数理解析研究所講究録
第 1555 巻 2007 年 1-8

On a generator of a nonstandard universe

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1. Nonstandard Universe

1.1. Superstructure

Given a set X, we define the *iterated power set* $V_n(X)$ by

$$V_0(X) = X,$$
$$V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)).$$

The *superstructure* $V(X)$ is the union

$$V(X) = \bigcup_{n<\omega} V_n(X).$$

A set X is said to be a *base set* if

$$\varnothing \notin X \quad \text{and} \quad \forall x \in X\ x \cap V(X) = \varnothing.$$

In a superstructure $V(X)$, the elements of $V(X) \setminus X$ are called *sets relative to* $V(X)$. We denote the structure $\langle V(X), \in\rangle$ for the language $\mathcal{L}_\in = \{\in\}$ of set theory by the same symbol $V(X)$.

1.2. Nonstandard Universe

A *nonstandard universe* is a triple $\langle V(X), V(Y), \star\rangle$ such that:

(1) X and Y are infinite base sets.

(2) **(Transfer Principle)** The map $\star$ is a bounded elementary embedding of $V(X)$ into $V(Y)$: $\star\colon V(X) \to V(Y)$,

$$V(X) \models \varphi(a, \bar{b}) \quad \text{iff} \quad V(Y) \models \varphi({}^{\star}a, {}^{\star}\bar{b}) \quad \text{for every } \Delta_0\text{-formula } \varphi(x, \bar{y}).$$

(3) ${}^{\star}X = Y$.

For $a \in V({}^{\star}X) = V(Y)$,
a is *standard* if $a = {}^{\star}x$ for some $x \in V(X)$ and
a is *internal* if $a \in {}^{\star}x$ for some $x \in V(X)$.
We denote the set of all internal elements n $V({}^{\star}X)$ by

$${}^{\star}V(X) = \{x \in V({}^{\star}X) \mid x \text{ is internal}\} = \bigcup_{n<\omega} {}^{\star}V_n(X).$$

The structure ${}^{\star}V(X)$ is transitive over Y. Then, we can simply denote by single ${}^{\star}V(X)$ nonstandard universe.

1.3. Invariants of nonstandard Universe

The *norm (of standardness)* $\mathrm{nos}(a)$ of an internal element a

$$\mathrm{nos}(a) = \min\{|x| \mid a \in {}^{\star}x\}.$$

The *radius* of ${}^{\star}V(X)$ is a cardinal defined by

$$\mathrm{rad}({}^{\star}V(X)) = \min\{\kappa \mid \forall y \in {}^{\star}V(X)\ \mathrm{nos}(y) < \kappa\}.$$

Let E be a subset of ${}^{\star}V(X)$. We denote

$$\mathrm{dcl}(E) = \{w(s) \mid w \in V(X),\ s \in E^{<\omega},\ s \in \mathrm{dom}\, w\}.$$

The *length* of ${}^{\star}V(X)$ is a cardinal defined by

$$\mathrm{len}({}^{\star}V(X)) = \min\{|E| \mid E \subseteq {}^{\star}V(X) \text{ and } \mathrm{dcl}(E/\star) = {}^{\star}V(X)\}.$$

${}^{\star}V(X)$ is *monogenic* if $\mathrm{len}({}^{\star}V(X)) = 1$. a is a *generator* of monogenic ${}^{\star}V(X)$ if ${}^{\star}V(X) = \mathrm{dcl}(\{a\})$.

From now on, we shall consider ${}^{\star}V(X)$ such that $\mathrm{rad}({}^{\star}V(X))$, $\mathrm{len}({}^{\star}V(X)) < |V(X)|$.

2. Examples of nonstandard universe

2.1. Bounded ultrapower

Let I be an index set. We define $\mathcal{P}(I)$*-valued universe* by

$$\widehat{V}(X)^I = \{u\colon I \to V(X) \mid \operatorname{ran} w \subseteq V_n(X) \text{ for some } n < \omega\}$$

with truth values

$$[\![u = v]\!] = \{i \in I \mid u(i) = v(i)\}, \qquad [\![u \in v]\!] = \{i \in I \mid u(i) \in v(i)\},$$
$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\phi]\!], \qquad [\![\neg\varphi]\!] = I \setminus [\![\varphi]\!],$$
$$[\![\exists x\, \varphi(x)]\!] = \bigcup \{[\![\varphi(u)]\!] \mid u \in \widehat{V}(X)^I\}.$$

Let $\mathcal{U}$ be an ultrafilter over I. We can define *Bounded ultrapower*

$$\widehat{V}(X)^I/\mathcal{U} = \{u/\mathcal{U} \mid u \in \mathcal{U}\},$$

where $u/\mathcal{U}$ is the equivalence class of the relation $[\![u = v]\!] \in \mathcal{U}$.

