On a generator of a nonstandard universe

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1. Nonstandard Universe

1.1. Superstructure

Given a set $X$, we define the \textit{iterated power set} $V_n(X)$ by

\begin{align*}
V_0(X) &= X, \\
V_{n+1}(X) &= V_n(X) \cup \mathcal{P}(V_n(X)).
\end{align*}

The \textit{superstructure} $V(X)$ is the union

$$V(X) = \bigcup_{n<\omega} V_n(X).$$

A set $X$ is said to be a \textit{base set} if

$$\emptyset \notin X \quad \text{and} \quad \forall x \in X \ x \cap V(X) = \emptyset.$$ 

In a superstructure $V(X)$, the elements of $V(X) \setminus X$ are called \textit{sets relative to $V(X)$}. We denote the structure $\langle V(X), \in \rangle$ for the language $\mathcal{L}_\in = \{\in\}$ of set theory by the same symbol $V(X)$. 
1.2. Nonstandard Universe

A nonstandard universe is a triple \((V(X), V(Y), \ast)\) such that:

1. \(X\) and \(Y\) are infinite base sets.

2. (Transfer Principle) The map \(\ast\) is a bounded elementary embedding of \(V(X)\) into \(V(Y)\): \(\ast : V(X) \rightarrow V(Y)\),

\[
V(X) \models \varphi(a, \bar{b}) \iff V(Y) \models \varphi(\ast a, \bar{b}) \quad \text{for every } \Delta_0\text{-formula } \varphi(x, \bar{y}).
\]

3. \(\ast X = Y\).

For \(a \in V(\ast X) = V(Y)\),

- \(a\) is standard if \(a = \ast x\) for some \(x \in V(X)\) and
- \(a\) is internal if \(a \in \ast x\) for some \(x \in V(X)\).

We denote the set of all internal elements in \(V(\ast X)\) by

\[
\ast V(X) = \{x \in V(\ast X) \mid x \text{ is internal}\} = \bigcup_{n<\omega} \ast V_n(X).
\]

The structure \(\ast V(X)\) is transitive over \(Y\). Then, we can simply denote by single \(\ast V(X)\) nonstandard universe.

1.3. Invariants of nonstandard Universe

The norm (of standardness) \(\text{nos}(a)\) of an internal element \(a\)

\[
\text{nos}(a) = \min \{|x| \mid a \in \ast x\}.
\]

The radius of \(\ast V(X)\) is a cardinal defined by

\[
\text{rad}(\ast V(X)) = \min \{\kappa \mid \forall y \in \ast V(X) \; \text{nos}(y) < \kappa\}.
\]

Let \(E\) be a subset of \(\ast V(X)\). We denote

\[
dcl(E) = \{w(s) \mid w \in V(X), s \in E^{<\omega}, s \in \text{dom } w\}.
\]

The length of \(\ast V(X)\) is a cardinal defined by

\[
\text{len}(\ast V(X)) = \min \{|E| \mid E \subseteq \ast V(X) \text{ and } dcl(E/\ast) = \ast V(X)\}.
\]

\(\ast V(X)\) is monogenic if \(\text{len}(\ast V(X)) = 1\). \(a\) is a generator of monogenic \(\ast V(X)\) if \(\ast V(X) = dcl(\{a\})\).

From now on, we shall consider \(\ast V(X)\) such that \(\text{rad}(\ast V(X)), \text{len}(\ast V(X)) < |V(X)|\).
2. Examples of nonstandard universe

