Quantifier Elimination for Products of Ordered Abelian Groups

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Introduction

Komori [1] and Weispfenning [2] gave an axiomatization for $\mathbb{Z} \times \mathbb{Q}$ in the language of Presburger, \{+, -, $<$, 0, 1, $\equiv_n | n \geq 1\}, and showed quantifier elimination, where $x \equiv_n y$ is interpreted as $\exists z(x - y = nz)$, and 1 as $(1, 0)$.

If $H$ is an ordered abelian group with extra relations and constants, the lexicographic product $H \times \mathbb{Q}$ can be naturally viewed as a structure for the language of $H$. Suzuki [3] generalized Komori’s result and proved that when $H$ admits quantifier elimination, so $H \times \mathbb{Q}$ does, by adding another relation $I = [0] \times \mathbb{Q}$ to $H \times \mathbb{Q}$.


Adapting their argument, we show quantifier elimination for $H \times \mathbb{Z}$ in the extended language by $I = [0] \times \mathbb{Z}$ and the language of Presburger.

1 Preliminaries

We write $\mathbb{N}^+$ for the natural numbers greater than 0, and $\bar{x}$ for the sequence of variables $(x_1, x_2, \cdots, x_n)$. Let $L_{og} = \{+, -, 0, <\}$, the language of the ordered abelian groups, and $L_{Pr} = L_{og} \cup \{1\} \cup \{\equiv_n | n \in \mathbb{N}^*\}$, the language of Presburger Arithmetic, where 1 is a constant symbol and $\equiv_n$ are binary relation symbols. The notation $nx$ stands for the term $x + \cdots + x$ (n times) for each $n \in \mathbb{N}^*$.

Let $H$ be an ordered abelian group as an $L_{og}$-structure and have an additional interpretation of $L_{RC}$, where $L_{RC}$ consists of relation and constant symbols. For simplicity, we define $L_{og^+} = L_{og} \cup L_{RC} \cup \{I\}$ and $L_{Pr^+} = L_{Pr} \cup L_{RC} \cup \{I\}$, where I is a unary relation symbol which does not appear in $L_{Pr} \cup L_{RC}$.

Here, we give a structure expanding the direct sum of two structures.

Definition 1.1 (Product interpretation) Suppose that $K$ is an ordered abelian group as an $L_{og}$-structure (resp., a model of Presburger arithmetic as an $L_{Pr}$-structure). We call $G$ the structure with the product interpretation of $H$ and $K$ for $L_{og^+}$ (resp., $L_{Pr^+}$) if $G$ is the direct product of $H$ and $K$ with the following interpretation of $L_{og^+}$ (resp., $L_{Pr^+}$):

1. $+, -$ are calculated coordinatewise,
2. $0^G := (0^H, 0^K),
3. (x_1, y_1) <^G (x_2, y_2)$ if and only if $x_1 <^H x_2$ or $(x_1 = x_2$ and $y_1 <^K y_2)$,
4. $c^G := (c^H, 0^K)$ for each constant symbol $c \in L_{RC}$,
5. $((x_1, y_1), \cdots, (x_n, y_n)) \in R^G$ if and only if $(x_1, \cdots, x_n) \in R^H$ for each relation symbol $R \in L_{RC}$, and
6. $f^G := [0^H] \times K.$

The rest clauses (7), (8) are additional for the product interpretation for $L_{Pr^+}$:

7. $\forall x(0 < x \rightarrow 1 \leq x),$
8. $\forall y(x \equiv_n y \leftrightarrow \exists z(x - y = nz))$ for each $n \in \mathbb{N}^*.$
2 Quantifier Elimination for the Product Structure with a Presburger Arithmetic

Suzuki[3] showed the fact below.

Fact 2.1 (Suzuki) Suppose that $H$ is an ordered abelian group in $L_{\text{og}}$ and admits quantifier elimination in $L_{\text{og}} \cup L_{\text{RC}}$. If $G$ is the structure with the product interpretation of $H \times \mathbb{Q}$ for $L_{\text{og}+}$, then $G$ admits quantifier elimination.

Tanaka and Yokoyama [4] gave a simpler proof for Fact 2.1. We prove quantifier elimination for structures with the product interpretation in a similar way.

