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Quantifier Elimination for Products of Ordered Abelian Groups

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Introduction

Komori [1] and Weispfenning [2] gave an axiomatization for $\mathbb{Z} \times \mathbb{Q}$ in the language of Presburger, $\{+, -, <, 0, 1, \equiv_n\}_{n\geq 1}$, and showed quantifier elimination, where $x \equiv_n y$ is interpreted as $\exists z(x-y = nz)$, and 1 as $(1, 0)$.

If H is an ordered abelian group with extra relations and constants, the lexicographic product $H \times \mathbb{Q}$ can be naturally viewed as a structure for the language of H. Suzuki [3] generalized Komori's result and proved that when H admits quantifier elimination, so $H \times \mathbb{Q}$ does, by adding another relation $I = \{0\} \times \mathbb{Q}$ to $H \times \mathbb{Q}$.

Suzuki's proof is model-theoretic but rather complicated. By incorpolating some syntactical argument, Tanaka and Yokoyama [4] gave a simpler one.

Adapting their argument, we show quantifier elimination for $H \times \mathbb{Z}$ in the extended language by $I = \{0\} \times \mathbb{Z}$ and the language of Presburger.

1 Preliminaries

We write $\mathbf{N}^+$ for the natural numbers greater than 0, and $\bar{x}$ for the sequence of variables $(x_1, x_2, \cdots, x_n)$. Let $L_{\mathrm{og}} = \{+, -, 0, <\}$, the language of the ordered abelian groups, and $L_{\mathrm{Pr}} = L_{\mathrm{og}} \cup \{1\} \cup \{\equiv_n\}_{n\in\mathbf{N}^+}$, the language of Presburger Arithmetic, where 1 is a constant symbol and $\equiv_n$ are binary relation symbols. The notation nx stands for the term $x + \cdots + x$ (n times) for each $n \in \mathbf{N}^+$.

Let H be an ordered abelian group as an L_{og}-structure and have an additional interpretation of L_{RC}, where L_{RC} consists of relation and constant symbols. For simplicity, we define $L_{\mathrm{og}+} = L_{\mathrm{og}} \cup L_{\mathrm{RC}} \cup \{I\}$ and $L_{\mathrm{Pr}+} = L_{\mathrm{Pr}} \cup L_{\mathrm{RC}} \cup \{I\}$, where I is a unary relation symbol which does not appear in $L_{\mathrm{Pr}} \cup L_{\mathrm{RC}}$.

Here, we give a structure expanding the direct sum of two structures.

Definition 1.1 (Product interpretation) Suppose that K is an ordered abelian group as an L_{og}-structure (resp., a model of Presburger arithmetic as an L_{Pr}-structure). We call G the structure with the *product interpretation* of H and K for $L_{\mathrm{og}+}$(resp., $L_{\mathrm{Pr}+}$) if G is the direct product of H and K with the following interpretation of $L_{\mathrm{og}+}$(resp., $L_{\mathrm{Pr}+}$):

(1) $+, -$ are calculated coordinatewise,
(2) $0^G := (0^H, 0^K)$,
(3) $(x_1, y_1) <^G (x_2, y_2)$ if and only if $x_1 <^H x_2$ or ($x_1 = x_2$ and $y_1 <^K y_2$),
(4) $c^G := (c^H, 0^K)$ for each constant symbol $c \in L_{\mathrm{RC}}$,
(5) $((x_1, y_1), \cdots, (x_n, y_n)) \in R^G$ if and only if $(x_1, \cdots, x_n) \in R^H$ for each relation symbol $R \in L_{\mathrm{RC}}$, and
(6) $I^G := \{0^H\} \times K$.

The rest clauses (7), (8) are additional for the product interpretation for $L_{\mathrm{Pr}+}$:
(7) $\forall x(0 < x \rightarrow 1 \leq x)$,
(8) $\forall xy(x \equiv_n y \leftrightarrow \exists z(x - y = nz))$ for each $n \in \mathbf{N}^+$.

2 Quantifier Elimination for the Product Structure with a Presburger Arithmetic

Suzuki[3] showed the fact below.

Fact 2.1 (Suzuki) *Suppose that H is an ordered abelian group in L_{og} and admits quantifier elimination in $L_{og} \cup L_{RC}$. If G is the structure with the product interpretation of $H \times \mathbb{Q}$ for L_{og+}, then G admits quantifier elimination.*

Tanaka and Yokoyama [4] gave a simpler proof for Fact 2.1. We prove quantifier elimination for structures with the product interpretation with $\mathbb{Z}$ in a similar way.

