretation of $L_{og+}$ (resp., $L_{Pr+}$):

(1) $+, -$ are calculated coordinatewise,
(2) $0^G := (0^H, 0^K),$
(3) $(x_1, y_1) <^G (x_2, y_2)$ if and only if $x_1 <^H x_2$ or $(x_1 = x_2$ and $y_1 <^K y_2),$
(4) $c^G := (c^H, 0^K)$ for each constant symbol $c \in L_{RC},$
(5) $((x_1, y_1), \cdots, (x_n, y_n)) \in R^G$ if and only if $(x_1, \cdots, x_n) \in R^H$ for each relation symbol $R \in L_{RC}$, and
(6) $I^G := \{0^H\} \times K.$

The rest clauses (7), (8) are additional for the product interpretation for $L_{Pr+}$:

(7) $\forall x (0 < x \rightarrow 1 \leq x),$
(8) $\forall xy(x \equiv y \leftrightarrow \exists z(x - y = nz))$ for each $n \in N^+$.
2 Quantifier Elimination for the Product Structure with a Presburger Arithmetic

Suzuki[3] showed the fact below.

**Fact 2.1 (Suzuki)** Suppose that $H$ is an ordered abelian group in $L_{og}$ and admits quantifier elimination in $L_{og} \cup L_{RC}$. If $G$ is the structure with the product interpretation of $H \times \mathbb{Q}$ for $L_{og+}$, then $G$ admits quantifier elimination.

Tanaka and Yokoyama [4] gave a simpler proof for Fact 2.1. We prove quantifier elimination for structures with the product interpretation with $\mathbb{Z}$ in a similar way.

**Lemma 2.2** For any quantifier-free $L_{og} \cup L_{RC}$-formula $\psi$, there exists a quantifier-free $L_{og+}$-formula $\psi'$ such that for all $g \in H \times \mathbb{Z}$, $H \models \psi(g) \iff H \times \mathbb{Z} \models \psi'(g)$

**Proof.** To obtain $\psi'$, replace all the occurrences of $0 < t$ with $0 < t \land \neg I(t)$, and $t = 0$ with $I(t)$. \qed

**Definition 2.3** For each term $t$, $t^1$ denotes the term obtained from $t$ by replacing 1 with 0, and $t^2$ by replacing $c$ with 0 for each constant symbol $c$ in $L_{RC}$.

**Theorem 2.4** Suppose that $H$ is an ordered abelian group in $L_{og}$ and admits quantifier elimination in $L_{og} \cup L_{RC}$. If $G$ is the structure with the product interpretation of $H$ and $\mathbb{Z}$ for $L_{Pr+}$, then $G$ admits quantifier elimination.

**Proof.** It suffices to eliminate each existential quantifier from the following two formulas with parameters $\vec{y}$:

Form 1:

$$\exists x \left\{ \begin{array}{l}
\neg s(g) < mx < t(g) \\
\land \bigwedge_{i \in \text{Pos}} mx =_{n_i} t_i(g) \land \bigwedge_{i \in \text{Neg}} mx \neq_{n_i} t_i(g) \\
\land \bigwedge_{i} R(mix + t_i(g)) \land \bigwedge_{i} \neg R(mix + t_i(g)) \\
\land \bigwedge_{i} I(mx + t_i(g)) \land \bigwedge_{i} \neg I(mx + t_i(g))
\end{array} \right. $$

Form 2:

$$\exists x \left\{ \begin{array}{l}
\neg s(g) = mx \\
\land \bigwedge_{i \in \text{Pos}} mx =_{n_i} t_i(g) \land \bigwedge_{i \in \text{Neg}} mx \neq_{n_i} t_i(g) \\
\land \bigwedge_{i} R(mix + t_i(g)) \land \bigwedge_{i} \neg R(mix + t_i(g)) \\
\land \bigwedge_{i} I(mx + t_i(g)) \land \bigwedge_{i} \neg I(mx + t_i(g))
\end{array} \right. $$

where $s(g)$, $t(g)$ and $t_i(g)$ are $L_{Pr+}$-terms with parameters $\vec{y}$ and $m$, $m_i$ and $n_i$ are in $\mathbb{N}^+$ in the respective formulas. For taking some coefficients commonly, use that

- $x = 0 \iff mx = 0$,
- $x \equiv_{n} 0 \iff mx \equiv_{mn} 0$ and
- $I(x) \iff I(mx)$

for each $m,n \in \mathbb{N}^+$.