Lemma 2.2 For any quantifier-free $L_{\text{og}} \cup L_{\text{RC}}$-formula $\psi$, there exists a quantifier-free $L_{\text{og}+}$-formula $\psi'$ such that for all $g \in H \times \mathbb{Z}$, $H \models \psi(g)$ implies $G \models \psi'(g)$

Proof. To obtain $\psi'$, replace all the occurrences of $0 < t$ with $0 < t \land \neg I(t)$, and $t = 0$ with $I(t)$. □

Definition 2.3 For each term $t$, $t^1$ denotes the term obtained from $t$ by replacing 1 with 0, and $t^2$ by replacing each constant symbol $c$ in $L_{\text{RC}}$ with 0.

Theorem 2.4 Suppose that $H$ is an ordered abelian group in $L_{\text{og}}$ and admits quantifier elimination in $L_{\text{og}} \cup L_{\text{RC}}$. If $G$ is the structure with the product interpretation of $H$ and $\mathbb{Z}$ for $L_{\text{Pr}+}$, then $G$ admits quantifier elimination.

Proof. It suffices to eliminate each existential quantifier from the following two formulas with parameters $g$:

Form 1:
\[
\exists x \left\{ s(g) < mx < t(g) \wedge \bigwedge_{i \in \text{Pos}} mx \equiv_n t_i(g) \wedge \bigwedge_{i \in \text{Neg}} mx \equiv_n t_i(g) \wedge R(mx + t_i(g)) \wedge \neg R(mx + t_i(g)) \wedge I(mx + t_i(g)) \wedge \neg I(mx + t_i(g)) \right\}
\]

Form 2:
\[
\exists x \left\{ s(g) = mx \wedge \bigwedge_{i \in \text{Pos}} mx \equiv_n t_i(g) \wedge \bigwedge_{i \in \text{Neg}} mx \equiv_n t_i(g) \wedge R(mx + t_i(g)) \wedge \neg R(mx + t_i(g)) \wedge I(mx + t_i(g)) \wedge \neg I(mx + t_i(g)) \right\}
\]

where $s(g), t(g)$ and $t_i(g)$ are $L_{\text{Pr}+}$-terms with parameters $g$ and $m, m_i$ and $n_i$ are in $\mathbb{N}^+$ in the respective formulas. For taking some coefficients commonly, use that
\[
x = 0 \iff mx = 0,
\]
\[
x \equiv_n 0 \iff mx \equiv_{mn} 0 \text{ and } I(x) \iff I(mx)
\]
for each $m, n \in \mathbb{N}^+$.

Henceforth we argue over form 1, because Form 2 can also be dealt in the same way. Considering that
\[
t^G(g) = (t^{1H}(g^1), t^{2\mathbb{Z}}(g^2))
\]
and
$G \models t(y) \equiv_n 0$ iff $H \models t^{1H}(y) \equiv_n 0$ and $\mathbb{Z} \models t^{2\mathbb{Z}}(y) \equiv_n 0$.

$G \models \text{Form1}$ is equivalent to the disjunction of (a)-(d).

(a) $G \models I(s(y) - t(y))$ and

$$
\exists \begin{array}{ll}
\vee \\
SCNeg
\end{array}
\begin{cases}
H \models \exists x \left( s^1(y) = mx = t^1(y) \right) \\
\wedge \bigwedge_{i \in Pos} mx \equiv_n t_i^1(y) \wedge \bigwedge_{i \in S} mx \not\equiv_n t_i^1(y) \\
\wedge \varphi_R(x, y) \\
s^2(y) < mx < t^2(y)
\end{cases}
$$

and

$$
\exists \begin{array}{ll}
\vee \\
SCNeg
\end{array}
\begin{cases}
Z \models \exists x \left( \bigwedge_{i \in Pos} mx \equiv_n t_i^2(y) \wedge \bigwedge_{i \in Neg \setminus S} mx \not\equiv_n t_i^2(y) \right) \\
\wedge \varphi_R(x, y)
\end{cases}
$$