For $a \in V(X)$ define $\check{a} \in \widehat{V}(X)^I$ by $\check{a}\colon I \to \{a\}$ and ${}^*a = \check{a}/\mathcal{U}$. Then $\widehat{V}(X)^I/\mathcal{U}$ is a nonstandard universe.

Theorem 1. *(1) If $|I| < |V(X)|$ then $\widehat{V}(X)^I/\mathcal{U}$ is monogenic.*
(2) Monogenic nonstandard universe ${}^*V(X)$ is isomorphic to a bounded ultrapower.

Proof. (1) Wlog I is a set relative to $V(X)$. Then $\mathrm{id}_I/\mathcal{U}$ is a generator of $\widehat{V}(X)^I/\mathcal{U}$.

(2) Let a be a generator of ${}^*V(X)$. Let I be a set relative to $V(X)$ such that $a \in {}^*I$. Define $\mathcal{U} = \{A \subseteq I \mid a \in {}^*A\}$ then ${}^*V(X)$ is isomorphic to $\widehat{V}(X)^I/\mathcal{U}$. □

Considering a generator, we have the theorem below.

Theorem 2. *If there is a bounded elementary embedding $e\colon \widehat{V}(X)^I/\mathcal{U} \to \widehat{V}(X)^J/\mathcal{V}$, then there is $h\colon J \to I$ such that $\mathcal{U} = \{A \subseteq I \mid h^{-1}``A \in \mathcal{V}\}$ and $e(u/\mathcal{U}) = (u \circ h)/\mathcal{V}$.*

Proof. Let $h/\mathcal{V} = e(\mathrm{id}_I/\mathcal{U})$. □

2.2. Bounded Boolean ultrapower

Let $\langle \mathcal{B}, \wedge, \vee, \neg, \mathbf{0}, \mathbf{1}\rangle$ be a cBa. We define *$\mathcal{B}$-valued universe* by

$$\widehat{V}(X)^{\langle \mathcal{B}\rangle} = \left\{ u\colon V(X) \to \mathcal{B} \;\middle|\; \begin{array}{l} u(x) \wedge u(y) = \mathbf{0} \text{ for } x \neq y, \\ \bigvee \operatorname{ran} u = \mathbf{1},\ \operatorname{supp} u \in V(X) \end{array} \right\},$$

where $\operatorname{supp} u = \{x \in V(X) \mid u(x) \neq \mathbf{0}\}$, with truth values

$$\begin{array}{ll} \llbracket u = v \rrbracket = \bigvee \{u(x) \wedge v(x) \mid x \in V(X)\} & \llbracket u \in v \rrbracket = \bigvee \{u(x) \wedge v(y) \mid x \in y\}, \\ \llbracket \varphi \wedge \psi \rrbracket = \llbracket \varphi \rrbracket \wedge \llbracket \phi \rrbracket, & \llbracket \neg\varphi \rrbracket = \neg \llbracket \varphi \rrbracket, \\ \llbracket \exists x\, \varphi(x) \rrbracket = \bigvee \{\llbracket \varphi(u) \rrbracket \mid u \in \widehat{V}(X)^{\langle \mathcal{B}\rangle}\}. & \end{array}$$

Let $\mathcal{U}$ be an ultrafilter of $\mathcal{B}$. As bounded ultrapower, we define *bounded Boolean ultrapower* $\widehat{V}(X)^{\langle \mathcal{B}\rangle}/\mathcal{U}$. Bounded ultrapower is a nonstandard universe. If $\mathcal{B}$ is atomless then $\widehat{V}(X)^{\langle \mathcal{B}\rangle}/\mathcal{U}$ is not monogenic by Theorem 1.

2.3. Bounded ultralimit

A set Λ of subsets of a Ba $\mathcal{B}$ is a *locally atomic complete algebra (LACA)* if

(1) $\bigcup \Lambda = \mathcal{B}$.

(2) If $S_1, S_2 \in \Lambda$ then $S_1 \cup S_2 \in \Lambda$.

