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On a generator of a nonstandard universe

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1. Nonstandard Universe

1.1. Superstructure

Given a set $X$, we define the *iterated power set* $V_n(X)$ by

\[ V_0(X) = X, \]
\[ V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)). \]

The *superstructure* $V(X)$ is the union

\[ V(X) = \bigcup_{n<\omega} V_n(X). \]

A set $X$ is said to be a *base set* if

\[ \emptyset \notin X \quad \text{and} \quad \forall x \in X \ x \cap V(X) = \emptyset. \]

In a superstructure $V(X)$, the elements of $V(X) \setminus X$ are called *sets relative to* $V(X)$. We denote the structure $\langle V(X), \in \rangle$ for the language $\mathcal{L}_\in = \{\in\}$ of set theory by the same symbol $V(X)$. 
1.2. Nonstandard Universe

A nonstandard universe is a triple $(V(X), V(Y), *)$ such that:

1. $X$ and $Y$ are infinite base sets.

2. (Transfer Principle) The map $*$ is a bounded elementary embedding of $V(X)$ into $V(Y)$: $*: V(X) \rightarrow V(Y)$,
   \[ V(X) \models \varphi(a, \overline{b}) \text{ iff } V(Y) \models \varphi(*)a, \overline{b} \text{ for every } \Delta_0\text{-formula } \varphi(x, \overline{y}). \]

3. $*X = Y$.

For $a \in V(*X) = V(Y)$,
- $a$ is standard if $a = *x$ for some $x \in V(X)$ and
- $a$ is internal if $a \in *x$ for some $x \in V(X)$.

We denote the set of all internal elements in $V(*X)$ by
\[ *V(X) = \{ x \in V(*X) \mid x \text{ is internal} \} = \bigcup_{n<\omega} *V_n(X). \]

The structure $*V(X)$ is transitive over $Y$. Then, we can simply denote by single $*V(X)$ nonstandard universe.

1.3. Invariants of nonstandard Universe

The norm (of standardness) $\text{nos}(a)$ of an internal element $a$
\[ \text{nos}(a) = \min \{ |x| \mid a \in *x \}. \]

The radius of $*V(X)$ is a cardinal defined by
\[ \text{rad}(*V(X)) = \min \{ \kappa \mid \forall y \in *V(X) \text{ nos}(y) < \kappa \}. \]

Let $E$ be a subset of $*V(X)$. We denote
\[ \text{dcl}(E) = \{ w(s) \mid w \in V(X), s \in E^{<\omega}, s \in \text{dom } w \}. \]

The length of $*V(X)$ is a cardinal defined by
\[ \text{len}(*V(X)) = \min \{ |E| \mid E \subseteq *V(X) \text{ and } \text{dcl}(E/*) = *V(X) \}. \]

$*V(X)$ is monogenic if $\text{len}(*V(X)) = 1$. $a$ is a generator of monogenic $*V(X)$ if $*V(X) = \text{dcl}(\{a\})$.

From now on, we shall consider $*V(X)$ such that $\text{rad}(*V(X)), \text{len}(*V(X)) < |V(X)|$. 

2. Examples of nonstandard universe

2.1. Bounded ultrapower

Let $I$ be an index set. We define \( \mathcal{P}(I) \)-valued universe by

\[
\hat{V}(X)^I = \{ u: I \to V(X) \mid \text{ran } u \subseteq V_n(X) \text{ for some } n < \omega \}
\]

with truth values

\[
[u = v] = \{ i \in I \mid u(i) = v(i) \}, \quad [u \in v] = \{ i \in I \mid u(i) \in v(i) \},
\]

\[
[\varphi \land \psi] = [\varphi] \cap [\psi], \quad [\lnot \varphi] = I \setminus [\varphi],
\]

\[
[\exists x \varphi(x)] = \bigcup \{ [\varphi(u)] \mid u \in \hat{V}(X)^I \}.
\]

Let \( \mathcal{U} \) be an ultrafilter over \( I \). We can define Bounded ultrapower

\[
\hat{V}(X)^I / \mathcal{U} = \{ u/\mathcal{U} \mid u \in \mathcal{U} \},
\]

where \( u/\mathcal{U} \) is the equivalence class of the relation \( [u = v] \in \mathcal{U} \).