Henceforth we argue over form 1, because Form 2 can also be dealt in the same way. Considering that

$$t^G(g) = (t_1^H(g^1), t_2^Z(g^2))$$

and
\( G \models t(\overline{y}) \equiv_{n} 0 \) iff \( H \models t^{1H}(\overline{y}^{1}) \equiv_{n} 0 \) and \( \mathbb{Z} \models t^{2Z}(\overline{y}^{2}) \equiv_{n} 0 \),

\( G \models (\text{Form} 1) \) is equivalent to the disjunction of (a)-(d).

(a) \( G \models I(s(\overline{y}) - t(\overline{y})) \) and

\[
\begin{align*}
H & \models \exists x \left( s^{1}(\overline{y}) = mx = t^{1}(\overline{y}) \land \bigwedge_{i \in \text{Pos}} mx \equiv_{n_{i}} t_{i}^{1}(\overline{y}) \land \bigwedge_{i \in \text{Neg} \setminus S} mx \neq n_{i} t_{i}^{1}(\overline{y}) \right) \\
S\text{CNeg} & \models \exists x \left( \bigwedge_{i \in \text{Pos}} mx \equiv_{n_{i}} t_{i}^{1}(\overline{y}) \land \bigwedge_{i \in \text{Neg} \setminus S} mx \neq n_{i} t_{i}^{1}(\overline{y}) \right),
\end{align*}
\]

(b) \( G \models \neg I(s(\overline{y}) - t(\overline{y})) \) and

\[
\begin{align*}
H & \models \exists x \left( s^{1}(\overline{y}) < mx < t^{1}(\overline{y}) \land \bigwedge_{i \in \text{Pos}} mx \equiv_{n_{i}} t_{i}^{1}(\overline{y}) \land \bigwedge_{i \in \text{Neg} \setminus S} mx \neq n_{i} t_{i}^{1}(\overline{y}) \right) \\
S\text{CNeg} & \models \exists x \left( \bigwedge_{i \in \text{Pos}} mx \equiv_{n_{i}} t_{i}^{1}(\overline{y}) \land \bigwedge_{i \in \text{Neg} \setminus S} mx \neq n_{i} t_{i}^{1}(\overline{y}) \right),
\end{align*}
\]

(c) \( G \models \neg I(s(\overline{y}) - t(\overline{y})) \) and

\[
\begin{align*}
H & \models \exists x \left( s^{1}(\overline{y}) = mx < t^{1}(\overline{y}) \land \bigwedge_{i \in \text{Pos}} mx \equiv_{n_{i}} t_{i}^{1}(\overline{y}) \land \bigwedge_{i \in \text{Neg} \setminus S} mx \neq n_{i} t_{i}^{1}(\overline{y}) \right) \\
S\text{CNeg} & \models \exists x \left( \bigwedge_{i \in \text{Pos}} mx \equiv_{n_{i}} t_{i}^{1}(\overline{y}) \land \bigwedge_{i \in \text{Neg} \setminus S} mx \neq n_{i} t_{i}^{1}(\overline{y}) \right),
\end{align*}
\]

(d) \( G \models \neg I(s(\overline{y}) - t(\overline{y})) \) and

\[
\begin{align*}
H & \models \exists x \left( s^{1}(\overline{y}) < mx = t^{1}(\overline{y}) \land \bigwedge_{i \in \text{Pos}} mx \equiv_{n_{i}} t_{i}^{1}(\overline{y}) \land \bigwedge_{i \in \text{Neg} \setminus S} mx \neq n_{i} t_{i}^{1}(\overline{y}) \right) \\
S\text{CNeg} & \models \exists x \left( \bigwedge_{i \in \text{Pos}} mx \equiv_{n_{i}} t_{i}^{1}(\overline{y}) \land \bigwedge_{i \in \text{Neg} \setminus S} mx \neq n_{i} t_{i}^{1}(\overline{y}) \right),
\end{align*}
\]

where \( \varphi_{R}(x, \overline{y}) \) is a quantifier-free formula with predicates \( R \) and \( I \).

We show that (a)-(d) are representable by a quantifier-free formula.