Lemma 2.2 *For any quantifier-free $L_{og} \cup L_{RC}$-formula ψ, there exists a quantifier-free L_{og+}-formula ψ' such that for all $\bar{y} \in H \times \mathbb{Z}$, $H \models \psi(\bar{y}^1) \Leftrightarrow H \times \mathbb{Z} \models \psi'(\bar{y})$*

Proof. To obtain ψ', replace all the occurrences of $0 < t$ with $0 < t \wedge \neg I(t)$, and $t = 0$ with $I(t)$. □

Definition 2.3 For each term t, t^1 denotes the term obtained from t by replacing 1 with 0, and t^2 by replacing c with 0 for each constant symbol c in L_{RC}.

Theorem 2.4 *Suppose that H is an ordered abelian group in L_{og} and admits quantifier elimination in $L_{og} \cup L_{RC}$. If G is the structure with the product interpretation of H and $\mathbb{Z}$ for L_{Pr+}, then G admits quantifier elimination.*

Proof. It suffices to eliminate each existential quantifier from the following two formulas with parameters $\bar{y}$:

Form 1:

$$\exists x \begin{cases} s(\bar{y}) < mx < t(\bar{y}) \\ \wedge \bigwedge_{i \in Pos} mx \equiv_{n_i} t_i(\bar{y}) \wedge \bigwedge_{i \in Neg} mx \not\equiv_{n_i} t_i(\bar{y}) \\ \wedge \bigwedge_i R(m_i x + t_i(\bar{y})) \wedge \bigwedge_i \neg R(m_i x + t_i(\bar{y})) \\ \wedge \bigwedge_i I(mx + t_i(\bar{y})) \wedge \bigwedge_i \neg I(mx + t_i(\bar{y})), \end{cases}$$

Form 2:

$$\exists x \begin{cases} s(\bar{y}) = mx \\ \wedge \bigwedge_{i \in Pos} mx \equiv_{n_i} t_i(\bar{y}) \wedge \bigwedge_{i \in Neg} mx \not\equiv_{n_i} t_i(\bar{y}) \\ \wedge \bigwedge_i R(m_i x + t_i(\bar{y})) \wedge \bigwedge_i \neg R(m_i x + t_i(\bar{y})) \\ \wedge \bigwedge_i I(mx + t_i(\bar{y})) \wedge \bigwedge_i \neg I(mx + t_i(\bar{y})), \end{cases}$$

where $s(\bar{y})$, $t(\bar{y})$ and $t_i(\bar{y})$ are L_{Pr+}-terms with parameters $\bar{y}$ and m, m_i and n_i are in $\mathbf{N}^+$ in the respective formulas. For taking some coefficients commonly, use that

$x = 0 \leftrightarrow mx = 0$,

$x \equiv_n 0 \leftrightarrow mx \equiv_{mn} 0$ and

$I(x) \leftrightarrow I(mx)$

for each $m, n \in \mathbf{N}^+$.

Henceforth we argue over form 1, because Form 2 can also be dealt in the same way. Considering that

$t^G(\bar{y}) = (t^{1H}(\bar{y}^1),\ t^{2\mathbb{Z}}(\bar{y}^2))$

and

$G \models t(\bar{y}) \equiv_n 0$ iff $H \models t^{1H}(\bar{y}^1) \equiv_n 0$ and $\mathbb{Z} \models t^{2\mathbb{Z}}(\bar{y}^2) \equiv_n 0$,
$G \models$ (Form 1) is equivalent to the disjunction of (a)-(d).

(a) $G \models I(s(\bar{y}) - t(\bar{y}))$ and

$$\bigvee_{S \subseteq Neg} \left\{ \begin{array}{l} H \models \exists x \left(\begin{array}{l} s^1(\bar{y}^1) = mx = t^1(\bar{y}^1) \\ \wedge \bigwedge_{i \in Pos} mx \equiv_{n_i} t_i^1(\bar{y}^1) \wedge \bigwedge_{i \in S} mx \not\equiv_{n_i} t_i^1(\bar{y}^1) \\ \wedge \varphi_{RI}(x, \bar{y}) \end{array} \right) \text{and} \\ \mathbb{Z} \models \exists x \left(\begin{array}{l} s^2(\bar{y}^2) < mx < t^2(\bar{y}^2) \\ \wedge \bigwedge_{i \in Pos} mx \equiv_{n_i} t_i^2(\bar{y}^2) \wedge \bigwedge_{i \in Neg \setminus S} mx \not\equiv_{n_i} t_i^2(\bar{y}^2) \end{array} \right), \end{array} \right.$$

