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# Bayesian Communication Leading to a Nash Equilibrium in Belief *

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**Abstract.** A Bayesian communication in the $p$-belief system is presented which leads to a Nash equilibrium of a strategic form game through messages as a Bayesian updating process. In the communication process each player predicts the other players' actions under his/her private information with probability at least his/her belief. The players communicate privately their conjectures through message according to the communication graph, where each player receiving the message learns and revises his/her conjecture. The emphasis is on that both any topological assumptions on the communication graph and any common-knowledge assumptions on the structure of communication are not required.
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## 1 Introduction

This article relates equilibria and distributed knowledge. In game theoretical situations among a group of players, the concept of mixed strategy Nash equilibrium has become central. Yet little is known the process by which players learn if they do. This article will give a protocol run by the mutual learning of their beliefs of players' actions, and it highlights an epistemic aspect of Bayesian updating process leading to a mixed strategy Nash equilibrium for a strategic form game.

As for as J.F. Nash [8]'s fundamental notion of strategic equilibrium is concerned, R.J. Aumann and A. Brandenburger [1] gives epistemic conditions for mixed strategy Nash equilibrium: They show that the common-knowledge of the predictions of the players having the partition information (that is, equivalently,

* This paper was presented in WINE 2005, 15-17 December 2005, Hong Kong, China ([6])

the **S5**-knowledge model) yields a Nash equilibrium of a game. However it is not clear just what learning process leads to the equilibrium. The present article aims to fill this gap from epistemic point of view.

Our real concern is with what Bayesian learning process leads to a mixed strategy Nash equilibrium of a finite strategic form game with emphasis on the epistemic point of view. We focus on the Bayesian belief revision through communication among group of players. We show that

**Main theorem.** *Suppose that the players in a strategic form game have the p-belief system with a common prior distribution. In a communication process of the game according to a protocol with revisions of their beliefs about the other players' actions, the profile of their future predictions induces a mixed strategy Nash equilibrium of the game in the long run.*

Let us consider the following protocol: The players start with the same prior distribution on a state-space. In addition they have private information given by a partition of the state space. Beliefs of players are posterior probabilities: A player *p-believes* ( simply, *believes*) an event with $0 < p \leq 1$ if the posterior probability of the event given his/her information is at least $p$. Each player predicts the other players' actions as his/her belief of the actions. He/she communicates privately their beliefs about the other players' actions through messages, and the receivers update their belief according to the messages. Precisely, the players are assumed to be rational and maximizing their expected utility according their beliefs at every stage. Each player communicates privately his/her belief about the others' actions as messages according to a protocol,[3] and the receivers update their private information and revise their belief.

The main theorem says that the players' predictions regarding the future beliefs converge in the long run, which lead to a mixed strategy Nash equilibrium of a game. The emphasis is on the two points: First that each player's prediction is not required to be common-knowledge among all players, and secondly that each player send to the another player not the exact information about his/her belief about the actions for the other players but the approximate information about the the other players' actions with probability at lest his/her belief of the others' actions.

This paper is organized as follows: In section 2 we give the formal model of the Bayesian communication on a game. Section 3 states explicitly our theorem and gives a sketch of the proof. In final section 4 we conclude some remarks. We are planning to present a small example to illustrate the theorem in our lecture presentation in the Kyoto Symposium 'Mathematical Economics.'