Case (a). First, eliminate the negative part from the formula below appeared in (a).

\[
Z \models \exists x \left( \bigwedge_{i \in \text{Pos}} mx \equiv_{n_{i}} t_{i}^{2}(\overline{y}^{2}) \land \bigwedge_{i \in \text{Neg} \setminus S} mx \neq n_{i} t_{i}^{2}(\overline{y}^{2}) \right) \quad \text{... (1)}
\]

As the negative part is equivalent in \( Z \) to

\[
\bigwedge_{i \in \text{Neg} \setminus S} \bigwedge_{j=1}^{n_{i}} (t_{i} + j)^{2}(\overline{y}^{2}),
\]

(1) can be rewritten by taking a disjunctive normal form as

\[
Z \models \exists x \left( s^{2}(\overline{y}^{2}) < mx < t^{2}(\overline{y}^{2}) \land \bigwedge_{i \in \text{Pos}} mx \equiv_{n_{i}} t_{i}^{2}(\overline{y}^{2}) \right) \quad \text{... (2)}.
\]

Furthermore, we can choose the \( n_{i} \) in (2) relatively prime to each other. Therefore, (a) is equivalent to the following form:
$(a)' G \models I(s(g) - t(ff))$

and

\[ s \sigma \bigvee_{Neg} \vee \{ \]

According to $S$, $H \vdash \exists x(s^{1} \bigwedge_{\wedge} \bigwedge_{i \epsilon Poe}^{(g^{1})} mx \equiv t_{i}^{1}(y^{1}) \wedge =mx_{\hslash_{i}}=t^{1}\overline{y}^{1}) \bigwedge_{i \epsilon S} mx \not\in n_{i}i_{i}^{1}(\overline{y}^{1}))$.

$\mathbb{Z} \vdash \exists x(s^{2}(g^{2})<mx<t^{2}(j^{2}) \wedge \wedge mx\equiv_{n_{i}}t_{i}^{2}())$


Remainder Theorem, there exists a unique solution modulo $\Pi n_{i}$ to the system

\[ \wedge v \equiv n_{i}t_{i}^{2}() \]

Thus, the formula

\[ Z \models \exists x(s^{2}(g^{2})<mx<t^{2}(j^{2}) \wedge \wedge mx\equiv_{n_{i}}t_{i}^{2}()) \]

can be replaced with

\[ Z \models \bigwedge_{i=1}^{\Pi n_{i}-1} \left( \bigwedge m(s(g) + i) \equiv_{n_{i}}t_{i}(y) \rightarrow s(g) + i < t(y) \right) \]

On the other hand, as $H$ admits quantifier elimination, there exists a quantifier-free $L_{\log} \cup L_{RC}$-formula $\psi$ such that $H \models \psi(g^{1})$ is equivalent to

\[ s^{1}(g^{1}) = mx = t^{1}(g^{1}) \]

and

\[ \left( \bigwedge_{i \epsilon Poe}^{(g^{1})} mx \equiv_{n_{i}}t_{i}^{1}(g^{1}) \wedge \bigwedge_{i \epsilon S} mx \equiv_{n_{i}}t_{i}^{1}(g^{1}) \right) \]

where $g \in H \times Z$.

By lemma(2.2), we have a formula $\psi'$ such that $H \models \psi'(g^{1})$ is equivalent to $H \times Z \models \psi'(y)$ for $y \in H \times Z$. This completes Case (a). Note that the formula obtained finally is determined uniquely by the theory of $H$ and $Z$.

Case (b). After eliminating some negative parts by the same procedure, eliminate the quantifier from

\[ \bigwedge_{i=1}^{\Pi n_{i}-1} \left( \bigwedge m(s(g) + i) \equiv_{n_{i}}t_{i}(y) \rightarrow s(g) + i < t(y) \right) \]

This case is simple. Because

\[ Z \models \exists x( \bigwedge_{i \epsilon S} mx \equiv_{n_{i}}t_{i}^{2}(g^{2}) ) \]

is always true, (b) is equivalent to a quantifier-free formula by lemma(2.2) and that $H$ has quantifier elimination.

(c), (d) are similar as (b).

As mentioned in the proof of Theorem 2.4, quantifier elimination for $H \times Z$ depends only on the theories of $H$ and $Q$.

**Corollary 2.5** Suppose that $H \equiv H'$ and $K \equiv Z$, and let $\varphi(x, y)$ and $\psi(y)$ be quantifier-free formulas. If $H \times Z \models \forall y(\exists x \varphi(x, y) \leftrightarrow \psi(y))$, then $H' \times K \models \forall y(\exists x \varphi(x, y) \leftrightarrow \psi(y))$. 
3 Acknowledgements

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References


