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<th>Derivative Nonlinear Schrodinger Equation with General Cubic Nonlinearity (Boundary value problems for partial differential equations)</th>
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1 Introduction and Main Theorem

We consider the Cauchy problem for the nonlinear Schrödinger equation which includes the first order derivatives of unknown function in its nonlinearity:

\[
\begin{cases}
i \partial_t u = -\frac{1}{2} \partial_x^2 u + \mathcal{N}(u, \partial_x u), \\
u(0, x) = u_0(x),
\end{cases}
\]

(1.1)

where $u$ is unknown function from $(t, x) \in \mathbb{R} \times \mathbb{R}$ to $\mathbb{C}$. The derivatives $\partial_t$ and $\partial_x$ denote $\partial/\partial t$ and $\partial/\partial x$, respectively. The nonlinearity $\mathcal{N}(u, q)$ consists of the cubic polynomial of $u, \overline{u}, q$ and $\overline{q}$, i.e.,

\[
\mathcal{N}(u, q) = \sum_{j_1 + j_2 + j_3 + j_4 = 3} C_{j_1 j_2 j_3 j_4} u^{j_1} \overline{u}^{j_2} q^{j_3} \overline{q}^{j_4},
\]

where $C_{j_1 j_2 j_3 j_4} \in \mathbb{C}$ and $j_1, \ldots, j_4$ are nonnegative integers.

When the nonlinear term contains the derivatives, it causes the regularity loss unless the special structure is imposed in the nonlinearity. Since the Schrödinger group $U_0(t) = \exp(it \partial_x^2/2)$ does not abose the derivatives in $L_x^\infty(L_T^2)$, we could not make use of contraction mapping principle simply in $L_x^\infty(L_T^2)$ framework, where $L_x^p(L_T^q)$ denotes the function space endowed with the norm $\|f\|_{L_x^p(L_T^q)} = \left( \int_0^T \|f(t, \cdot)\|_{L_x^q}^p dt \right)^{1/p}$. Of course, if we impose the special structure on $\mathcal{N}(u, q)$, it is possible to derive a priori estimate so that the energy method works. For the general nonlinearity as in the present case, we refer to Kenig-Ponce-Vega’s work [2]. In [2], they derived the crucial smoothing property of $U(t)$ in the new function space $L_x^\infty(L_T^2)$:

\[
\|\partial_x \int_0^t U(t - t') F(t') dt'\|_{L_x^2 L_T^2} \leq C \|F\|_{L_T^1(L_x^2)},
\]

where $\|u\|_{L_x^p(L_T^q)} = \sup_x (\int_0^T |u(t, x)|^q dt)^{1/q}$ and $\|u\|_{L_x^2(L_T^2)} = \|(u(\cdot, x))_{L_T^2}\|_{L_x^2}$. This linear estimate recovers the regularity loss in the nonlinearity and the contraction mapping principle is applicable via the integral equation and obtain the local well-posedness of the solution. In their work, however, one requires the size restriction on the initial data. This
is because the estimate $L^{2}_{x}(L^{p}_{\nu}) \cdot L^{2}_{x}(L^{p}_{\nu}) \cdot L^{\infty}_{x}(L^{2}_{x}) \subset L^{1}_{x}(L^{2}_{x})$ is applied to the nonlinear term and the quantity $\|u\|_{L^{2}_{x}(L^{p}_{\nu})}$ does not expect to be small even when $T \downarrow 0$.

To remove this size restriction, Hayashi-Ozawa [1] applied a nonlinear transformation of unknown function so that the nonlinear component causing the regularity loss is eliminated. They showed that the energy method is still applicable to the general nonlinear case. In [1], they obtained the existence and uniqueness of the solution by assuming that $u_{0} \in H^{3}_{x}$ (the sophisticated estimate likely relaxes this regularity condition into $H^{4}_{x}$ with $s > 5/2$ since the regularity of $u_{0}$ is determined by the estimate of $\|\partial_{x}^{2}u(t)\|_{L^{\infty}_{x}}$), where $H^{s}_{x} = \{ u; \|u\|_{H^{s}_{x}} = \|(D_{x})^{s}u\|_{L^{2}_{x}} < \infty \}$ with $(D_{x})^{s} = F^{-1}(\xi)^{s}F$ with $(\xi) = (1 + \xi^{2})^{1/2}$.

More recently, Kenig-Ponce-Vega [4] have studied how to remove the size restriction of $u_{0}$ and obtained the local well-posedness of the solution. In [4], they write (1.1) as

$$
i \partial_{t}u^{(k)} = -\frac{1}{2} \partial_{x}^{2}u^{(k)} + N(u, \partial_{x}u)\partial_{x}u^{(k)} + N_{\nu}(u, \partial_{x}u)\partial_{x}\bar{u}^{(k)} + \text{(remainder)}$$

where $u^{(k)} = \partial_{x}^{k}u, N_{\nu}(u, q) = \partial_{q}N(u, q), N_{\nu}(u, q) = \partial_{q}N(u, q)$ and the remainder consists of at most $k$-th order derivatives together with $\partial_{x}(u - u_{0})\partial_{x}u^{(k)}$ etc. They derived the smoothing property of the linear solution to $L v = F$ in the time-space norm, where

$$L v = i \partial_{t}v + \frac{1}{2} \partial_{x}^{2}v - N_{\nu}(u, \partial_{x}u)\partial_{x}v - N_{\nu}(u, \partial_{x}u)\partial_{x}\bar{v}.$$  

The merit arising from the representation (1.2) is that $\|\partial_{x}(u - u_{0})\|_{L^{2}_{x}(L^{p}_{\nu})}$ or $\|u - u_{0}\|_{L^{2}_{x}(L^{p}_{\nu})}$ included in the remainder is regarded as negligible quantity by taking $T > 0$ sufficiently small. Hence, one can apply the contraction mapping principle via the integral equation. In their argument, the theory of pseudo-differential operators is the key to the estimate of $v$. This suggests that one requires the large regularity of $u_{0}$.

Our aim in this work is to minimize the regularity of $u_{0}$ without any size restriction and to obtain the local well-posedness of the solution. The idea is based on a gauge transformation different from Hayashi-Ozawa type and a priori estimate in terms of the smoothing properties of $U(t)$ due to Kenig-Ponce-Vega [2]. Concretely speaking, we first modify (1.1) by the following regularization:

$$\begin{cases}
i \partial_{t} u_{\nu} = -\frac{1}{2} \partial_{x}^{2} u_{\nu} + N(u_{\nu}, \partial_{x} \eta_{\nu} \ast u_{\nu}), \\
u_{\nu}(0, x) = u_{0}(x),
\end{cases}$$

where $\eta_{\nu}(x) = \nu^{-1} \eta(x/\nu)$ and $\int \eta(x)dx = 1$ with $\eta \in C_{0}^{\infty}(\mathbb{R})$ and $\nu \in (0, 1)$. Since $\eta_{\nu} \ast$ provides the regularizing property like $\|\partial_{x} \eta_{\nu} \ast u_{\nu}\|_{L^{2}_{x}} \leq C \nu^{-1} \|u_{\nu}\|_{L^{2}_{x}}$, a convenient local solution to (1.3) is constructed via the integral equation. Let $T_{\nu} \in (0, \infty)$ be the upper time bound for the existence of the solution. To realize the solution to (1.1) by taking $\nu \downarrow 0$, we require the lower uniform bound of $T_{\nu}$. For this purpose, we derive an a priori estimate in the Banach space $Y_{T}$ with the norm:

$$\|u\|_{Y_{T}} = \|u\|_{L^{p}_{x}(H^{1}_{x})} + \|(D_{x})^{\nu/2}(\partial_{x}^{2}u)\|_{L^{p}_{x}(L^{2}_{x})} + \max_{j = 0, 1} \|(D_{x})^{\mu}(\partial_{x}^{2}u)\|_{L^{2}_{x}(L^{p}_{\nu})},$$

where $s > 0$ will be specified later and $\mu > 0$ is small. This is the remarkably different point from the usual energy method. To seek for the a priori estimate, we apply the gauge
transformation given by the pseudo-differential operator and, roughly speaking, eliminate the heavy term in the nonlinearity of (1.2) after diagonalizing the system of \(u_\nu = (u_\nu, \bar{u}_\nu)^t\) (see section 2). This kind of elimination is available especially in one space dimension. In our argument, the regularity condition on \(u_0\) are essentially given by (so-called) the estimate of maximal function, i.e., \(\|\partial_x U(t)u_0\|_{L^2_x(L^p_t)} \leq C\|u_0\|_{H^s_x}\), where \(s > 3/2\). Our main theorem in this article is