## 2 The Model

Let $\Omega$ be a non-empty *finite* set called a *state-space*, $N$ a set of finitely many *players* $\{1, 2, \ldots n\}$ at least two ($n \geq 2$), and let $2^{\Omega}$ be the family of all subsets

[3] When a player communicates with another, the other players are not informed about the contents of the message.

of $\Omega$. Each member of $2^{\Omega}$ is called an *event* and each element of $\Omega$ called a *state*. Let $\mu$ be a probability measure on $\Omega$ which is common for all players. For simplicity it is assumed that $(\Omega, \mu)$ is a *finite* probability space with $\mu$ *full support.*[4]

### 2.1 $p$-Belief System[5]

Let $p$ be a real number with $0 < p \leq 1$. The *p-belief system* associated with the partition information structure $(\Pi_i)_{i\in N}$ is the tuple $\langle N, \Omega, \mu, (\Pi_i)_{i\in N}, (B_i(*, p))_{i\in N}\rangle$ consisting of the following structures and interpretations: $(\Omega, \mu)$ is a finite probability space, and $i$'s *p-belief operator* $B_i(*; p)$ is the operator on $2^{\Omega}$ such that $B_i(E, p)$ is the set of states of $\Omega$ in which $i$ *p-believes* that $E$ has occurred with probability at least $p$ ; that is, $B_i(E; p) := \{\omega \in \Omega \mid \mu(E \mid \Pi_i(\omega)) \geq p\}$.

*Remark 1.* When $p = 1$ the 1-belief operator $B_i(*; 1)$ becomes knowledge operator.

### 2.2 Game on $p$-Belief System[6]

By a *game G* we mean a *finite* strategic form game $\langle N, (A_i)_{i\in N}, (g_i)_{i\in N}\rangle$ with the following structure and interpretations: $N$ is a finite set of players $\{1, 2, \dots, i, \dots n\}$ with $n \geq 2$, $A_i$ is a finite set of $i$'s *actions* (or $i$'s pure strategies) and $g_i$ is an $i$'s *payoff function* of $A$ into $\mathbb{R}$, where $A$ denotes the product $A_1 \times A_2 \times \cdots \times A_n$, $A_{-i}$ the product $A_1 \times A_2 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n$. We denote by $g$ the $n$-tuple $(g_1, g_2, \dots g_n)$ and by $a_{-i}$ the $(n-1)$-tuple $(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ for $a$ of $A$. Furthermore we denote $a_{-I} = (a_i)_{i\in N\setminus I}$ for each $I \subset N$.

A probability distribution $\phi_i$ on $A_{-i}$ is said to be $i$'s *overall conjecture* (or simply $i$'s *conjecture*). For each player $j$ other than $i$, this induces the marginal distribution on $j$'s actions; we call it $i$'s *individual conjecture* about $j$ (or simply $i$'s conjecture *about* $j$.) Functions on $\Omega$ are viewed like random variables in the probability space $(\Omega, \mu)$. If $\mathbf{x}$ is a such function and $x$ is a value of it, we denote by $[\mathbf{x} = x]$ (or simply by $[x]$) the set $\{\omega \in \Omega | \mathbf{x}(\omega) = x\}$.

The information structure $(\Pi_i)$ with a common prior $\mu$ yields the distribution on $A \times \Omega$ defined by $\mathbf{q}_i(a, \omega) = \mu([\mathbf{a} = a] | \Pi_i(\omega))$; and the $i$'s overall conjecture defined by the marginal distribution $\mathbf{q}_i(a_{-i}, \omega) = \mu([\mathbf{a}_{-i} = a_{-i}] | \Pi_i(\omega))$ which is viewed as a random variable of $\phi_i$. We denote by $[\mathbf{q}_i = \phi_i]$ the intersection $\bigcap_{a_{-i}\in A_{-i}}[\mathbf{q}_i(a_{-i}) = \phi_i(a_{-i})]$ and denote by $[\phi]$ the intersection $\bigcap_{i\in N}[\mathbf{q}_i = \phi_i]$. Let $\mathbf{g}_i$ be a random variable of $i$'s payoff function $g_i$ and $\mathbf{a}_i$ a random variable of an $i$'s action $a_i$. Where we assume that $\Pi_i(\omega) \subseteq [a_i] := [\mathbf{a}_i = a_i]$ for all $\omega \in [a_i]$ and for every $a_i$ of $A_i$. $i$'s action $a_i$ is said to be *actual* at a state $\omega$ if $\omega \in [\mathbf{a}_i = a_i]$; and the profile $a_I$ is said to be actually played at $\omega$ if $\omega \in [\mathbf{a}_I = a_I] := \bigcap_{i\in I}[\mathbf{a}_i = a_i]$ for $I \subset N$. The pay off functions $g = (g_1, g_2, \dots, g_n)$ is said to be *actually*

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[4] That is; $\mu(\omega) \neq 0$ for every $\omega \in \Omega$.