2.1. Bounded ultrapower

Let $I$ be an index set. We define $\mathcal{P}(I)$-valued universe by

$$\hat{V}(X)^I = \{u: I \to V(X) \mid \text{ran } u \subseteq \mathcal{V}_n(X) \text{ for some } n < \omega\}$$

with truth values

$$[u = v] = \{i \in I \mid u(i) = v(i)\}, \quad [u \in v] = \{i \in I \mid u(i) \in v(i)\}, \quad [\varphi \land \psi] = [\varphi] \cap [\psi], \quad [-\varphi] = I \setminus [\varphi],$$

$$[\exists x \varphi(x)] = \bigcup \{[\varphi(u)] \mid u \in \hat{V}(X)^I\}.$$  

Let $\mathcal{U}$ be an ultrafilter over $I$. We can define Bounded ultrapower

$$\hat{V}(X)^I/\mathcal{U} = \{u/\mathcal{U} \mid u \in \mathcal{U}\},$$

where $u/\mathcal{U}$ is the equivalence class of the relation $[u = v] \in \mathcal{U}$.

For $a \in V(X)$ define $\check{a} \in \hat{V}(X)^I$ by $\check{a}: I \to \{a\}$ and $\star a = \check{a}/\mathcal{U}$. Then $\hat{V}(X)^I/\mathcal{U}$ is a nonstandard universe.

**Theorem 1.** (1) If $|I| < |V(X)|$ then $\hat{V}(X)^I/\mathcal{U}$ is monogenic.

(2) Monogenic nonstandard universe $\ast V(X)$ is isomorphic to a bounded ultrapower.

**Proof.** (1) Wlog $I$ is a set relative to $V(X)$. Then $\text{id}_I/\mathcal{U}$ is a generator of $\hat{V}(X)^I/\mathcal{U}$.

(2) Let $a$ be a generator of $\ast V(X)$. Let $I$ be a set relative to $V(X)$ such that $a \in \ast I$. Define $\mathcal{U} = \{A \subseteq I \mid a \in \ast A\}$ then $\ast V(X)$ is isomorphic to $\hat{V}(X)^I/\mathcal{U}$. \qed

Considering a generator, we have the theorem below.

**Theorem 2.** If there is a bounded elementary embedding $e: \hat{V}(X)^I/\mathcal{U} \to \hat{V}(X)^J/\mathcal{V}$, then there is $h: J \to I$ such that $\mathcal{U} = \{A \subseteq I \mid h^{-1}A \in \mathcal{V}\}$ and $e(u/\mathcal{U}) = (u \circ h)/\mathcal{V}$.

**Proof.** Let $h/\mathcal{V} = e(\text{id}_I/\mathcal{U})$. \qed
2.2. Bounded Boolean ultrapower

Let $\langle \mathfrak{B}, \wedge, \vee, \neg, 0, 1 \rangle$ be a cBa. We define $B$-valued universe by

$$\hat{V}(X)^{(B)} = \left\{ u: V(X) \to \mathfrak{B} \mid u(x) \land u(y) = 0 \text{ for } x \neq y, \sqrt{\text{ran } u = 1, \text{ supp } u \in V(X)} \right\},$$

where $\text{supp } u = \{ x \in V(X) \mid u(x) \neq 0 \}$, with truth values

$$[u = v] = \bigvee \{ u(x) \land v(x) \mid x \in V(X) \} \quad [u \in v] = \bigvee \{ u(x) \land v(y) \mid x \in y \},$$

$$[\varphi \land \psi] = [\varphi] \land [\psi], \quad \neg \varphi = \neg [\varphi],$$

$$[\exists x \varphi(x)] = \bigvee \{ [[\varphi(u)] \mid u \in \hat{V}(X)^{(B)}] \}.$$  

Let $\mathcal{U}$ be an ultrafilter of $\mathfrak{B}$. As bounded ultrapower, we define bounded Boolean ultrapower $\hat{V}(X)^{(B)}/\mathcal{U}$. Bounded ultrapower is a nonstandard universe. If $\mathfrak{B}$ is atomless then $\hat{V}(X)^{(B)}/\mathcal{U}$ is not monogenic by Theorem 1.

2.3. Bounded ultralimit

A set $\Lambda$ of subsets of a Ba $\mathfrak{B}$ is a locally atomic complete algebra (LACA) if

1. $\cup \Lambda = \mathfrak{B}$.