**Theorem 1.1** Let \(u_0 \in H^s_x\) with \(s > 3/2\). Then, we have the following assertions.

1. For some \(T > 0\), there exists a unique solution \(u\) to (1.1) such that \(u \in C([0,T];H^s_x)\cap Y_T\).

2. Let \(u'\) be the solution to (1.1) with initial data \(u'_0 \in B_\rho(u_0) \equiv \{v_0; \|v_0 - u_0\|_{H^s_x} < \rho\}\) where \(\rho > 0\) is sufficiently small. Then, for some \(T' \in (0,T)\), we have

\[
\|u' - u\|_{L^p_x(H^s_x)} \leq C\|u'_0 - u_0\|_{H^s_x},
\]

\[
\|(D_x)^{-3/2}\partial_x^2(u' - u)\|_{L^p_x(L^p_x)} \leq C\|u'_0 - u_0\|_{H^s_x}.
\]

We now close this section by introducing several notations. The quantity \(\|\cdot\|_X\) denotes the norm of a Banach space \(X\). Let \(B(X;Y)\) be the set of bounded operators from \(X\) to \(Y\). When \(X = Y\), we simply write \(B(X;X) = B(X)\). The summation space is defined by \(X + Y = \{x + y; x \in X \text{ and } y \in Y\}\) with the norm \(\|f\|_{X+Y} = \inf\{\|x\|_{X} + \|y\|_{Y}; f = x + y, x \in X \text{ and } y \in Y\}\). Let \(L^p_x(L^p_y)\) and \(L^p_T(L^p_x(R))\) be the function spaces \(L^p(R;L^p[0,T])\) and \(L^p([0,T];L^p_x)\), respectively. The fractional order differentiation \(D^s_x\) stands for \(\mathcal{F}^{-1}[\xi^s\mathcal{F}]\). We sometimes use \(\hat{f}\) or \(\mathcal{F}f\) for the Fourier transform. Throughout this paper, \(C\) denotes a positive constant which is independent of \(\nu \in (0,1]\) and does not diverge as \(\varphi \rightarrow u_0\) in \(H^s_x\). Also, \(C_\rho\) denotes a positive constant which is independent of \(\nu \in (0,1]\) but may possibly diverge as \(\varphi \rightarrow u_0\) in \(H^s_x\).

## 2 Deformation of (1.3)

In this section, we deform (1.1) by using a gauge transformation defined by a pseudo-differential operator so that the uniform bound of \(\|u_\nu\|_{Y_\nu}\) \((0 < \nu \leq 1)\) is derived. Let \(u^{(1)}_\nu = \partial_x u_\nu\). Then, \(u^{(1)}_\nu\) satisfies

\[
i\partial_t u^{(1)}_\nu = -\frac{1}{2}\partial_x^2 u^{(1)}_\nu + \mathcal{N}_u(u_\nu, \eta_\nu \ast u^{(1)}_\nu)\partial_x \eta_\nu \ast u^{(1)}_\nu + \mathcal{N}_\eta(u_\nu, \eta_\nu \ast u^{(1)}_\nu)\partial_x \eta_\nu \ast \bar{u}^{(1)}_\nu
\]

\[
+ \mathcal{N}_q(u_\nu, \eta_\nu \ast u^{(1)}_\nu)\eta_\nu \ast u^{(1)}_\nu + \mathcal{N}_\bar{q}(u_\nu, \eta_\nu \ast u^{(1)}_\nu)\eta_\nu \ast \bar{u}^{(1)}_\nu,
\]

where \(\mathcal{N}_u\) and \(\mathcal{N}_\eta\) stand for the partial derivatives of \(\mathcal{N}(u, q)\) with respective to \(u\) and \(\bar{u}\). Since \(\partial_x \bar{u}^{(1)}_\nu\) does not vanish by the gauge transformation, we first eliminate it by the diagonalization. To this end, we employ the systemized representation of the above equation. Namely, let \(\bar{u}^{(1)}_\nu = (u^{(1)}_\nu, \bar{u}^{(1)}_\nu)^t\) and write

\[
i\partial_t \bar{u}^{(1)}_\nu = -\frac{1}{2}A\partial_x^2 \bar{u}^{(1)}_\nu + B_\nu(u_\nu)\partial_x \eta_\nu \ast \bar{u}^{(1)}_\nu + \bar{F}_\nu(u_\nu),
\]

(2.1)
where \( A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), \( B_{\nu}(u) = \begin{pmatrix} N_{q}(u, \partial_{x} \eta_{\nu} * u) & N_{q}(u, \partial_{x} \eta_{\nu} * u) \\ -N_{q}(u, \partial_{x} \eta_{\nu} * u) & -N_{q}(u, \partial_{x} \eta_{\nu} * u) \end{pmatrix} \) and \( \tilde{P}_{\nu}(u) \) is

\[
\tilde{P}_{\nu}(u) = \begin{pmatrix} N_{q}(u, \partial_{x} \eta_{\nu} * u) \partial_{x} \eta_{\nu} * u + N_{q}(u, \partial_{x} \eta_{\nu} * u) \partial_{x} \eta_{\nu} * \tilde{u} \\ -N_{q}(u, \partial_{x} \eta_{\nu} * u) \partial_{x} \eta_{\nu} * u - N_{u}(u, \partial_{x} \eta_{\nu} * u) \partial_{x} \eta_{\nu} * \tilde{u} \end{pmatrix}
\]

(Step 1) Diagonalization. Let \( \varphi(x) \in C_{0}^{\infty}(\mathbb{R}) \) (which will be taken sufficiently close to \( u_{0} \) in \( X^{s} \) so that \( u_{\nu}(t) - \varphi \) is small when \( t \downarrow 0 \)). We write (2.1) as

\[
i \partial_{t} \tilde{u}_{\nu}^{(1)} = -\frac{1}{2} A \partial_{x}^{2} \tilde{u}_{\nu}^{(1)} + B_{\nu}(\varphi) \partial_{x} \eta_{\nu} * \vec{u}_{\nu}^{(1)} + \vec{P}_{\nu}(u_{\nu}),
\]

Some readers might wander why we do not take \( \varphi = u_{0} \). The answer to this question will be shown at the end of this section. Let

\[
\vec{v}_{\nu} = (I - J_{\nu}(D_{x})^{-2} \partial_{x} \eta_{\nu} *) \vec{u}_{\nu}^{(1)},
\]

where \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( J_{\nu} = \begin{pmatrix} 0 & -N_{q}(\varphi, \partial_{x} \eta_{\nu} * \varphi) \\ -\overline{N_{q}(\varphi, \partial_{x} \eta_{\nu} * \varphi)} & \overline{\partial_{x} \eta_{\nu} * \varphi} \end{pmatrix} \). By the commutator relation like

\[
[(I - J_{\nu}(D_{x})^{-2} \partial_{x} \eta_{\nu} *), -\frac{1}{2} A \partial_{x}^{2} ] = \begin{pmatrix} 0 & -N_{q}(\varphi, \partial_{x} \eta_{\nu} * \varphi) \\ -\overline{N_{q}(\varphi, \partial_{x} \eta_{\nu} * \varphi)} & \overline{\partial_{x} \eta_{\nu} * \varphi} \end{pmatrix} (D_{x})^{-2} \partial_{x}^{3} \eta_{\nu} * - A((\partial_{x} J_{\nu}) (D_{x})^{-2} \partial_{x}^{3} \eta_{\nu} * + \frac{1}{2} (\partial_{x}^{2} J_{\nu}) (D_{x})^{-2} \partial_{x}^{2} \eta_{\nu} *),
\]