(b) $G \models \neg I(s(y) - t(y))$ and

$$
\exists \begin{array}{ll}
\vee \\
SCNeg
\end{array}
\begin{cases}
H \models \exists x \left( s^1(y) < mx < t^1(y) \right) \\
\wedge \bigwedge_{i \in Pos} mx \equiv_n t_i^1(y) \wedge \bigwedge_{i \in S} mx \not\equiv_n t_i^1(y) \\
\wedge \varphi_R(x, y) \\
s^2(y) < mx
\end{cases}
$$

and

$$
\exists \begin{array}{ll}
\vee \\
SCNeg
\end{array}
\begin{cases}
Z \models \exists x \left( \bigwedge_{i \in Pos} mx \equiv_n t_i^2(y) \wedge \bigwedge_{i \in Neg \setminus S} mx \not\equiv_n t_i^2(y) \right)
\end{cases}
$$

(c) $G \models \neg I(s(y) - t(y))$ and

$$
\exists \begin{array}{ll}
\vee \\
SCNeg
\end{array}
\begin{cases}
H \models \exists x \left( s^1(y) = mx < t^1(y) \right) \\
\wedge \bigwedge_{i \in Pos} mx \equiv_n t_i^1(y) \wedge \bigwedge_{i \in S} mx \not\equiv_n t_i^1(y) \\
\wedge \varphi_R(x, y) \\
s^2(y) < mx
\end{cases}
$$

and

$$
\exists \begin{array}{ll}
\vee \\
SCNeg
\end{array}
\begin{cases}
Z \models \exists x \left( \bigwedge_{i \in Pos} mx \equiv_n t_i^2(y) \wedge \bigwedge_{i \in Neg \setminus S} mx \not\equiv_n t_i^2(y) \right)
\end{cases}
$$

(d) $G \models \neg I(s(y) - t(y))$ and

$$
\exists \begin{array}{ll}
\vee \\
SCNeg
\end{array}
\begin{cases}
H \models \exists x \left( s^1(y) < mx = t^1(y) \right) \\
\wedge \bigwedge_{i \in Pos} mx \equiv_n t_i^1(y) \wedge \bigwedge_{i \in S} mx \not\equiv_n t_i^1(y) \\
\wedge \varphi_R(x, y) \\
xm < t^2(y)
\end{cases}
$$

and

$$
\exists \begin{array}{ll}
\vee \\
SCNeg
\end{array}
\begin{cases}
Z \models \exists x \left( \bigwedge_{i \in Pos} mx \equiv_n t_i^2(y) \wedge \bigwedge_{i \in Neg \setminus S} mx \not\equiv_n t_i^2(y) \right)
\end{cases}
$$

where $\varphi_R(x, y)$ is a quantifier-free formula with predicates $R$ and $I$.

We show that (a)-(d) are representable by a quantifier-free formula.

Case (a). First, eliminate the negative part from the formula below appeared in (a).

$$
\exists \begin{array}{ll}
\vee \\
SCNeg
\end{array}
\begin{cases}
Z \models \exists x \left( \bigwedge_{i \in Pos} mx \equiv_n t_i^1(y) \wedge \bigwedge_{i \in Neg \setminus S} mx \not\equiv_n t_i^1(y) \right)
\end{cases}

\ldots (1)
$$

As the negative part is equivalent in $\mathbb{Z}$ to

$$
\bigwedge_{i \in Neg \setminus S} \bigvee_{n_i} \left\{ t_i + j y \right\}
$$

(1) can be rewritten by taking a disjunctive normal form as

$$
\exists x \bigvee \left( s^2(y) < mx < t^2(y) \wedge \bigwedge_{i \in Neg \setminus S} mx \equiv_n t_i^2(y) \right) \ldots (2).
$$

Furthermore, we can choose the $n_i$ in (2) relatively prime to each other. Therefore, (a) is equivalent to the following form:
(a)' $G \models I(s(y) - t(y))$ and
\[ \bigvee_{S \subseteq N_{eg}} \bigvee_{S \subseteq \text{Neg}} \left\{ \begin{array}{l}
H \models \exists x \left( s^{i}(g^{1}) = mx = t^{i}(g^{1}) \\
\wedge \bigwedge_{i \in \text{Po}e} mx \equiv_{n_{i}} t^{i}_{1}(g^{1}) \wedge \bigwedge_{i \in S} mx \not\equiv_{n_{i}} t^{i}_{1}(g^{1}) \wedge \varphi_{R}(x, \bar{y}) \right) \\
Z \models \exists x \left( s^{2}(g^{2}) < mx < t^{2}(g^{2}) \wedge \bigcap_{i \in S} mx \equiv_{n_{i}} t^{2}_{i}(g^{2}) \right).
\end{array} \right. \]