2. If $S_1, S_2 \in \Lambda$ then $S_1 \cup S_2 \in \Lambda$.

3. If $S \in \Lambda$ and $T \subseteq S$ then $T \in \Lambda$.

4. For every $S \in \Lambda$, there is an atomic complete regular subalgebra $C$ of $\mathfrak{B}$ such that $S \subseteq C \in \Lambda$.

We say the Boolean algebra $\cup \Lambda$ is base Boolean algebra of $\Lambda$ denoted by $\mathcal{B}(\Lambda)$.

We define $\mathcal{B}(\Lambda)$-valued universe by

$$\hat{V}(X)^{\langle \Lambda \rangle} = \left\{ u: V(X) \to \mathcal{B}(\Lambda) \mid u(x) \land u(y) = 0 \text{ for } x \neq y, \text{ ran } u \in \Lambda, \sqrt{\text{ran } u = 1, \text{ supp } u \in V(X)} \right\}$$

with truth value assignment as that of $B$-valued universe, where $\mathcal{B}(\Lambda)$ is a completion of $\mathcal{B}(\Lambda)$. 
Lemma 3. Let $\varphi$ be a statement of $\hat{V}(X)^{\langle\Lambda\rangle}$ then $[\varphi] \in \mathcal{B}(\Lambda)$. So $\hat{V}(X)^{\langle\Lambda\rangle}$ is $\mathcal{B}(\Lambda)$-valued.

Let $\mathcal{U}$ be an ultrafilter of $\mathcal{B}(\Lambda)$. As bounded ultrapower and bounded Boolean ultrapower, we define bounded Boolean ultralimit $\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}$. Bounded ultralimit is a nonstandard universe.

Theorem 4 (representation theorem). For every nonstandard universe $^*V(X)$, there is a Bounded ultralimit isomorphic to $^*V(X)$.

3. Looking for a generator outside $^*V(X)$

If $^*V(X)$ is not monogenic, there is not a generator in $^*V(X)$. We are looking for a 'generator' outside $^*V(X)$.

3.1. Ultrasheaf

Let $V^{(B)}$ be a cBa $\mathcal{B}$-valued universe of set theory: $V^{(B)} = \bigcup_\alpha V^{(B)}_\alpha$,

$V^{(B)}_\alpha = \left\{ u : \text{dom } u \rightarrow \mathcal{B} \mid \text{dom } u \subseteq \bigcup \{ V^{(B)}_\beta \mid \beta < \alpha \} \right\}$,

$\check{c} : \{ \check{x} \mid x \in c \} \rightarrow \{ 1 \}$

with truth values

$[u \in v] = \bigvee \{ v(x) \wedge [x = u] \mid x \in \text{dom } v \}$,

$[u = v] = \bigwedge \{ [x \in u] \leftrightarrow [x \in v] \mid x \in \text{dom } u \cup \text{dom } v \}$.

Inside $V^{(B)}$, we consider iterated power sets $V_n(\check{X})$, and let

$\hat{V}(X)^{(B)} = \bigcup_{n<\omega} V_n(\check{X})$. 
Let $\mathcal{U}$ be an ultrafilter of $\mathcal{B}$. As bounded ultrapower and bounded Boolean ultrapower, we define bounded Boolean sheaf $\hat{V}(X)^{(\mathfrak{B})}/\mathcal{U}$.

**Theorem 5.** The map $\star: V(X) \to \hat{V}(X)^{(\mathfrak{B})}/\mathcal{U}$ $\star a = \check{a}/\mathcal{U}$ is a bounded elementary embedding and $\mathcal{V}(X) = \bigcup_{n<\omega} \mathcal{V}_n(X)$ is isomorphic to the Boolean ultrapower $\hat{V}(X)^{(\mathfrak{B})}/\mathcal{U}$. If $\mathcal{B}$ is atomless, $\mathcal{V}(X) \neq \hat{V}(X)^{(\mathfrak{B})}/\mathcal{U}$.