(b) $G \models \neg I(s(\bar{y}) - t(\bar{y}))$ and

$$\bigvee_{S \subseteq Neg} \left\{ \begin{array}{l} H \models \exists x \left(\begin{array}{l} s^1(\bar{y}^1) < mx < t^1(\bar{y}^1) \\ \wedge \bigwedge_{i \in Pos} mx \equiv_{n_i} t_i^1(\bar{y}^1) \wedge \bigwedge_{i \in S} mx \not\equiv_{n_i} t_i^1(\bar{y}^1) \\ \wedge \varphi_{RI}(x, \bar{y}) \end{array} \right) \text{and} \\ \mathbb{Z} \models \exists x \left(\bigwedge_{i \in Pos} mx \equiv_{n_i} t_i^2(\bar{y}^2) \wedge \bigwedge_{i \in Neg \setminus S} mx \not\equiv_{n_i} t_i^2(\bar{y}^2) \right), \end{array} \right.$$

(c) $G \models \neg I(s(\bar{y}) - t(\bar{y}))$ and

$$\bigvee_{S \subseteq Neg} \left\{ \begin{array}{l} H \models \exists x \left(\begin{array}{l} s^1(\bar{y}^1) = mx < t^1(\bar{y}^1) \\ \wedge \bigwedge_{i \in Pos} mx \equiv_{n_i} t_i^1(\bar{y}^1) \wedge \bigwedge_{i \in S} mx \not\equiv_{n_i} t_i^1(\bar{y}^1) \\ \wedge \varphi_{RI}(x, \bar{y}) \end{array} \right) \text{and} \\ \mathbb{Z} \models \exists x \left(\begin{array}{l} s^2(\bar{y}^2) < mx \\ \wedge \bigwedge_{i \in Pos} mx \equiv_{n_i} t_i^2(\bar{y}^2) \wedge \bigwedge_{i \in Neg \setminus S} mx \not\equiv_{n_i} t_i^2(\bar{y}^2) \end{array} \right), \end{array} \right.$$

(d) $G \models \neg I(s(\bar{y}) - t(\bar{y}))$ and

$$\bigvee_{S \subseteq Neg} \left\{ \begin{array}{l} H \models \exists x \left(\begin{array}{l} s^1(\bar{y}^1) < mx = t^1(\bar{y}^1) \\ \wedge \bigwedge_{i \in Pos} mx \equiv_{n_i} t_i^1(\bar{y}^1) \wedge \bigwedge_{i \in S} mx \not\equiv_{n_i} t_i^1(\bar{y}^1) \\ \wedge \varphi_{RI}(x, \bar{y}) \end{array} \right) \text{and} \\ \mathbb{Z} \models \exists x \left(\begin{array}{l} mx < t^2(\bar{y}^2) \\ \wedge \bigwedge_{i \in Pos} mx \equiv_{n_i} t_i^2(\bar{y}^2) \wedge \bigwedge_{i \in Neg \setminus S} mx \not\equiv_{n_i} t_i^2(\bar{y}^2) \end{array} \right), \end{array} \right.$$

where $\varphi_{RI}(x, \bar{y})$ is a quantifier-free formula with predicates R and I.
We show that (a)-(d) are representable by a quantifier-free formula.

Case (a). First, eliminate the negative part from the formula below appeared in (a).

$$\mathbb{Z} \models \exists x \left(\begin{array}{l} s^2(\bar{y}^2) < mx < t^2(\bar{y}^2) \\ \wedge \bigwedge_{i \in Pos} mx \equiv_{n_i} t_i^2(\bar{y}^2) \wedge \bigwedge_{i \in Neg \setminus S} mx \not\equiv_{n_i} t_i^2(\bar{y}^2) \end{array} \right) \quad \cdots (1)$$

As the negative part is equivalent in $\mathbb{Z}$ to

$$\bigwedge_{i \in Neg \setminus S} \bigvee_{j=1}^{n_i} mx \equiv_{n_i} (t_i + j)^2(\bar{y}^2),$$

(1) can be rewritten by taking a disjunctive normal form as

$$\mathbb{Z} \models \exists x \bigvee \left(s^2(\bar{y}^2) < mx < t^2(\bar{y}^2) \wedge \bigwedge mx \equiv_{n_i} t_i^2(\bar{y}^2) \right) \quad \cdots (2).$$

Furthermore, we can choose the n_i in (2) relatively prime to each other. Therefore, (a) is equivalent to the following form:

(a)′ $G \models I(s(\bar{y}) - t(\bar{y}))$ and

$$\bigvee_{S\subseteq Neg} \bigvee \left\{ \begin{array}{l} H \models \exists x \left(\begin{array}{l} s^1(\bar{y}^1) = mx = t^1(\bar{y}^1) \\ \wedge \bigwedge_{i\in Pos} mx \equiv_{n_i} t_i^1(\bar{y}^1) \wedge \bigwedge_{i\in S} mx \not\equiv_{n_i} t_i^1(\bar{y}^1) \\ \wedge\, \varphi_{RI}(x,\bar{y}) \end{array} \right) \text{and} \\ \mathbb{Z} \models \exists x \left(s^2(\bar{y}^2) < mx < t^2(\bar{y}^2) \wedge \bigwedge mx \equiv_{n_i} t_i^2(\bar{y}^2) \right). \end{array} \right.$$

According to Sunzi's(Chinese) Remainder Theorem, there exists a unique solution modulo Πn_i to the system

$$\bigwedge v \equiv_{n_i} t_i^2(\bar{y}^2).$$

Thus, the formula

$$\mathbb{Z} \models \exists x \left(s^2(\bar{y}^2) < mx < t^2(\bar{y}^2) \wedge \bigwedge mx \equiv_{n_i} t_i^2(\bar{y}^2) \right)$$

can be replaced with

$$\mathbb{Z} \models \bigwedge_{i=1}^{\Pi n_i - 1} \left(\bigwedge m(s^2(\bar{y}^2) + i) \equiv_{n_i} t_i^2(\bar{y}^2) \to s^2(\bar{y}) + i < t^2(\bar{y}^2) \right). \qquad \cdots (3)$$

Notice that each step of transformations above depends only on the theory of $\mathbb{Z}$. Under the condition $G \models I(s(\bar{y}) - t(\bar{y}))$ in (a)′, (3) is equivalent to

$$H \times \mathbb{Z} \models \bigwedge_{i=1}^{\Pi n_i - 1} \left(\bigwedge m(s(\bar{y}) + i) \equiv_{n_i} t_i(\bar{y}) \to s(\bar{y}) + i < t(\bar{y}) \right).$$

On the other hand, as H admits quantifier elimination, there exists a quantifier-free $L_{og} \cup L_{RC}$-formula ψ such that $H \models \psi(\bar{y}^1)$ is equivalent to

$$H \models \exists x \left(\begin{array}{l} s^1(\bar{y}^1) = mx = t^1(\bar{y}^1) \\ \wedge \bigwedge_{i\in Pos} mx \equiv_{n_i} t_i^1(\bar{y}^1) \wedge \bigwedge_{i\in S} mx \not\equiv_{n_i} t_i^1(\bar{y}^1) \\ \wedge\, \varphi_{RI}(x,\bar{y}^1) \end{array} \right),$$

where $\bar{y} \in H \times \mathbb{Z}$.

By lemma(2.2), we have a formula ψ' such that $H \models \psi(\bar{y}^1)$ is equivalent to $H \times \mathbb{Z} \models \psi'(\bar{y})$ for $\bar{y} \in H \times \mathbb{Z}$. This completes Case (a). Note that the formula obtained finally is determined uniquely by the theory of H and $\mathbb{Z}$.

Case (b). After eliminating some negative parts by the same procedure, eliminate the quantifier from

$$\bigvee_{S\subseteq Neg} \bigvee \left\{ \begin{array}{l} H \models \exists x \left(\begin{array}{l} s^1(\bar{y}^1) < mx < t^1(\bar{y}^1) \\ \wedge \bigwedge_{i\in Pos} mx \equiv_{n_i} t_i^1(\bar{y}^1) \wedge \bigwedge_{i\in S} mx \not\equiv_{n_i} t_i^1(\bar{y}^1) \\ \wedge\, \varphi_{RI}(x,\bar{y}) \end{array} \right) \text{and} \\ \mathbb{Z} \models \exists x \left(\bigwedge mx \equiv_{n_i} t_i^2(\bar{y}^2) \right). \end{array} \right.$$

This case is simple. Because

$$\mathbb{Z} \models \exists x (\bigwedge mx \equiv_{n_i} t_i^2(\bar{y}^2))$$

is always true, (b) is equivalent to a quantifier-free formula by lemma(2.2) and that H has quantifier elimination.

(c), (d) are similar as (b). □

As mentioned in the proof of Theorem 2.4, quantifier elimination for $H \times \mathbb{Z}$ depends only on the theories of H and $\mathbb{Q}$.

Corollary 2.5 *Suppose that $H \equiv H'$ and $K \equiv \mathbb{Z}$, and let $\varphi(x,\bar{y})$ and $\psi(\bar{y})$ be quantifier-free formulas. If $H \times \mathbb{Z} \models \forall \bar{y}(\exists x \varphi(x,\bar{y}) \leftrightarrow \psi(\bar{y}))$, then $H' \times K \models \forall \bar{y}(\exists x \varphi(x,\bar{y}) \leftrightarrow \psi(\bar{y}))$.*

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References

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