[5] Monderer and Samet [7].

[6] Aumann and Brandenburger [1]

*played* at a state $\omega$ if $\omega \in [\mathbf{g} = g] := \bigcap_{i\in N}[\mathbf{g}_i = g_i]$. Let **Exp** denote the expectation defined by $\mathbf{Exp}(g_i(b_i, \mathbf{a}_{-i}); \omega) := \sum_{a_{-i}\in A_{-i}} g_i(b_i, a_{-i})\ \mathbf{q}_i(a_{-i}, \omega)$.

A player $i$ is said to be *rational* at $\omega$ if each $i$'s actual action $a_i$ maximizes the expectation of his actually played pay off function $g_i$ at $\omega$ when the other players actions are distributed according to his conjecture $\mathbf{q}_i(\cdot\,;\omega)$. Formally, letting $g_i = \mathbf{g}_i(\omega)$ and $a_i = \mathbf{a}_i(\omega)$, $\mathbf{Exp}(g_i(a_i, \mathbf{a}_{-i}); \omega) \geq \mathbf{Exp}(g_i(b_i, \mathbf{a}_{-i}); \omega)$ for every $b_i$ in $A_i$. Let $R_i$ denote the set of all of the states at which $i$ is rational.

## 2.3 Protocol [7]

We assume that the players communicate by sending *messages*. Let $T$ be the time horizontal line $\{0, 1, 2, \cdots t, \cdots\}$. A *protocol* is a mapping $\mathrm{Pr} : T \to N \times N, t \mapsto (s(t), r(t))$ such that $s(t) \neq r(t)$. Here $t$ stands for *time* and $s(t)$ and $r(t)$ are, respectively, the *sender* and the *receiver* of the communication which takes place at time $t$. We consider the protocol as the directed graph whose vertices are the set of all players $N$ and such that there is an edge (or an arc) from $i$ to $j$ if and only if there are infinitely many $t$ such that $s(t) = i$ and $r(t) = j$.

A protocol is said to be *fair* if the graph is strongly-connected; in words, every player in this protocol communicates directly or indirectly with every other player infinitely often. It is said to contain a *cycle* if there are players $i_1, i_2, \ldots, i_k$ with $k \geq 3$ such that for all $m < k$, $i_m$ communicates directly with $i_{m+1}$, and such that $i_k$ communicates directly with $i_1$. The communications is assumed to proceed in *rounds*[8]

## 2.4 Communication on $p$-Belief System

A *Bayesian belief communication process* $\pi(G)$ with revisions of players' conjectures $(\phi_i^t)_{(i,t)\in N\times T}$ according to a protocol for a game $G$ is a tuple

$$\pi(G) = \langle \mathrm{Pr}, (\Pi_i^t)_{i\in N}, (B_i^t)_{i\in N}, (\phi_i^t)_{(i,t)\in N\times T} \rangle$$

with the following structures: the players have a common prior $\mu$ on $\Omega$, the protocol Pr among $N$, $\mathrm{Pr}(t) = (s(t), r(t))$, is fair and it satisfies the conditions that $r(t) = s(t+1)$ for every $t$ and that the communications proceed in rounds. The revised information structure $\Pi_i^t$ at time $t$ is the mapping of $\Omega$ into $2^\Omega$ for player $i$. If $i = s(t)$ is a sender at $t$, the message sent by $i$ to $j = r(t)$ is $M_i^t$. An $n$-tuple $(\phi_i^t)_{i\in N}$ is a revision process of individual conjectures. These structures are inductively defined as follows:

- Set $\Pi_i^0(\omega) = \Pi_i(\omega)$.
- Assume that $\Pi_i^t$ is defined. It yields the distribution $\mathbf{q}_i^t(a, \omega) = \mu([\mathbf{a} = a] | \Pi_i^t(\omega))$. Whence

[7] C.f.: Parikh and Krasucki [9]

[8] There exists a time $m$ such that for all $t$, $\mathrm{Pr}(t) = \mathrm{Pr}(t + m)$. The *period* of the protocol is the minimal number of all $m$ such that for every $t$, $\mathrm{Pr}(t + m) = \mathrm{Pr}(t)$.

- $R_i^t$ denotes the set of all the state $\omega$ at which $i$ is *rational* according to his conjecture $\mathbf{q}_i^t(\cdot\,;\omega)$; that is, each $i$'s actual action $a_i$ maximizes the expectation of his pay off function $g_i$ being actually played at $\omega$ when the other players actions are distributed according to his conjecture $\mathbf{q}_i^t(\cdot\,;\omega)$ at time $t$.[9]
- The message $M_i^t : \Omega \to 2^\Omega$ sent by the sender $i$ at time $t$ is defined by

$$M_i^t(\omega) = \bigcap_{a_{-i}\in A_{-i}} B_i^t([a_{-i}]; \mathbf{q}_i^t(a_{-i},\omega)),$$

  where $B_i^t : 2^\Omega \to 2^\Omega$ is defined by

$$B_i^t(E;p) = \{\omega \in \Omega \mid \mu(E \mid \Pi_i^t(\omega)) \geq p\ \}.$$

  Then:

- The revised partition $\Pi_i^{t+1}$ at time $t+1$ is defined as follows:
  - $\Pi_i^{t+1}(\omega) = \Pi_i^t(\omega) \cap M_{s(t)}^t(\omega)$ if $i = r(t)$;
  - $\Pi_i^{t+1}(\omega) = \Pi_i^t(\omega)$ otherwise,
- The revision process $(\phi_i^t)_{(i,t)\in N\times T}$ of conjectures is inductively defined as follows:
  - Let $\omega_0 \in \Omega$, and set $\phi_{s(0)}^0(a_{-s(0)}) := \mathbf{q}_{s(0)}^0(a_{-s(0)},\omega_0)$
  - Take $\omega_1 \in M_{s(0)}^0(\omega_0) \cap B_{r(0)}([g_{s(0)}] \cap R_{s(0)}^0; p)$,[10] and set $\phi_{s(1)}^1(a_{-s(1)}) := \mathbf{q}_{s(1)}^1(a_{-s(1)},\omega_1)$
  - Take $\omega_{t+1} \in M_{s(t)}^t(\omega_t) \cap B_{r(t)}([g_{s(t)}] \cap R_{s(t)}^t; p)$, and set $\phi_{s(t+1)}^{t+1}(a_{-s(t+1)}) := \mathbf{q}_i^{t+1}(a_{-s(t+1)},\omega_{t+1})$.

The specification is that a sender $s(t)$ at time $t$ informs the receiver $r(t)$ his/her individual conjecture about the other players' actions with a probability greater than his/her belief. The receiver revises her/his information structure under the information. She/he predicts the other players action at the state where the player $p$-believes that the sender $s(t)$ is rational, and she/he informs her/his the predictions to the other player $r(t+1)$.

We denote by $\infty$ a sufficient large $\tau$ such that for all $\omega \in \Omega$, $\mathbf{q}_i^\tau(\cdot\,;\omega) = \mathbf{q}_i^{\tau+1}(\cdot\,;\omega) = \mathbf{q}_i^{\tau+2}(\cdot\,;\omega) = \cdots$. Hence we can write $\mathbf{q}_i^\tau$ by $\mathbf{q}_i^\infty$ and $\phi_i^\tau$ by $\phi_i^\infty$.