For \( a \in V(X) \) define \( \check{a} \in \hat{V}(X)^I \) by \( \check{a}: I \to \{ a \} \) and \( \star a = \check{a}/\mathcal{U} \). Then \( \hat{V}(X)^I / \mathcal{U} \) is a nonstandard universe.

**Theorem 1.** (1) If \( |I| < |V(X)| \) then \( \hat{V}(X)^I / \mathcal{U} \) is monogenic.

(2) Monogenic nonstandard universe \( \star V(X) \) is isomorphic to a bounded ultrapower.

**Proof.** (1) Wlog \( I \) is a set relative to \( V(X) \). Then \( \text{id}_I / \mathcal{U} \) is a generator of \( \hat{V}(X)^I / \mathcal{U} \).

(2) Let \( a \) be a generator of \( \star V(X) \). Let \( I \) be a set relative to \( V(X) \) such that \( a \in \star I \). Define \( \mathcal{U} = \{ A \subseteq I \mid a \in \star A \} \) then \( \star V(X) \) is isomorphic to \( \hat{V}(X)^I / \mathcal{U} \). \( \square \)

Considering a generator, we have the theorem below.

**Theorem 2.** If there is a bounded elementary embedding \( e : \hat{V}(X)^I / \mathcal{U} \to \hat{V}(X)^J / V \), then there is \( h: J \to I \) such that \( \mathcal{U} = \{ A \subseteq I \mid h^{-1} A \in V \} \) and \( e(u/\mathcal{U}) = (u \circ h)/V \).

**Proof.** Let \( h/V = e(\text{id}_I / \mathcal{U}) \). \( \square \)
2.2. Bounded Boolean ultrapower

Let \( \langle \mathfrak{B}, \land, \lor, \neg, 0, 1 \rangle \) be a cBa. We define \( \mathcal{B} \)-valued universe by

\[
\hat{V}(X)^{\langle \mathfrak{B} \rangle} = \left\{ u: V(X) \to \mathfrak{B} \mid u(x) \land u(y) = 0 \text{ for } x \neq y, \sqrt{\text{ran } u = 1}, \text{ supp } u \in V(X) \right\},
\]

where \( \text{supp } u = \{ x \in V(X) \mid u(x) \neq 0 \} \), with truth values

\[
[u = v] = \bigvee \{ u(x) \land v(x) \mid x \in V(X) \}, \quad [u \in v] = \bigvee \{ u(x) \land v(y) \mid x \in y \},
\]

\[
[\varphi \land \psi] = [\varphi] \land [\psi], \quad [\neg \varphi] = \neg [\varphi],
\]

\[
[\exists x \varphi(x)] = \bigvee \{ [[\varphi(u)] \mid u \in \hat{V}(X)^{\langle \mathfrak{B} \rangle} \}.
\]

Let \( \mathcal{U} \) be an ultrafilter of \( \mathfrak{B} \). As bounded ultrapower, we define bounded Boolean ultrapower \( \hat{V}(X)^{\langle \mathfrak{B} \rangle}/\mathcal{U} \). Bounded ultrapower is a nonstandard universe. If \( \mathfrak{B} \) is atomless then \( \hat{V}(X)^{\langle \mathfrak{B} \rangle}/\mathcal{U} \) is not monogenic by Theorem 1.

2.3. Bounded ultralimit

A set \( \Lambda \) of subsets of a Ba \( \mathfrak{B} \) is a locally atomic complete algebra (LACA) if

1. \( \cup \Lambda = \mathfrak{B} \).