(3) If $S \in \Lambda$ and $T \subseteq S$ then $T \in \Lambda$.

(4) For every $S \in \Lambda$, there is an atomic complete regular subalgebra C of $\mathcal{B}$ such that $S \subseteq C \in \Lambda$.

We say the Boolean algebra $\bigcup \Lambda$ is *base Boolean algebra* of Λ denoted by $\mathcal{B}(\Lambda)$.

We define *$\overline{\mathcal{B}(\Lambda)}$-valued universe* by

$$\widehat{V}(X)^{\langle\!\langle \Lambda \rangle\!\rangle} = \left\{ u\colon V(X) \to \mathcal{B}(\Lambda) \;\middle|\; \begin{array}{l} u(x) \wedge u(y) = \mathbf{0} \text{ for } x \neq y,\ \operatorname{ran} u \in \Lambda \\ \bigvee \operatorname{ran} u = \mathbf{1},\ \operatorname{supp} u \in V(X) \end{array} \right\}$$

with thuth value assignment as that of $\mathcal{B}$-valued universe, where $\overline{\mathcal{B}(\Lambda)}$ is a completion of $\mathcal{B}(\Lambda)$.

Lemma 3. *Let φ be a statement of $\widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}$ then $[\![\varphi]\!] \in \mathcal{B}(\Lambda)$. So $\widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}$ is $\mathcal{B}(\Lambda)$-valued.*

Let $\mathcal{U}$ be an ultrafilter of $\mathcal{B}(\Lambda)$. As bounded ultrapower and bounded Boolean ultrapower, we define *bounded Boolean ultralimit* $\widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathcal{U}$. Bounded ultralimit is a nonstandard universe.

Theorem 4 (representatioin theorem). *For every nonstandard universe $^*V(X)$, there is a Bounded ultralimit isomorphic to $^*V(X)$.*

3. Looking for a generator outside $^*V(X)$

If $^*V(X)$ is not monogenic, there is not a generator in $^*V(X)$. We are looking for a 'generator' outside $^*V(X)$.

3.1. Ultrasheaf

Let $\mathbf{V}^{(\mathcal{B})}$ be a cBa $\mathcal{B}$-valued universe of set theory: $\mathbf{V}^{(\mathcal{B})} = \bigcup_\alpha \mathbf{V}_\alpha^{(\mathcal{B})}$,

$$\mathbf{V}_\alpha^{(\mathcal{B})} = \left\{u\colon \operatorname{dom} u \to \mathcal{B} \;\middle|\; \operatorname{dom} u \subseteq \bigcup\{\mathbf{V}_\beta^{(\mathcal{B})} \mid \beta < \alpha\}\right\},$$
$$\check{c}\colon \{\check{x} \mid x \in c\} \to \{\mathbf{1}\}$$

with truth values

$$[\![u \in v]\!] = \bigvee \left\{v(x) \wedge [\![x = u]\!] \mid x \in \operatorname{dom} v\right\},$$
$$[\![u = v]\!] = \bigwedge \left\{[\![x \in u]\!] \Leftrightarrow [\![x \in v]\!] \mid x \in \operatorname{dom} u \cup \operatorname{dom} v\right\}.$$

Inside $\mathbf{V}^{(\mathcal{B})}$, we consider iterated power sets $V_{\check{n}}(\check{X})$, and let

$$\widehat{V}(X)^{(\mathcal{B})} = \bigcup_{n<\omega} V_{\check{n}}(\check{X}).$$

Let $\mathcal{U}$ be an ultrafilter of $\mathcal{B}$. As bounded ultrapower and bounded Boolean ultrapower, we define *bounded Boolean sheaf* $\hat{V}(X)^{(\mathcal{B})}/\mathcal{U}$.

Theorem 5. *The map* $\star\colon V(X) \to \hat{V}(X)^{(\mathcal{B})}/\mathcal{U} \quad {}^\star a = \check{a}/\mathcal{U}$ *is a bounded elementary embedding and* ${}^\star V(X) = \bigcup_{n<\omega} {}^\star V_n(X)$ *is isomorphic to the Boolean ultrapower* $\hat{V}(X)^{(\mathcal{B})}/\mathcal{U}$. *If* $\mathcal{B}$ *is atomless,* ${}^\star V(X) \neq \hat{V}(X)^{(\mathcal{B})}/\mathcal{U}$.