*Remark 2.* The Bayesian belief communication is a modification of the communication model introduced by Ishikawa [3].

[9] Formally, letting $g_i = \mathbf{g}_i(\omega)$, $a_i = \mathbf{a}_i(\omega)$, the expectation at time $t$, $\mathbf{Exp}^t$, is defined by $\mathbf{Exp}^t(g_i(a_i,\mathbf{a}_{-i});\omega) := \sum_{a_{-i}\in A_{-i}} g_i(a_i,a_{-i})\ \mathbf{q}_i^t(a_{-i},\omega)$. An player $i$ is said to be rational according to his conjecture $\mathbf{q}_i^t(\cdot,\omega)$ at $\omega$ if for all $b_i$ in $A_i$, $\mathbf{Exp}^t(g_i(a_i,\mathbf{a}_{-i});\omega) \geq \mathbf{Exp}^t(g_i(b_i,\mathbf{a}_{-i});\omega)$.

[10] We denote $[g_i] := [\mathbf{g}_i = g_i]$

## 3 The Result

We can now state the main theorem :

**Theorem 1.** *Suppose that the players in a strategic form game $G$ have the p-belief system with $\mu$ a common prior. In the Bayesian belief communication process $\pi(G)$ according to a protocol among all players in the game with revisions of their conjectures $(\phi_i^t)_{(i,t)\in N\times T}$ there exists a time $\infty$ such that for each $t \geq \infty$, the n-tuple $(\phi_i^t)_{i\in N}$ induces a mixed strategy Nash equilibrium of the game.*

The proof is based on the below proposition:

**Proposition 1.** *Suppose that the players in a strategic form game have the p-belief system with $\mu$ a common prior. In the Bayesian belief communication process $\pi(G)$ in a game $G$ with revisions of their conjectures, if the protocol has no cycle then both the conjectures $\mathbf{q}_i^\infty$ and $\mathbf{q}_j^\infty$ on $A \times \Omega$ must coincide; that is, $\mathbf{q}_i^\infty(a;\omega_\infty) = \mathbf{q}_j^\infty(a;\omega_{\infty+t})$ for $(i,j) = (s(\infty), s(\infty+t))$ and for any $t = 1,2,3,\cdots$.*

*Proof.* Let us first consider the case that $(i,j) = (s(\infty), r(\infty))$. We denote

$$W_i^\infty(\omega) = \{\xi \in \Omega | M_i^\infty(\xi) = M_i^\infty(\omega)\}.$$

In view of the construction of $\{\Pi_i^t\}_{t\in T}$ we can observe that

$$\Pi_j^\infty(\xi) \subseteq W_i^\infty(\omega) \quad \text{for all } \xi \in W_i^\infty(\omega). \tag{1}$$

It immediately follows that $W_i^\infty(\omega)$ is decomposed into a disjoint union of components $\Pi_j^\infty(\xi)$ for $\xi \in \Pi_i^\infty(\omega)$;

$$W_i^\infty(\omega) = \bigcup_{k=1,2,\dots,m} \Pi_j^\infty(\xi_k) \quad \text{where } \xi_k \in W_i^\infty(\omega). \tag{2}$$