According to Sunzi's (Chinese) Remainder Theorem, there exists a unique solution modulo $\Pi n_{i}$ to the system
\[ \bigwedge_{i=1}^{\Pi n_{i}-1} (s(g) + l \equiv_{n_{i}} t_{i}(j) \rightarrow s(g) + i < t(j)). \]

Thus, the formula
\[ Z \models \exists x \left( s^{2}(g^{2}) < mx < t^{2}(g^{2}) \wedge \bigcap_{i \in S} mx \equiv_{n_{i}} t^{2}_{i}(g^{2}) \right) \]
can be replaced with
\[ Z \models \bigwedge_{i=1}^{\Pi n_{i}-1} \left( \bigwedge_{i \in \text{Po}e} mx \equiv_{n_{i}} t^{2}_{i}(g^{2}) \rightarrow s^{2}(g) + i < t^{2}(g^{2}) \right). \]

Notice that each step of transformations above depends only on the theory of $Z$. Under the condition $G \models I(s(y) - t(y))$ in (a)', (3) is equivalent to
\[ H \times Z \models \bigwedge_{i=1}^{\Pi n_{i}-1} \left( \bigwedge_{i \in \text{Po}e} mx \equiv_{n_{i}} t^{2}_{i}(g^{2}) \rightarrow s(g) + i < t(y) \right). \]

On the other hand, as $H$ admits quantifier elimination, there exists a quantifier-free $L_{\text{og}} \cup L_{\text{RC}}$-formula $\psi$ such that $H \models \psi(g^{1})$ is equivalent to
\[ H \models \exists x \left( \bigwedge_{i \in \text{Po}e} mx \equiv_{n_{i}} t^{2}_{i}(g^{1}) \wedge \varphi_{R}(x, \bar{y}) \right), \]
where $\bar{y} \in H \times Z$.

By lemma (2.2), we have a formula $\psi'$ such that $H \models \psi'(g^{1})$ is equivalent to $H \times Z \models \psi'(y)$ for $y \in H \times Z$. This completes Case (a). Note that the formula obtained finally is determined uniquely by the theory of $H$ and $Z$.

Case (b). After eliminating some negative parts by the same procedure, eliminate the quantifier from
\[ \bigvee_{S \subseteq N_{eg}} \bigvee_{S \subseteq \text{Neg}} \left\{ \begin{array}{l}
H \models \exists x \left( s^{i}(g^{1}) < mx < t^{i}(g^{1}) \\
\wedge \bigwedge_{i \in \text{Po}e} mx \equiv_{n_{i}} t^{i}_{1}(g^{1}) \wedge \bigwedge_{i \in S} mx \not\equiv_{n_{i}} t^{i}_{1}(g^{1}) \wedge \varphi_{R}(x, \bar{y}) \right) \\
Z \models \exists x \left( \bigwedge_{i \in S} mx \equiv_{n_{i}} t^{2}_{i}(g^{2}) \right).
\end{array} \right. \]

This case is simple. Because
\[ Z \models \exists x \left( \bigwedge_{i \in S} mx \equiv_{n_{i}} t^{2}_{i}(g^{2}) \right) \]
is always true, (b) is equivalent to a quantifier-free formula by lemma (2.2) and that $H$ has quantifier elimination.

(c), (d) are similar as (b). \qed

As mentioned in the proof of Theorem 2.4, quantifier elimination for $H \times Z$ depends only on the theories of $H$ and $Q$.

Corollary 2.5 Suppose that $H \equiv H'$ and $K \equiv Z$, and let $\varphi(x, y)$ and $\psi(y)$ be quantifier-free formulas. If $H \times Z \models \forall y (\exists x \varphi(x, y) \leftrightarrow \psi(y))$, then $H' \times K \models \forall y (\exists x \varphi(x, y) \leftrightarrow \psi(y))$. 
3 Acknowledgements

I would like to express my gratitude to Prof. Hirotaka Kikyo for giving me these problems and valuable suggestions.

References