we see that

\[
i \partial_{t} \vec{v}_{\nu} = -\frac{1}{2} A \partial_{x}^{2} \vec{v}_{\nu} + B_{\nu, diag}(\varphi) \partial_{x} \eta_{\nu} * \vec{v}_{\nu} + (B_{\nu}(u_{\nu}) - B_{\nu}(\varphi)) \partial_{x} \eta_{\nu} * \vec{u}_{\nu} + \vec{Q}_{\nu}(\varphi, u_{\nu}),
\]

where \( \vec{u}_{\nu} = (u_{\nu}, \overline{u}_{\nu})^{t} \) and \( B_{\nu, diag}(\varphi) \) denotes the diagonal part of \( B_{\nu}(\varphi) \) and

\[
\vec{Q}_{\nu}(\varphi, u) = -J_{\nu}(D_{x})^{-2} \partial_{x} \eta_{\nu} * B_{\nu}(u) \partial_{x} \eta_{\nu} * \vec{u} + (I - J_{\nu}(D_{x})^{-2} \partial_{x} \eta_{\nu} *) \tilde{P}_{\nu}(u) + B_{\nu, diag}(\varphi) \partial_{x} \eta_{\nu} * (J_{\nu}(D_{x})^{-2} \partial_{x} \eta_{\nu} * \vec{u}) - B_{\nu, off}(\varphi)(I + (D_{x})^{-2} \partial_{x}^{2}) \partial_{x}^{2} \eta_{\nu} * \vec{u}
\]

\[
- A((\partial_{x} J_{\nu})(D_{x})^{-2} \partial_{x}^{2} \eta_{\nu} * \vec{u} + \frac{1}{2} (\partial_{x}^{2} J_{\nu})(D_{x})^{-2} \partial_{x}^{2} \eta_{\nu} * \vec{u})
\]

with \( \vec{u} = (u, \overline{u})^{t} \) and \( B_{\nu, off}(\varphi) = B_{\nu}(\varphi) - B_{\nu, diag} \).

(Step 2) Gauge Transformation. To eliminate \( B_{\nu, diag}(\eta_{\nu} * \varphi) \eta_{\nu} * \vec{v}_{\nu} \) on the right hand side of (2.4), we set \( \vec{w}_{\nu} = K_{\nu}(x, i^{-1} \partial_{x}) \vec{v}_{\nu} = K_{\nu} \vec{v}_{\nu} \) where \( K_{\nu}(x, i^{-1} \partial_{x}) \) is the pseudodifferential operator with the symbol:

\[
K_{\nu}(x, \xi) = \begin{pmatrix} \exp(-\eta(\nu \xi) \partial_{x}^{-1} N_{q}(\varphi, \partial_{x} \eta_{\nu} * \varphi)) & 0 \\ 0 & \exp(-\eta(\nu \xi) \partial_{x}^{-1} N_{q}(\varphi, \partial_{x} \eta_{\nu} * \varphi)) \end{pmatrix},
\]
where $\partial_x^{-1} f$ denotes $\int_{-\infty}^{x} f(y) \, dy$. This transformation yields
\[
\iota \partial_t \vec{w}_\nu = -\frac{1}{2} A \partial_x^2 \vec{w}_\nu + K_\nu (B_\nu(u_\nu) - B_\nu(\varphi)) \eta_\nu * \partial_x^2 \vec{u}_\nu + \vec{R}_\nu(\varphi, u_\nu),
\]
which we denote by $\partial_x^2 \vec{K}_\nu(x, \xi)$.

Since the remainder $\tilde{R}_\nu(\varphi, u_\nu)$ contains the large order derivatives of $\varphi$, we could not replace $\varphi$ by $u_0$.

### 3 Preliminaries

In this section, we introduce several key estimates frequently used in our argument. In what follows, we employ the brief notation $GF$ for $\int_0^t U(t-t')F(t') \, dt'$. The smoothing property of $U(t)$ and $G$ plays an important role to recover the regularity loss arising from the nonlinearity. Hereafter, we assume that $0 < T < 1$.

**Lemma 3.1** Let $p \in [2, \infty]$ and $q \in [2, \infty)$. Then, we have
\[
\|D_x^{1/2} U(t) \phi\|_{L^p_x(L^2_T)} \leq C \|\phi\|_{L^2_x},
\]
(3.1)
\[
\|\partial_x GF\|_{L^q_x(L^p_T)} \leq C \|F\|_{L^1_x(L^2_T)},
\]
(3.2)
\[
\|D_x^{1/2} GF\|_{L^q_x(L^p_T)} \leq C \|F\|_{L^1_x(L^2_T)}. \tag{3.3}
\]

**Proof of Lemma 3.1.** All the estimates in Lemma 3.1 are given in [3; Theorem 2.3, Corollary 2.3]. □

Let us call $\|f(\cdot, x)\|_{L^p_x}$ "the maximal function of $f(t, x)$". We next give the estimates for the maximal function. Remark that the estimate (3.5) essentially determines the regularity constraint of the initial data.

**Lemma 3.2** Let $\sigma > 1/2$. Then, we have
\[
\|U(t) \phi\|_{L^2_x(L^p_T)} \leq C \|\phi\|_{H^2_x}, \tag{3.4}
\]
\[
\|GF\|_{L^2_x(L^p_T)} \leq CT^{1/4}(1 + T)^{\sigma/2-1/4} \|D_x^{\sigma-1/2} F\|_{L^1_x(L^2_T)}. \tag{3.5}
\]

**Proof of Lemma 3.2.** For the estimate (3.4), see [5]. The estimate (3.5) is proved in [6], where the estimate of maximal function is derived for the linearized Benjamin-Ono equation but the derivation in [6] is similarly applied to the Schrödinger equation. In (3.5), the power of $T$ is extracted by the normal scaling argument. □

When we apply the fractional order derivative to the nonlinear term, we often use Leibniz' type rule described in the following.
Lemma 3.3 (1) Let \( \sigma \in (0, 1) \), \( \sigma_1, \sigma_2 \in [0, \sigma] \) with \( \sigma = \sigma_1 + \sigma_2 \). Also, let \( p, r \in (1, \infty) \) and \( p_1, p_2, r_1, r_2 \in (1, \infty) \) with \( 1/p = 1/p_1 + 1/p_2 \) and \( 1/r = 1/r_1 + 1/r_2 \). Then, we have
\[
\|D_x^\sigma(fg) - (D_x^\sigma f)g - f(D_x^\sigma g)\|_{L_x^p(L_T^r)} \leq C \|D_x^{\sigma_1} f\|_{L_x^{p_1}(L_T^{r_1})} \|D_x^{\sigma_2} g\|_{L_x^{p_2}(L_T^{r_2})}.
\] (3.6)
Moreover, for \( \sigma_1 = 0 \), the value \( r_1 = \infty \) is allowed.

(2) Let \( \sigma, \sigma_1, \sigma_2 \) as in (1). Also, \( p, r \in (1, \infty) \) and \( p_1, p_2, r_1, r_2 \in (1, \infty) \) with \( 1/p = 1/p_1 + 1/p_2 \) and \( 1/r = 1/r_1 + 1/r_2 \). Then, we have
\[
\|D_x^\sigma(fg) - (D_x^\sigma f)g - f(D_x^\sigma g)\|_{L_x^1(L_T^2)} \leq C \|D_x^{\sigma_1} f\|_{L_x^{p_1}(L_T^{r_1})} \|D_x^{\sigma_2} g\|_{L_x^{p_2}(L_T^{r_2})}.
\] (3.7)

Proof of Lemma 3.3. See [4; Appendix]. \( \square \)

In the nonlinear estimate, we often encounter the lower order derivatives like \( D_x^{s-3/2} \partial_x u \) and \( \partial_x^2 u \) etc. The following interpolation helps us estimate these quantities. In particular, we require the end point case, i.e., \( p_0 = 1, p_1 = \infty, r_0 = \infty \) and \( r_1 = 2 \).