It can be observed that

$$\mu([\mathbf{a}=a] \| W_i^\infty(\omega)) = \sum_{k=1}^{m} \lambda_k \mu([\mathbf{a}=a] \| \Pi_j^\infty(\xi_k)) \tag{3}$$

for some $\lambda_k > 0$ with $\sum_{k=1}^m \lambda_k = 1$.[11] Since $\Pi_i(\omega) \subseteq [a_i]$ for all $\omega \in [a_i]$, we can observe that $\mathbf{q}_i^\infty(a_{-i};\omega) = \mathbf{q}_i^\infty(a;\omega)$. On noting that $W_i^\infty(\omega)$ is decomposed into a disjoint union of components $\Pi_i^\infty(\xi)$ for $\xi \in \Pi_i^\infty(\omega)$, we can obtain $\mathbf{q}_i^\infty(a;\omega) = \mu([\mathbf{a}=a] \| W_i^\infty(\omega)) = \mu([\mathbf{a}=a] \| \Pi_i^\infty(\xi_k))$ for any $\xi_k \in W_i^\infty(\omega)$. It follows by (3) that, for each $\omega \in \Omega$ there exists a state $\xi_\omega \in \Pi_i^\infty(\omega)$ such that $\mathbf{q}_i^\infty(a;\omega) \leq \mathbf{q}_j^\infty(a;\xi_\omega)$ for $(i,j) = (s(\infty), t(\infty))$.

On continuing this process according to the *fair* protocol, the below facts can be plainly verified: For each $\omega \in \Omega$ and for sufficient large $\tau \geq 1$,

[11] This property is called the *convexity* for the conditional probability $\mu(X|*)$ in Parikh and Krasucki [9].

1. For any $t \geq 1$, $\mathbf{q}^{\infty}_{s(\infty)}(a;\omega) \leq \mathbf{q}^{\infty}_{s(\infty+t)}(a;\xi_t)$ for some $\xi_t \in \Omega$; and
2. $\mathbf{q}^{\infty}_i(a;\omega) \leq \mathbf{q}^{\infty}_i(a;\xi) \leq \mathbf{q}^{\infty}_i(a;\zeta) \leq \cdots$ for some $\xi, \zeta, \cdots \in \Omega$.

Since $\Omega$ is finite it can be obtained that $\mathbf{q}^{\infty}_i(a;\omega_\infty) = \mathbf{q}^{\infty}_j(a;\omega_{\infty+t})$ for $(i,j) = (s(\infty), s(\infty+t))$ for every $a$, in completing the proof.

**Proof of Theorem 1:** We denote by $\Gamma(i)$ the set of all the players who directly receive the message from $i$ on $N$; i.e., $\Gamma(i) = \{\, j \in N \mid (i,j) = \Pr(t) \text{ for some } t \in T\}$. Let $F_i$ denote $[\phi^{\infty}_i] := \bigcap_{a_{-i}\in A_i}[\mathbf{q}^{\infty}_i(a_{-i};*) = \phi^{\infty}_i(a_{-i})]$. It is noted that $F_i \cap F_j \neq \emptyset$ for each $i \in N$, $j \in \Gamma(i)$.

We observe the first point that for each $i \in N$, $j \in \Gamma(i)$ and for every $a \in A$, $\mu([\mathbf{a}_{-j} = a_{-j}] \mid F_i \cap F_j) = \phi^{\infty}_j(a_{-j})$. Then summing over $a_{-i}$, we can observe that $\mu([\mathbf{a}_i = a_i] \mid F_i \cap F_j) = \phi^{\infty}_j(a_i)$ for any $a \in A$. In view of Proposition 1 it can be observed that $\phi^{\infty}_j(a_i) = \phi^{\infty}_k(a_i)$ for each $j, k, \neq i$; i.e., $\phi^{\infty}_j(a_i)$ is independent of the choices of every $j \in N$ other than $i$. We set the probability distribution $\sigma_i$ on $A_i$ by $\sigma_i(a_i) := \phi^{\infty}_j(a_i)$, and set the profile $\sigma = (\sigma_i)$.