2. If \( S_1, S_2 \in \Lambda \) then \( S_1 \cup S_2 \in \Lambda \).

3. If \( S \in \Lambda \) and \( T \subseteq S \) then \( T \in \Lambda \).

4. For every \( S \in \Lambda \), there is an atomic complete regular subalgebra \( C \) of \( \mathfrak{B} \) such that \( S \subseteq C \in \Lambda \).

We say the Boolean algebra \( \cup \Lambda \) is base Boolean algebra of \( \Lambda \) denoted by \( \mathcal{B}(\Lambda) \).

We define \( \overline{\mathcal{B}(\Lambda)} \)-valued universe by

\[
\hat{V}(X)^{\langle \Lambda \rangle} = \left\{ u: V(X) \to \mathcal{B}(\Lambda) \mid u(x) \land u(y) = 0 \text{ for } x \neq y, \sqrt{\text{ran } u = 1}, \text{ supp } u \in V(X) \right\}
\]

with thuth value assignment as that of \( \mathcal{B} \)-valued universe, where \( \overline{\mathcal{B}(\Lambda)} \) is a completion of \( \mathcal{B}(\Lambda) \).
Lemma 3. Let $\varphi$ be a statement of $\hat{V}(X)^{\langle\Lambda\rangle}$ then $[\varphi] \in \mathcal{B}(\Lambda)$. So $\hat{V}(X)^{\langle\Lambda\rangle}$ is $\mathcal{B}(\Lambda)$-valued.

Let $\mathcal{U}$ be an ultrafilter of $\mathcal{B}(\Lambda)$. As bounded ultrapower and bounded Boolean ultrapower, we define bounded Boolean ultralimit $\hat{V}(X)^{\langle\Lambda\rangle}/\mathcal{U}$. Bounded ultralimit is a nonstandard universe.

Theorem 4 (representation theorem). For every nonstandard universe $^*V(X)$, there is a Bounded ultralimit isomorphic to $^*V(X)$.

3. Looking for a generator outside $^*V(X)$

If $^*V(X)$ is not monogenic, there is not a generator in $^*V(X)$. We are looking for a 'generator' outside $^*V(X)$.

3.1. Ultrasheaf

Let $V^{(B)}$ be a cBa $\mathcal{B}$-valued universe of set theory: $V^{(B)} = \bigcup_{\alpha} V^{(B)}_{\alpha}$,

$$V^{(B)}_{\alpha} = \{ u: \text{dom } u \rightarrow \mathcal{B} \mid \text{dom } u \subseteq \bigcup \{ V^{(B)}_{\beta} \mid \beta < \alpha \} \},$$

$$\check{c}: \{ x \mid x \in c \} \rightarrow \{ 1 \}$$

with truth values

$$[u \in v] = \bigvee \{ v(x) \land [x = u] \mid x \in \text{dom } v \},$$

$$[u = v] = \bigwedge \{ [x \in u] \leftrightarrow [x \in v] \mid x \in \text{dom } u \cup \text{dom } v \}.$$ 

Inside $V^{(B)}$, we consider iterated power sets $V_\alpha(X)$, and let

$$\hat{V}(X)^{(B)} = \bigcup_{n < \omega} V_\alpha(X).$$
Let $\mathcal{U}$ be an ultrafilter of $\mathcal{B}$. As bounded ultrapower and bounded Boolean ultrapower, we define bounded Boolean sheaf $\hat{V}(X)^{(\mathcal{B})}/\mathcal{U}$.

**Theorem 5.** The map $\star: V(X) \to \hat{V}(X)^{(\mathcal{B})}/\mathcal{U}$ $\star a = \check{a}/\mathcal{U}$ is a bounded elementary embedding and $\mathcal{V}(X) = \bigcup_{n<\omega} \mathcal{V}_n(X)$ is isomorphic to the Boolean ultrapower $\hat{V}(X)^{(\mathcal{B})}/\mathcal{U}$. If $\mathcal{B}$ is atomless, $\mathcal{V}(X) \neq \hat{V}(X)^{(\mathcal{B})}/\mathcal{U}$.