Wlog, these inclusions hold:

$$\{{}^\star x \mid x \in V(X)\} \subsetneq {}^\star V(X) \subsetneq \hat{V}(X)^{(\mathcal{B})}/\mathcal{U} \subsetneq V({}^\star X)$$

Suppose $\mathcal{B}$ is a set relative to $V(X)$. *Canonial generic filter* $\mathbb{G} \in \hat{V}(X)^{(\mathcal{B})}$ is defined by

$$\operatorname{dom}\mathbb{G} = \check{\mathcal{B}}, \quad \mathbb{G}(\check{b}) = b.$$

Theorem 6. *For every* $a \in {}^\star V(X)$, *there is* $w \in V(X)$ *such that* $w\colon \mathcal{B} \to V(X) \setminus X$ *and* $a = \bigcup \bigcap {}^\star w``\mathbb{G}/\mathcal{U}$. $\hat{V}(X)^{(\mathcal{B})}/\mathcal{U}$ *is the least transitive substructure of* $V({}^\star X)$ *that contains* ${}^\star V(X) \cup \{\mathbb{G}/\mathcal{U}\}$.

Theorem 7. *If there is a bounded elementary embedding*

$$e\colon \hat{V}(X)^{(\mathcal{A})}/\mathcal{U} \to \hat{V}(X)^{(\mathcal{B})}/\mathcal{V},$$

then there is a cBa homomorphism $h\colon \mathcal{A} \to \mathcal{B}$ *such that* $\mathcal{U} = h^{-1}``\mathcal{V}$ *and* $e(u/\mathcal{U}) = (h \circ u)/\mathcal{V}$.

Proof. Since $\mathbb{G}/\mathcal{U}$ is ${}^\star\mathcal{P}(\mathcal{A})$-complete ultrafilter of $\mathcal{A}$, there is a $\mathcal{P}(\mathcal{A})^\checkmark$-complete ultrafilter H of $\check{\mathcal{A}}$ inside $V(X)^{(\mathcal{B})}$ such that $H/\mathcal{V} = e(\mathbb{G}/\mathcal{U})$. Then, we have the homomorphism $h(a) = [\![\check{a} \in H]\!]$. □

Compare Theorem 2 with Theorem 7.

3.2. Generator of bounded ultralimit

Let $\widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathfrak{U}$ be a bounded ultralimit. Suppose $\mathfrak{B}(\Lambda)$ is a cBa in $V(X)$. Define *generator* Γ of $\widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathfrak{U}$ by

$$\Gamma = \left\{ \mathrm{id}_P /\mathfrak{U} \;\middle|\; P \in \Lambda \text{ is a partition of unity} \right\} \in V({}^{\star}X).$$

Theorem 8. *The generator Γ of $\widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathfrak{U}$ is ${}^{\star}\Lambda$-complete ultrafilter of ${}^{\star}\mathfrak{B}(\Lambda)$. For every $a \in \widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathfrak{U}$, there is $w \in V(X)$ such that $w\colon \mathfrak{B}(\Lambda) \to V(X) \setminus X$ and $a = \bigcup\bigcap {}^{\star}w``\Gamma$. If Λ is the largest LACA on a cBa $\mathfrak{B}$ then $\Gamma = \mathbb{G}/\mathfrak{U}$.*

Lemma 9. *Let Λ be an LACA. There is the least LACA $\overline{\Lambda}$ such that $\Lambda \subseteq \overline{\Lambda}$ and $\mathfrak{B}(\overline{\Lambda}) = \overline{\mathfrak{B}(\Lambda)}$. If $\mathfrak{U} \subseteq \overline{\mathfrak{U}}$ then $\widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathfrak{U} \cong \widehat{V}(X)^{\langle\langle\overline{\Lambda}\rangle\rangle}/\overline{\mathfrak{U}}$.*

4. Questions

Let $\widehat{V}(X)^{(\langle\Lambda\rangle)}/\mathfrak{U}$ be the least transitive substructure which contains ${}^{\star}V(X) \cup \{\Gamma\}$.

Suppose $\widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathfrak{U}_1 = {}^{\star}V(X) = \widehat{V}(X)^{\langle\langle\Lambda\rangle\rangle}/\mathfrak{U}_2$. Dose $\widehat{V}(X)^{(\langle\Lambda\rangle)}/\mathfrak{U}_1$ coincide with $\widehat{V}(X)^{(\langle\Lambda\rangle)}/\mathfrak{U}_2$?

References

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