Lemma 3.4 Let \( \sigma = (1 - \theta)\sigma_0 + \theta \sigma_1 \), \( 1/p = (1 - \theta)/p_0 + \theta/p_1 \) and \( 1/r = (1 - \theta)/r_0 + \theta/r_1 \) with \( \theta \in [0, 1] \) and \( p_0, p_1, r_0, r_1 \in [1, \infty] \). Then, for \( f \in S(\mathbb{R}; C^\infty[0, T]) \), we have
\[
\|D_x^\sigma f\|_{L_x^p(L_T^r)} \leq \sup_{\lambda \in \mathbb{R}} (e^{-\lambda^2} \|D_x^{\sigma_0 + i\lambda(\sigma_1 - \sigma_0)} f\|_{L_x^{p_0}(L_T^{r_0})})^{1-\theta} \times \sup_{\lambda \in \mathbb{R}} (e^{1-\lambda^2} \|D_x^{\sigma_1 + i\lambda(\sigma_1 - \sigma_0)} f\|_{L_x^{p_1}(L_T^{r_1})})^\theta.
\] (3.8)

Proof of Lemma 3.4. Let \( f, g \in C_0^\infty(\mathbb{R}; C^\infty[0, T]) \) and
\[
g_z(t, x) = ||g(\cdot, x)||_{L_x^{p'}(L_T^{r'})}^{(1-z)(p'/p_0'-r'/r_0') + z(p'/p_1'-r'/r_1')} |g(t, x)|^{(1-z)r'/r_0' + zr'/r_1'} \text{sgn} g(t, x)
\] with \( z \in \mathbb{C} \) and \( 1/p + 1/p' = 1/r + 1/r' = 1 \). By the three line theorem on the strip \( \{z; 0 \leq \text{Re} z \leq 1\} \), we see that
\[
|e^{z^2}((g_z, D_x^{(1-z)\sigma_0 + z\sigma_1} f))| \leq \sup_{\lambda \in \mathbb{R}} |e^{-\lambda^2} ((g_{i\lambda}, D_x^{\sigma_0 + i\lambda(\sigma_1 - \sigma_0)} f))|^{1-\text{Re} z} \times \sup_{\lambda \in \mathbb{R}} |e^{(1+i\lambda)^2} ((g_{1+i\lambda}, D_x^{\sigma_1 + i\lambda(\sigma_1 - \sigma_0)} f))|^{\text{Re} z},
\] (3.9)
where \( ((\cdot, \cdot)) \) denotes the integration of time-space variables. Take \( z = \theta \). Then, Hölder's inequality gives the bound of the right hand side of (3.9) like
\[
\|g\|_{L_x^{p'}(L_T^{r'})} \sup_{\lambda} (e^{-\lambda^2} \|D_x^{\sigma_0 + i\lambda(\sigma_1 - \sigma_0)} f\|_{L_x^{p_0}(L_T^{r_0})})^{1-\theta} \sup_{\lambda} (e^{1-\lambda^2} \|D_x^{\sigma_1 + i\lambda(\sigma_1 - \sigma_0)} f\|_{L_x^{p_1}(L_T^{r_1})})^\theta.
\]
Then, the duality argument yields Lemma 3.4. \( \square \)

We next show the estimate of the gauge transform \( K_{\nu}(z, i^{-1} \partial_x) \).
Lemma 3.5 Let $p, r \in [1, \infty]$ and $\sigma \in [0, 1)$. Then, we have

\[
\|D_x^\sigma K_\nu(x, i^{-1}\partial_x)\hat{f}\|_{L_x^p(L_T^r)} \leq C \exp(C\|\varphi\|_{H_x^\infty}) \|\langle D_x\rangle^\sigma\hat{f}\|_{L_x^p(L_T^r)}.
\]

(3.10)

In the above inequality, we may replace $\| \cdot \|_{L_x^p(L_T^r)}$ by $\| \cdot \|_{L_x^p}$.

Proof of Lemma 3.5. It suffices to consider the pseudo-differential operator with the symbol like $k_\nu(x, \xi) = \exp(\hat{\eta}(\nu \xi) \psi(x))$, where $\psi = \partial_x^{-1}N_q(\varphi, \eta_\nu * \partial_x \varphi)$ or $\partial_x^{-1}N_q(\varphi, \eta_\nu * \partial_x \varphi)$.

We first show that $k_\nu(x, i^{-1}\partial_x) \in \mathcal{B}(L_x^pL_T^r)$. Note that $k_\nu(x, i^{-1}\partial_x) - I$ has the integral kernel given by

\[
[k_\nu(x, i^{-1}\partial_x) - I](x, y) = |(2\pi\nu)^{-1} \int \{\exp(\hat{\eta}(\xi) \psi(x)) - 1\} e^{i\xi(x-y)/\nu} d\xi| \leq C_N \exp(C\|\psi\|_{L_x^\infty}) \nu^{-1} \langle (x-y)/\nu \rangle^{-N},
\]

where the last inequality in the above follows from the integration by parts. Therefore, Young's inequality yields $k_\nu(x, i^{-1}\partial_x) - I \in \mathcal{B}(L_x^p(L_T^r))$.

We next show that $\{D_x\}^\sigma, k_\nu(x, i^{-1}\partial_x) \in \mathcal{B}(L_x^p(L_T^r))$ and its operator norm is bounded by $C\|\partial_x\psi\|_{L_x^\infty} \exp(C\|\psi\|_{L_x^\infty})$. Note that the integral kernel of $[\{D_x\}, k_\nu(x, i^{-1}\partial_x)]$ is given by the oscillatory integral like

\[
L(x, y) \equiv (2\pi)^{-2} \iiint e^{i(x-z)\xi}(\xi)^\sigma \times e^{i(z-y)\zeta}(k_\nu(z, \zeta) - k_\nu(x, \zeta)) d\xi d\zeta dz
\]

or

\[
= (2\pi)^{-2} \iiint e^{i(x-z)\xi} i^{-1}\partial_\xi(\xi)^\sigma \times e^{i(z-y)\zeta} \int_0^1 \partial_x k_\nu(\theta z + (1-\theta)x, \zeta) d\theta d\xi d\zeta dz.
\]

Since

\[
\left| \int e^{i(x-z)\xi} \partial_\xi(\xi)^\sigma d\xi \right| \leq C_N |x-z|^{-\sigma} \langle x-z \rangle^{-N}
\]

and

\[
\left| \int e^{i(z-y)\zeta} \int_0^1 \partial_x k_\nu(\theta z + (1-\theta)x, \zeta) d\theta d\zeta \right| \leq C_N \|\partial_x\psi\|_{L_x^\infty} \exp(C\|\psi\|_{L_x^\infty}) \nu^{-1} \langle (z-y)/\nu \rangle^{-N},
\]

we see that

\[
|L(x, y)| \leq C_N \|\partial_x\psi\|_{L_x^\infty} \exp(C\|\psi\|_{L_x^\infty}) \int |x-z|^{-\sigma} \langle x-z \rangle^{-N} \nu^{-1} \langle (z-y)/\nu \rangle^{-N} dz
\]

\[
\leq C_N \|\partial_x\psi\|_{L_x^\infty} \exp(C\|\psi\|_{L_x^\infty}) |x-y|^{-\sigma} \langle x-y \rangle^{-N}.
\]

Thus, Young's inequality yields $[\{D_x\}^\sigma, k_\nu(x, i^{-1}\partial_x)] \in \mathcal{B}(L_x^p(L_T^r))$. Since $\{D_x\}^\sigma - D_x^\sigma \in \mathcal{B}(L_x^p(L_T^r))$, we obtain Lemma 3.5. □
4 Nonlinear Estimates

When we apply Lemma 3.1 to the nonlinearity, we require the nonlinear estimates given in the following two lemmas. In what follows, we only consider the case $s \in (3/2, 2)$.