Wlog, these inclusions hold:

$$\{x | x \in V(X)\} \subseteq \mathcal{V}(X) \subseteq \hat{V}(X)^{(\mathcal{B})}/\mathcal{U} \subseteq V(\star X)$$

Suppose $\mathcal{B}$ is a set relative to $V(X)$. Canonical generic filter $G \in \hat{V}(X)^{(\mathcal{B})}$ is defined by

$$\text{dom } G = \check{\mathcal{B}}, \quad G(\check{b}) = b.$$ 

**Theorem 6.** For every $a \in \mathcal{V}(X)$, there is $w \in V(X)$ such that $w: \mathcal{B} \to V(X) \setminus X$ and $a = \bigcup \bigcap \mathcal{V}^w G/\mathcal{U}$. $\hat{V}(X)^{(\mathcal{B})}/\mathcal{U}$ is the least transitive substructure of $V(\star X)$ that contains $\mathcal{V}(X) \cup \{G/\mathcal{U}\}$.

**Theorem 7.** If there is a bounded elementary embedding

$$e: \hat{V}(X)^{(\mathcal{A})}/\mathcal{U} \to \hat{V}(X)^{(\mathcal{B})}/\mathcal{V},$$

then there is a $cBa$ homomorphism $h: \mathcal{A} \to \mathcal{B}$ such that $\mathcal{U} = h^{-1}\mathcal{V}$ and $e(u/\mathcal{U}) = (h \circ u)/\mathcal{V}$.

**Proof.** Since $G/\mathcal{U}$ is $\mathcal{P}(\mathcal{A})$-complete ultrafilter of $\mathcal{A}$, there is a $\mathcal{P}(\mathcal{A})^\vee$-complete ultrafilter $H$ of $\hat{\mathcal{A}}$ inside $V(X)^{(\mathcal{B})}$ such that $H/\mathcal{V} = e(G/\mathcal{U})$. Then, we have the homomorphism $h(a) = [a \in H]$.

Compare Theorem 2 with Theorem 7.
3.2. Generator of bounded ultralimit

Let $\hat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}$ be a bounded ultralimit. Suppose $\mathcal{B}(\Lambda)$ is a cBa in $V(X)$. Define generator $\Gamma$ of $\hat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}$ by

$$\Gamma = \{ \text{id}_P /\mathcal{U} | P \in \Lambda \text{ is a partition of unity} \} \in V(\star X).$$

**Theorem 8.** The generator $\Gamma$ of $\hat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}$ is $\star \Lambda$-complete ultrafilter of $\mathcal{B}(\Lambda)$. For every $a \in \hat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}$, there is $w \in V(X)$ such that $w: \mathcal{B}(\Lambda) \to V(X) \setminus X$ and $a = \bigcup \cap \text{"}w" \Gamma$. If $\Lambda$ is the largest LACA on a cBa $\mathfrak{B}$ then $\Gamma = G/\mathcal{U}$.

**Lemma 9.** Let $\Lambda$ be an LACA. There is the least LACA $\overline{\Lambda}$ such that $\Lambda \subseteq \overline{\Lambda}$ and $\mathcal{B}(\overline{\Lambda}) = \overline{\mathcal{B}(\Lambda)}$. If $\mathcal{U} \subseteq \overline{\mathcal{U}}$ then $\hat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U} \cong \hat{V}(X)^{\langle \overline{\Lambda} \rangle}/\overline{\mathcal{U}}$.

4. Questions

Let $\hat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}$ be the least transitive substructure which contains $\mathcal{V}(X) \cup \{ \Gamma \}$.

Suppose $\hat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}_1 = \mathcal{V}(X) = \hat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}_2$. Does $\hat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}_1$ coincide with $\hat{V}(X)^{\langle \Lambda \rangle}/\mathcal{U}_2$?
References