We observe the second point that for every $a \in \prod_{i\in N} \mathrm{Supp}(\sigma_i)$, $\phi^{\infty}_i(a_{-i}) = \sigma_1(a_1)\cdots\sigma_{i-1}(a_{i-1})\sigma_{i+1}(a_{i+1})\cdots\sigma_n(a_n)$ : In fact, viewing the definition of $\sigma_i$ we shall show that $\phi^{\infty}_i(a_{-i}) = \prod_{k\in N\setminus\{i\}} \phi^{\infty}_i(a_k)$. To verify this it suffices to show that for every $k = 1, 2, \cdots, n$, $\phi^{\infty}_i(a_{-i}) = \phi^{\infty}_i(a_{-I_k})\prod_{k\in I_k\setminus\{i\}} \phi^{\infty}_i(a_k)$ : We prove it by induction on $k$. For $k = 1$ the result is immediate. Suppose it is true for $k \geq 1$. On noting the protocol is fair, we can take the sequence of sets of players $\{I_k\}_{1\leq k\leq n}$ with the following properties:

(a) $I_1 = \{i\} \subset I_2 \subset \cdots \subset I_k \subset I_{k+1} \subset \cdots \subset I_m = N$ :
(b) For every $k \in N$ there is a player $i_{k+1} \in \bigcup_{j\in I_k} \Gamma(j)$ with $I_{k+1} \setminus I_k = \{i_{k+1}\}$.

We let take $j \in I_k$ such that $i_{k+1} \in \Gamma(j)$. Set $H_{i_{k+1}} := [\mathbf{a}_{i_{k+1}} = a_{i_{k+1}}] \cap F_j \cap F_{i_{k+1}}$. It can be verified that $\mu([\mathbf{a}_{-j-i_{k+1}} = a_{-j-i_{k+1}}] \mid H_{i_{k+1}}) = \phi^{\infty}_{-j-i_{k+1}}(a_{-j})$. Dividing $\mu(F_j \cap F_{i_{k+1}})$ yields that

$$\mu([\mathbf{a}_{-j} = a_{-j}] \mid F_j \cap F_{i_{k+1}}) = \phi^{\infty}_{i_{k+1}}(a_{-j})\mu([\mathbf{a}_{i_{k+1}} = a_{i_{k+1}}] \mid F_j \cap F_{i_{k+1}}).$$

Thus $\phi^{\infty}_j(a_{-j}) = \phi^{\infty}_{i_{k+1}}(a_{-j-i_{k+1}})\phi^{t}_j(a_{i_{k+1}})$; then summing over $a_{I_k}$ we obtain $\phi^{\infty}_j(a_{-I_k}) = \phi^{\infty}_{i_{k+1}}(a_{-I_k-i_{k+1}})\phi^{\infty}_j(a_{i_{k+1}})$. It immediately follows from Proposition 1 that $\phi^{\infty}_i(a_{-I_k}) = \phi^{\infty}_i(a_{-I_k-i_{k+1}})\phi^{\infty}_i(a_{i_{k+1}})$, as required.

Furthermore we can observe that all the other players $i$ than $j$ agree on the same conjecture $\sigma_j(a_j) = \phi^{\infty}_i(a_j)$ about $j$. We conclude that each action $a_i$ appearing with positive probability in $\sigma_i$ maximizes $g_i$ against the product of the distributions $\sigma_l$ with $l \neq i$. This implies that the profile $\sigma = (\sigma_i)_{i\in N}$ is a mixed strategy Nash equilibrium of $G$, in completing the proof. □

## 4 Concluding remarks

We have observed that in a communication process with revisions of players' beliefs about the other actions, their predictions induces a mixed strategy Nash equilibrium of the game in the long run. Matsuhisa [4] established the same assertion in the **S4**-knowledge model. Furthermore Matsuhisa [5] showed a similar result for $\varepsilon$-mixed strategy Nash equilibrium of a strategic form game in the **S4**-knowledge model, which gives an epistemic aspect in Theorem of E. Kalai and E. Lehrer [2]. This article highlights the Bayesian belief communication with missing some information, and shows that the convergence to an exact Nash equilibrium is guaranteed even in such the communication on approximate information.

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