Lemma 4.1 Let $s$ as in Theorem 1.1 and $\mu \in (0, 1)$. Then, there exist $C > 0$ and $\theta \in (0, 1)$ such that

$$
\|\langle D_x \rangle^{s-3/2}(fg\partial_x h)\|_{L^1_x(L^2_T)} \leq C \|f\|_{L^2_x(L^\infty_T)} \|g\|_{L^2_x(L^\infty_T)} \|\langle D_x \rangle^{s-3/2}\partial_x h\|_{L^2_x(L^2_T)}
$$

$$
+ C \|D_x^{\mu}f\|_{L^p_x(L^\tilde{p}_T)} \|\langle D_x \rangle^{s-3/2}\partial_x f\|_{L^2_x(L^2_T)} \|g\|_{L^2_x(L^\infty_T)}
$$

$$
+ C \|f\|_{L^2_x(L^\infty_T)} \|D_x^{\mu}g\|_{L^2_x(L^\infty_T)} \|\langle D_x \rangle^{s-3/2}\partial_x g\|_{L^2_x(L^2_T)}
$$

$$
+ C \|f\|_{L^2_x(L^\infty_T)} \|D_x^{\mu}f\|_{L^2_x(L^\infty_T)} \|\langle D_x \rangle^{s-3/2}\partial_x f\|_{L^2_x(L^2_T)}
$$

$$
(4.1)
$$

$$
\|\langle D_x \rangle^{s-3/2}(fg\partial_x h)\|_{L^2_x(L^2_T)} \leq C T^{1/2} \|f\|_{L^\infty_T(L^1_x)} (\|g\|_{L^2_x(L^2_T)} + \|g\|_{L^\infty_T(H^{-1}_x)})
$$

$$
\times (\|\langle D_x \rangle^{s-3/2}\partial_x h\|_{L^2_x(L^2_T)} + \|h\|_{L^\infty_T(H^{-1}_x)}).
$$

(4.2)

Lemma 4.2 Let $\tilde{R}_\nu(\varphi, u_\nu)$ defined in section 2 and $s' < s$. Then, we have

$$
\|\tilde{R}_\nu(\varphi, u_\nu)\|_{L^1_x(H^{-1}_x)} \leq C_{\varphi} T(\|u_\nu\|_{Y_T} + \|u_\nu\|_{Y_T}),
$$

(4.3)

$$
\|\tilde{R}_\nu(\varphi, u_\nu) - \tilde{R}_\nu'(\varphi, u_\nu')\|_{L^1_x(H^{-1}_x)} \leq C_{\varphi} T(1 + \|u_\nu\|_{Y_T} + \|u_\nu\|_{Y_T}) \|u_\nu - u_\nu'\|_{Y_T}
$$

$$
+ C_{\varphi} (\nu^\beta + \nu'^\beta)(1 + \|u_\nu\|_{Y_T} + \|u_\nu\|_{Y_T})^3.
$$

(4.4)

Proof of Lemma 4.1. Applying $\langle D_x \rangle^{s-3/2} - D_x^{s-3/2} \in \mathcal{B}(L^1_x(L^2_T))$ and Lemma 3.3, we see that

$$
\|\langle D_x \rangle^{s-3/2}(fg\partial_x h)\|_{L^1_x(L^2_T)} \leq \|f\|_{L^2_x(L^\infty_T)} \|g\|_{L^2_x(L^\infty_T)} \|\langle D_x \rangle^{s-3/2}\partial_x h\|_{L^2_x(L^2_T)}
$$

$$
+ C \|D_x^{s-3/2}(fg)\|_{L^2_x(L^2_T)} \|\partial_x h\|_{L^2_x(L^2_T)}
$$

$$
+ C \|fg\partial_x h\|_{L^2_x(L^2_T)},
$$

where $1/p = (1 - \theta)/2 + 1/\infty$, $1/r = (1 - \theta)/2 + \theta/2$, $1/p + 1/\tilde{p} = 1$ and $1/r + 1/\tilde{r} = 1/2$ together with $1 = (1 - \theta)\mu/2 + \theta (s - 1/2 - \mu/2)$. Using Lemma 3.4, we have

$$
\|\partial_x h\|_{L^2_x(L^\tilde{r}_T)} \leq \sup_{\lambda} e^{-\lambda^2} \|D_x^{\mu/2 + 1/2 - \mu\lambda} 1_{x<s-1/2-\mu/2} \mathcal{F}^{-1} \mathcal{F}h\|_{L^2_x(L^\tilde{r}_T)}
$$

$$
\times (\sup_{\lambda} e^{-1/2 + 1/2 + \lambda\lambda} \|\partial_x h\|_{L^2_x(L^\tilde{r}_T)})^{\theta}
$$

$$
\leq C \|\langle D_x \rangle^{\mu}h\|_{L^2_x(L^\tilde{r}_T)} \|\langle D_x \rangle^{s-3/2}\partial_x h\|_{L^2_x(L^\tilde{r}_T)},
$$

(4.5)
where we made use of
\[\|D_x^{s-3/2} (fg)\|_{L_T^1(L_x^2)} \leq C \|D_x^{s-3/2} g\|_{L_T^2(L_x^2)} \|\partial_x h\|_{L_T^2(L_x^2)} + T \|D_x^{s-3/2} (fg)\|_{L_T^2(L_x^{2+4/\epsilon})} \|\partial_x h\|_{L_T^2(L_x^2)}\]
with \(N\) sufficiently large. By the similar argument to derive (4.5), we have
\[\|D_x^{s-3/2} (fg)\|_{L_T^1(L_x^2)} \leq C \|\langle D_x\rangle^{\mu} f\|_{L_T^\infty(L_x^2)} \|\langle D_x\rangle^\epsilon \partial_x h\|_{L_T^\infty(L_x^2)} \|g\|_{L_T^\infty(L_x^2)},\]
(4.6)
Also, we can show that
\[\|f g\partial_x h\|_{L_T^1(L_x^2)} \leq C \|f\|_{L_T^\infty(L_x^2)} \|g\|_{L_T^\infty(L_x^2)} \|\partial_x h\|_{L_T^\infty(L_x^2)} + C \|\langle D_x\rangle^{\mu} f\|_{L_T^\infty(L_x^2)} \|\langle D_x\rangle^\epsilon \partial_x h\|_{L_T^\infty(L_x^2)} \|g\|_{L_T^\infty(L_x^2)},\]
(4.7)
Combining (4.5)-(4.7), we obtain (4.1). To prove (4.2), we apply the Leibniz’ rule for fractional order derivatives. We have
\[\|D_x^{s-3/2} (fg\partial_x h)\|_{L_T^1(L_x^2)} \leq \|f g\partial_x h\|_{L_T^1(L_x^2)} + T \|D_x^{s-3/2} (fg)\|_{L_T^1(L_x^{2+4/\epsilon})} \|\partial_x h\|_{L_T^1(L_x^{2+4/\epsilon})} \equiv I_1 + I_2.\]
By Hölder’s inequality and Sobolev’s embedding, \(I_1\) is estimated as
\[I_1 \leq T^{1/2} \|f\|_{L_T^2(L_x^2)} \|g\|_{L_T^2(L_x^2)} \|\partial_x h\|_{L_T^2(L_x^2)} \]
\[\leq CT^{1/2} \|f\|_{L_T^2(L_x^2)} \|g\|_{L_T^2(L_x^2)} \|\partial_x h\|_{L_T^2(L_x^2)}\]
As for \(I_2\), Leibniz’ rule and Sobolev’s embedding yield
\[I_2 \leq CT \|f\|_{L_T^2(H_x^{s-1})} \|g\|_{L_T^2(H_x^{s-1})} \|h\|_{L_T^2(H_x^{s-1})}\]
Hence, we obtain (4.2).