Wlog, these inclusions hold:

$$\{\star x | x \in V(X)\} \subseteq \mathcal{V}(X) \subseteq \hat{V}(X)^{(\mathfrak{B})}/\mathcal{U} \subseteq V(\star X)$$

Suppose $\mathcal{B}$ is a set relative to $V(X)$. Canonical generic filter $G \in \hat{V}(X)^{(\mathfrak{B})}$ is defined by

$$\text{dom } G = \check{\mathcal{B}}, \ G(\check{b}) = b.$$ 

**Theorem 6.** For every $a \in \mathcal{V}(X)$, there is $w \in V(X)$ such that $w: B \to V(X) \setminus X$ and $a = \bigcup \bigcap w^{\mathcal{U}} G / \mathcal{U}$. $\hat{V}(X)^{(B)}/\mathcal{U}$ is the least transitive substructure of $V(\star X)$ that contains $\mathcal{V}(X) \cup \{G / \mathcal{U}\}$.

**Theorem 7.** If there is a bounded elementary embedding

$$e: \hat{V}(X)^{(\mathcal{A})}/\mathcal{U} \to \hat{V}(X)^{(\mathfrak{B})}/\mathcal{V},$$

then there is a cBa homomorphism $h: \mathcal{A} \to \mathcal{B}$ such that $\mathcal{U} = h^{-1} \mathcal{V}$ and $e(u / \mathcal{U}) = (h \circ u) / \mathcal{V}$.

**Proof.** Since $G / \mathcal{U}$ is $\mathcal{P}(\mathcal{A})$-complete ultrafilter of $\mathcal{A}$, there is a $\mathcal{P}(\mathcal{A})^V$-complete ultrafilter $H$ of $\check{\mathcal{A}}$ inside $V(X)^{(\mathfrak{B})}$ such that $H / \mathcal{V} = e(G / \mathcal{U})$. Then, we have the homomorphism $h(a) = [\check{a} \in H]$. 

Compare Theorem 2 with Theorem 7.
3.2. Generator of bounded ultralimit

Let $\widehat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}$ be a bounded ultralimit. Suppose $\mathcal{B}(\Lambda)$ is a cBa in $V(X)$. Define generator $\Gamma$ of $\widehat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}$ by

$$\Gamma = \{ \text{id}_P /\mathcal{U} \mid P \in \Lambda \text{ is a partition of unity} \} \in V(\star X).$$

**Theorem 8.** The generator $\Gamma$ of $\widehat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}$ is $^\star \Lambda$-complete ultrafilter of $^\star \mathcal{B}(\Lambda)$.

For every $a \in \widehat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}$, there is $w \in V(X)$ such that $w: \mathcal{B}(\Lambda) \rightarrow V(X) \setminus X$ and $a = \bigcup \bigcap ^w " \Gamma"$. If $\Lambda$ is the largest LACA on a cBa $\mathcal{B}$ then $\Gamma = G/\mathcal{U}$.

**Lemma 9.** Let $\Lambda$ be an LACA. There is the least LACA $\overline{\Lambda}$ such that $\Lambda \subseteq \overline{\Lambda}$ and $\mathcal{B}(\Lambda) = \mathcal{B}(\overline{\Lambda})$. If $\mathcal{U} \subseteq \overline{\mathcal{U}}$ then $\widehat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U} \cong \widehat{V}(X)^{\langle \overline{\Lambda} \rangle}/\overline{\mathcal{U}}$.

4. Questions

Let $\widehat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}$ be the least transitive substructure which contains $^\star V(X) \cup \{ \Gamma \}$.

Suppose $\widehat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}_1 = ^\star V(X) = \widehat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}_2$. Dose $\widehat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}_1$ coincide with $\widehat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}_2$?
References