Proof of Lemma 4.2. By the \(H_x^{s-1}\)-boundedness of \(K_{\nu}\), we see that
\[\|\tilde{R}_{\nu}(\varphi, u_{\nu})\|_{L_T^1(H_x^{s-1})} \leq C_{\nu} \|u_{\nu}\|_{L_T^2(H_x^2)} + \|\tilde{Q}_{\nu}(\varphi, u_{\nu})\|_{L_T^1(H_x^{s-1})}.\]
To estimate \(\|\tilde{Q}_{\nu}(\varphi, u_{\nu})\|_{L_T^1(H_x^{s-1})}\), it suffice to consider
\[\|(D_x)^{s-1-\beta} J_{\nu}(D_x)^{-2} \partial_x \eta_{\nu} * B_{\nu}(u_{\nu}) \|_{L_T^1(L_x^2)} \]
\[\leq C \|(D_x)^{s-3} \partial_x B_{\nu}(u_{\nu}) \|_{L_T^2(L_x^2)} \]
\[\leq C \|B_{\nu}(u_{\nu}) \|_{L_T^1(L_x^2)} \]
\[\leq C T^{1/2} \|u_{\nu}\|_{L_T^2(L_x^2)} + C \|\eta_{\nu} - \eta_{\nu'}\|_{L_T^1(L_x^2)} \]
The proof of (4.4) likewise follows. We note that \(\nu^\beta + \nu'^\beta\) arises from the estimates of \(K_{\nu} - K_{\nu'}, J_{\nu} - J_{\nu'}\) and \((\eta_{\nu} - \eta_{\nu'})^*\) which cause the slight loss of regularity. \(\Box\)
5 A priori estimate in $Y_T$ and convergence of $u_\nu$

To obtain the a priori estimate of $u_\nu$ for $\nu \in (0, 1]$, we use the following integral representations:

\[ \overline{w}_\nu = U(t)\overline{w}_{\nu,0} - iGK_\nu(B_\nu(u_\nu) - B_\nu(\varphi)) \eta_\nu * \partial_x^2 u_\nu \]
\[ = -iG \tilde{F}_\nu(\varphi, u_\nu), \]
\[ u_\nu = U(t)u_0 - iGN(u_\nu, \partial_x u_\nu), \]

where $U(t) = \exp(itA\partial_x^2/2)$, $G\tilde{F} = \int_0^t U(t-\tau)\tilde{F}(\tau)d\tau$ and $\overline{w}_{\nu,0} = K_\nu(\partial_x \overline{u}_0 + J_\nu \eta_\nu * \overline{u}_0)$ with $\overline{u}_0 = (u_0, \overline{u}_0)^t$. The construction of the approximating solution $u_\nu$ in $Y_T$ is simple. In fact, by applying Lemma 3.1, 3.2 to (5.2) and in virtue of the regularization due to $\eta_\nu$ together with Lemma 3.3, the nonlinear term is, for instance, estimated as

\[ \|D_x^{s-3/2} \partial_x N(u_\nu, \partial_x \eta_\nu * u_\nu)\|_{L^1_t(L^2_x)} \]
\[ \leq C \nu^{-N} T^{1/2} (\max_{j=0,1} \|D_x^\mu \partial_x^j u_\nu\|_{L^2_t(L^\infty_x)}) \|u_\nu\|_{L^\infty_t(L^2_x)}. \]

Thus, by taking $T > 0$ sufficiently small, the contraction mapping principle successfully works in $Y_T$. The local solution $u_\nu$ is continued as long as $\|u_\nu(t)\|_{H^s_x}$ is finite. Note that $\|u_\nu\|_{Y_T}$ is continuous with respect to $T$.

For brief description, we define several norms as follows

\[ \|u\|_{Y_T} = \|u\|_{L^\infty_t(H^s_x)} + \|D_x^{s-3/2} \partial_x^2 u\|_{L^\infty_t(L^2_x)} + \max_{j=0,1} \|D_x^\mu \partial_x^j u\|_{L^2_t(L^\infty_x)} \]
\[ \equiv \|u\|_{\text{initial}} + \|u\|_{\text{smooth}} + \|u\|_{\text{maxim}}. \]

To ensure the convergence of the nonlinearity as $\nu \downarrow 0$, we require the Cauchy property of $\{u_\nu\}_{\nu \in (0, 1]}$. Note that the proof fails when we consider $\|u_\nu - u_\nu\|_{Y_T}$, since the estimate $\|(\eta_\nu - \eta_\nu')u_\nu\|_{H^s_x} \leq C(\nu^\beta + \nu'^\beta)\|u_\nu\|_{H^{s+\beta}}$ indicates the regularity loss. Therefore, we employ the function space slightly weaker than $Y_T$, i.e.,

\[ \|u\|_{Z_T} = \|u\|_{L^\infty_t(H^s_x)} + \|D_x^{s-3/2} \partial_x^2 u\|_{L^\infty_t(L^2_x)} + \max_{j=0,1} \|D_x^\mu \partial_x^j u\|_{L^2_t(L^\infty_x)}, \]

where $s' < s$ and $\mu' < \mu$. The key proposition to obtain our main theorem is

**Proposition 5.1 (a priori estimate)** The following assertions hold.

1. **Let** $T_\nu = \sup\{T'; \|u_\nu\|_{Y_T} < 2C_0\delta_0 \text{ for } 0 < \tau < T'\}$. Then, $\liminf_{\nu \downarrow 0} T_\nu = T_0 > 0$,

2. **Let** $\|u_0\|_{H^s_x} \leq \delta_0$ and $T \in (0, T_0]$ sufficiently small. Then, we have

\[ \|u_\nu\|_{Y_T} \leq 2C_0\delta_0, \]
\[ \|u_\nu - u_\nu\|_{Z_T} \leq C_\varphi (\nu^\beta + \nu'^\beta) (1 + 4C_0\delta_0)^3, \]

where $C_0$ and $C_\varphi$ do not depend on $\nu \in (0, 1]$ but $C_\varphi$ may diverge as $\varphi \to u_0$ in $H^s_x$.

To prove Proposition 5.1, we need two lemmas. The first one indicates that the estimates of $u_\nu$ is replaced by those of $\overline{w}_\nu$. 

Lemma 5.2 Let $s > s' > 3/2$ and $\nu, \nu' > 0$ sufficiently small. Then, we have

$$\|u_{\nu}\|_{L^{p}(H^{s}_{\nu})} \leq C(\|\overline{w}_{\nu}\|_{L^{p}(H^{s-1}_{\nu})} + \|u_{\nu}\|_{L^{p}(L^{2}_{\nu})}),$$

(5.5)

$$\|\langle D_{x}\rangle^{s-3/2}\partial_{x}^{2}u_{\nu}\|_{L^{p}(L^{2}_{\nu})} \leq C\|\langle D_{x}\rangle^{s-3/2}\partial_{x}\overline{w}_{\nu}\|_{L^{p}(L^{2}_{\nu})} + C_{\nu}T^{1/2}\|u_{\nu}\|_{L^{p}(H^{s}_{\nu})},$$

(5.6)

$$\|u_{\nu} - u_{\nu'}\|_{L^{p}(H^{s}_{\nu'})} \leq C(\|\overline{w}_{\nu} - \overline{w}_{\nu'}\|_{L^{p}(H^{s}_{\nu'})} + \|u_{\nu} - u_{\nu'}\|_{L^{p}(L^{2}_{\nu})} + C_{\nu}(\nu^{\beta} + \nu'^{\beta})\|u_{\nu}\|_{Y_{T}}),$$

(5.7)

$$\|\langle D_{x}\rangle^{s'-3/2}\partial_{x}^{2}(u_{\nu} - u_{\nu'})\|_{L^{p}(L^{2}_{\nu})} \leq C(\|\langle D_{x}\rangle^{s'-3/2}\partial_{x}(\overline{w}_{\nu} - \overline{w}_{\nu'})\|_{L^{p}(L^{2}_{\nu})} + C_{\nu}T^{1/2}\|u_{\nu} - u_{\nu'}\|_{L^{p}(H^{s}_{\nu'})})$$

+ C_{\nu}(\nu^{\beta} + \nu'^{\beta})\|u_{\nu}\|_{Y_{T}}),$$

(5.8)

where $\beta$ is a small positive constant.

Proof of Lemma 5.2. Since $\overline{w}_{\nu} = K_{\nu}(\partial_{x}\overline{w}_{\nu} + J_{\nu}\eta_{\nu} * \overline{w}_{\nu})$, we see that

$$\langle D_{x}\rangle^{s}\partial_{x}^{-1}\overline{w}_{\nu} = K_{\nu}(\langle D_{x}\rangle^{s}\partial_{x}^{-1}\overline{w}_{\nu} + \langle D_{x}\rangle^{s}\partial_{x}^{-1}K_{\nu}J_{\nu}\eta_{\nu} * \overline{w}_{\nu}) = \tilde{K}_{\nu}(\langle D_{x}\rangle^{s}\partial_{x}^{-1}K_{\nu}\overline{w}_{\nu} + \langle D_{x}\rangle^{s}\partial_{x}^{-1}K_{\nu}J_{\nu}\eta_{\nu} * \overline{w}_{\nu})$$

(5.9)

Let $\tilde{K}_{\nu} = \tilde{K}_{\nu}(x, i^{-1}\partial_{x})$ be the pseudo-differential operator of the symbol:

$$\tilde{K}_{\nu}(x, \xi) = \left(\begin{array}{cc}
\exp(\hat{\eta}(\nu \xi)\partial_{x}^{-1}\mathcal{N}_{q}(\varphi, \partial_{x}\eta_{\nu} * \varphi)) & 0 \\
0 & \exp(\hat{\eta}(\nu \xi)\partial_{x}^{-1}\mathcal{N}_{q}(\varphi, \partial_{x}\eta_{\nu} * \varphi))
\end{array}\right).$$

Note that $\tilde{K}_{\nu}$ plays a role like the inverse of $K_{\nu}$. Then, from (5.9), it follows that

$$\langle D_{x}\rangle^{s}\partial_{x}^{2}u_{\nu} = \tilde{K}_{\nu}(\langle D_{x}\rangle^{s}\partial_{x}^{2}u_{\nu}) - (\tilde{K}_{\nu}K_{\nu} - I)\langle D_{x}\rangle^{s}\partial_{x}^{2}u_{\nu}$$

Taking $s = s - 1$ and $j = 1$ in (5.10) and applying Lemma 3.5–3.3 together with $[(D_{x})^{s}, K_{\nu}] \in B(L^{2}_{x}, H_{x}^{-(1-s)})$ uniformly in $\nu \in (0, 1]$, we have

$$\|u_{\nu}\|_{L^{p}(H^{s}_{\nu})} \leq C\|\overline{w}_{\nu}\|_{L^{p}(H^{s-1}_{\nu})} + C_{\nu}\nu^{\beta}\|u_{\nu}\|_{L^{p}(H^{s}_{\nu})} + C_{\nu}\|u_{\nu}\|_{L^{p}(H^{s-1}_{\nu})}.$$

Taking $\nu > 0$ so that $C_{\nu}\nu^{\beta} < 1/4$ and applying $\|u_{\nu}\|_{L^{p}(H^{s-1}_{\nu})} \leq \varepsilon\|u_{\nu}\|_{L^{p}(H^{s}_{\nu})}$, we obtain (5.5). To prove (5.6), we let $s = s - 2/3$ and $j = 2$ in (5.10). Then, it follows that

$$\|\langle D_{x}\rangle^{s-3/2}\partial_{x}^{2}u_{\nu}\|_{L^{p}(L^{2}_{\nu})} \leq C\|\langle D_{x}\rangle^{s-3/2}\partial_{x}\overline{w}_{\nu}\|_{L^{p}(L^{2}_{\nu})} + C_{\nu}\nu^{\beta}\|\langle D_{x}\rangle^{s-3/2}\partial_{x}^{2}u_{\nu}\|_{L^{p}(L^{2}_{\nu})}$$

Taking $\nu, \varepsilon > 0$ small and applying $\|u_{\nu}\|_{L^{p}(L^{2}_{\nu})} \leq T^{1/2}\|u_{\nu}\|_{L^{p}(L^{p}(H^{s}_{\nu})}$, we obtain (5.6). The estimates (5.7) and (5.8) follow from the description:

$$\langle D_{x}\rangle^{s}\partial_{x}^{2}(\overline{w}_{\nu} - \overline{w}_{\nu'}) = \tilde{K}_{\nu}(\langle D_{x}\rangle^{s}\partial_{x}^{2}(\overline{w}_{\nu} - \overline{w}_{\nu'})) - \tilde{K}_{\nu}(\langle D_{x}\rangle^{s}\partial_{x}(\overline{w}_{\nu} - \overline{w}_{\nu'}))$$

$$- \tilde{K}_{\nu}(\langle D_{x}\rangle^{s}\partial_{x}^{2}K_{\nu}(\partial_{x}\overline{w}_{\nu} + J_{\nu}\eta_{\nu} * \overline{w}_{\nu}))$$

$$- \tilde{K}_{\nu}(\langle D_{x}\rangle^{s}\partial_{x}^{2}K_{\nu}(J_{\nu}\eta_{\nu} * \overline{w}_{\nu} - J_{\nu}\eta_{\nu} * \overline{w}_{\nu'})).$$
Note that the coefficient $\nu^\beta + \nu'^\beta$ appears in the estimates of $K_{\nu} - K_{\nu'}$, $J_{\nu} - J_{\nu'}$ and $(\eta_{\nu} - \eta_{\nu'})\ast$. □

The second lemma shows that one can make $\|u_{\nu} - \varphi\|_{\text{maxim}}$ and $\|u_{\nu}\|_{\text{smooth}}$ (appearing in the nonlinear estimates) small enough by taking $\varphi$ close to $u_0$ and $T > 0$ small.

**Lemma 5.3** There exist $\beta > 0$ and $\theta \in (0, 1)$ such that

\[
\begin{align*}
\|u_{\nu} - \varphi\|_{\text{maxim}} & \leq C\|u_0 - \varphi\|_{H^s_T} + C\|\nu\|_{Y^w_T}^3, \\
\|u_{\nu}\|_{\text{smooth}} & \leq C\|u_0 - \varphi\|_{H^s_T} \\
& \quad + C(\|u_{\nu} - \varphi\|_{\text{maxim}} + \|u_{\nu} - \varphi\|_{\text{maxim}}^{1-\theta}\|u_{\nu}\|_{Y^w_T})^2 \\
& \quad + C\|\nu\|_{Y^w_T}^3.
\end{align*}
\] (5.12)

**Proof of Lemma 5.3.** From the integral equation (5.2), it follows that

\[
\|u_{\nu} - \varphi\|_{\text{maxim}} \leq \|U(t)u_0 - \varphi\|_{\text{maxim}} + \|\mathcal{N}\mathcal{N}(u_{\nu}, \partial_x u_{\nu})\|_{\text{maxim}} \\
\equiv I_1 + I_2.
\] (5.13)

Note that, by Lemma 3.2,

\[
I_1 \leq \|U(t)(u_0 - \varphi)\|_{\text{maxim}} + \|U(t)\varphi - \varphi\|_{\text{maxim}} \\
\leq C\|u_0 - \varphi\|_{H^s_T} + CT\|\varphi\|_{H^s_T},
\] (5.14)

where $\sigma > 0$ is sufficiently large. As for the estimate of $I_2$, we only consider the case $\mathcal{N}(u_{\nu}, \partial_x \eta_{\nu}\ast u_{\nu}) = (\partial_x \eta_{\nu}\ast u_{\nu})^3$ and $j = 1$ in the definition of $\|\cdot\|_{\text{maxim}}$. Lemma 3.2, 3.3 and 4.1 yield

\[
I_2 \leq C\|\nu\|_{Y^w_T}^3.
\] (5.15)

Combining (5.13)–(5.15), we obtain (5.11). To prove (5.12), we use (5.1). Then, Lemma 3.1 yields

\[
\|\langle D_x\rangle^{s-3/2}\partial_x \overline{G}\nu\|_{L^2_T(L^2_x)} \leq \|\langle D_x\rangle^{s-3/2}\partial_x U(t)\overline{w}_{\nu,0}\|_{L^2_T(L^2_x)} \\
+ C\|\langle D_x\rangle^{s-3/2}\mathcal{K}_{\nu}(\mathcal{B}_\nu(u_{\nu}) - \mathcal{B}_\nu(\varphi))\|_{L^2_T(L^2_x)} \\
+ C\|\overline{R}_\nu(\varphi, u_{\nu})\|_{L^2_T(H^{-1}_x)} \\
\equiv I_1' + I_2' + I_3'.
\] (5.16)

Note that, to get $I_3'$, we apply Lemma 3.1 (3.1) in the following way:

\[
\|\langle D_x\rangle^{s-3/2}\partial_x \overline{G}\nu\|_{L^2_T(L^2_x)} \leq \int_0^T \|\langle D_x\rangle^{s-3/2}\partial_x U(\cdot)\overline{R}_\nu\|_{L^2_T(L^2_x)} \, dt' \\
\leq C\|\langle D_x\rangle^{s-3/2}\overline{D}_x^{1/2}\overline{R}_\nu\|_{L^2_T(L^2_x)}.
\]
Let \( \vec{\varphi}_{\nu} = K_{\nu}(\partial_{x}\vec{\varphi} + J_{\nu}\eta_{\nu} \ast \vec{\varphi}) \) with \( \vec{\varphi} = (\varphi, \overline{\varphi})^{t} \). Then, Lemma 3.1 (3.1) gives

\[
I_{1}' \leq \| (D_{x})^{s-3/2}\partial_{x}U(t)(\vec{w}_{\nu,0} - \vec{\varphi}_{\nu}) \|_{L_{x}^{\infty}(L_{T}^{2})} + \| (D_{x})^{s-3/2}\partial_{x}U(t)\vec{\varphi}_{\nu} \|_{L_{x}^{\infty}(L_{T}^{2})}
\]
\[
\leq C\| \vec{w}_{\nu,0} - \vec{\varphi}_{\nu} \|_{H_{x}^{s-1,0}} + C_{\varphi}T^{1/2}
\]
\[
\leq C\| u_{0} - \varphi \|_{H_{x}^{s}} + C_{\varphi}T^{1/2}.
\]

We next consider the estimate of \( I_{2}' \). By Lemma 3.5 and 4.1,

\[
I_{2}' \leq C\| (D_{x})^{s-3/2}(B_{\nu}(u_{\nu}) - B_{\nu}(\varphi)) \eta_{\nu} \ast \partial_{x}^{2}u_{\nu} \|_{L_{x}^{1}(L_{T}^{2})}
\]
\[
\leq C\| (D_{x})^{s-3/2}(B_{\nu}(u_{\nu}) - B_{\nu}(\varphi)) \eta_{\nu} \ast \partial_{x}^{2}u_{\nu} \|_{L_{x}^{1}(L_{T}^{2})}
\]
\[
+ C\| (D_{x})^{s-3/2} - D_{x}^{s-3/2}(B_{\nu}(u_{\nu}) - B_{\nu}(\varphi)) \eta_{\nu} \ast \partial_{x}^{2}u_{\nu} \|_{L_{x}^{1}(L_{T}^{2})}
\]
\[
\leq C(\| u_{\nu} - \varphi \|_{\text{maxim}} + \| u_{\nu} - \varphi \|_{\text{maxim}}^{\theta} \| u_{\nu} \|_{Y_{T}}^{1-\theta})(1 + \| u_{\nu} \|_{Y_{T}})^{2},
\]

where we used \( (D_{x})^{s-3/2} - D_{x}^{s-3/2} \in \mathcal{B}(L_{x}^{1}(L_{T}^{2})) \) and \( \| \partial_{x}^{2}u_{\nu} \|_{L_{x}(L_{T}^{2})} \leq \| u_{\nu} \|_{\text{smooth}} \). Since \( \| u_{\nu} - \varphi \|_{\text{smooth}} \leq \| u_{\nu} \|_{\text{smooth}} + C_{\varphi}T^{1/2} \), we have

\[
I_{2}' \leq C(\| u_{\nu} - \varphi \|_{\text{maxim}} + \| u_{\nu} - \varphi \|_{\text{maxim}}^{\theta} \| u_{\nu} \|_{Y_{T}}^{1-\theta})(1 + \| u_{\nu} \|_{Y_{T}})^{2} + C_{\varphi}T^{1/2}(1 + \| u_{\nu} \|_{Y_{T}})^{2}.
\]

As for \( I_{3}' \), we apply Lemma 4.2 and observe that

\[
I_{3}' \leq C\varphi T(1 + \| u_{\nu} \|_{Y_{T}})^{3}.
\]

Combining (5.16)–(5.18) and Lemma 5.2(5.6), we obtain (5.12). \( \square \)

We are ready for the proof of Proposition 5.1.

**Proof of Proposition 5.1.** Applying Lemma 3.1, 4.1 and 4.2 to (5.1), we see that

\[
\| \vec{w}_{\nu} \|_{L_{x}^{\infty}(H_{x}^{s-1,0})} + \| (D_{x})^{s-3/2}\partial_{x}\vec{w}_{\nu} \|_{L_{x}^{\infty}(L_{T}^{2})}
\]
\[
\leq C\| u_{0} \|_{H_{x}^{s}} + C(\| u_{\nu} - \varphi \|_{\text{maxim}} + \| u_{\nu} - \varphi \|_{\text{maxim}}^{\theta} \| u_{\nu} \|_{Y_{T}}^{1-\theta})(1 + \| u_{\nu} \|_{Y_{T}})^{2}
\]
\[
+ C\varphi T^{\theta}(1 + \| u_{\nu} \|_{Y_{T}})^{2}.
\]

By Lemma 5.2,

\[
\| u_{\nu} \|_{\text{initial}} + \| u_{\nu} \|_{\text{smooth}}
\]
\[
\leq C\| u_{0} \|_{H_{x}^{s}} + C(\| u_{\nu} - \varphi \|_{\text{maxim}} + \| u_{\nu} - \varphi \|_{\text{maxim}}^{\theta} \| u_{\nu} \|_{Y_{T}}^{1-\theta})(1 + \| u_{\nu} \|_{Y_{T}})^{2}
\]
\[
+ C\varphi T^{\theta}(1 + \| u_{\nu} \|_{Y_{T}})^{2}.
\]

Also, applying Lemma 3.2 and 4.1 to (5.2), we have

\[
\| u_{\nu} \|_{\text{maxim}} \leq C\| u_{0} \|_{H_{x}^{s}} + CT^{1/4} \| u_{\nu} \|_{Y_{T}}^{3}.
\]

From (5.19)–(5.20), it follows that

\[
\| u_{\nu} \|_{Y_{T}} \leq C_{0}\delta_{0}
\]
\[
+ C(\| u_{\nu} - \varphi \|_{\text{maxim}} + \| u_{\nu} - \varphi \|_{\text{maxim}}^{\theta} \| u_{\nu} \|_{Y_{T}}^{1-\theta})(1 + \| u_{\nu} \|_{Y_{T}})^{2}
\]
\[
+ C\varphi T^{\theta}(1 + \| u_{\nu} \|_{Y_{T}})^{2}.
\]
Taking $T \uparrow T_{\nu}$ in (5.21) if $T_{\nu} < \infty$, we have

$$2C_0\delta_0 \leq C_0\delta_0 + C\left(\|u_{\nu} - \varphi\|_{\text{maxim}} + \|u_{\nu} - \varphi\|_{\text{maxim}}^2 (2C_0\delta_0)^{1 - \theta}\right) \cdot (1 + 2C_0\delta_0)^2 + C_\varphi T_{\nu}^\beta (1 + 2C_0\delta_0)^3.$$  (5.22)

Assume here that $\liminf_{\nu \downarrow 0} T_{\nu} = 0$. Then, this is the contradiction. Indeed, by taking $\varphi$ sufficiently close to $u_0$ in $H_x^s$, Lemma 5.3 and (5.22) yield $2C_0\delta_0 \leq 3/2C_0\delta_0$. Hence, $T_{\nu} \geq T_0 > 0$ and (5.3) follows. We next prove (5.4). By the integral equation (5.1) and Lemma 3.1, we see that

$$\|\langle D_x \rangle^{s'-3/2} \partial_x (\varpi_{\nu} - \varpi_{\nu'})\|_{L_x^p(L_T^2)} \leq C\|\langle D_x \rangle^{s'-3/2} (K_{\nu} - K_{\nu'}) (B_{\nu}(u_{\nu}) - B_{\nu}(\varphi)) \eta_\nu * \partial_x^2 \bar{u}_{\nu}\|_{L_x^1(L_T^2)}$$

$$+ C\|\langle D_x \rangle^{s'-3/2} K_{\nu'} (B_{\nu}(u_{\nu}) - B_{\nu'}(u_{\nu'})) \eta_\nu * \partial_x^2 \bar{u}_{\nu}\|_{L_x^1(L_T^2)}$$

$$+ C\|\langle D_x \rangle^{s'-3/2} K_{\nu'} (B_{\nu}(u_{\nu}) - B_{\nu'}(\varphi)) \eta_\nu * \partial_x^2 (\bar{u}_{\nu} - \bar{u}_{\nu'})\|_{L_x^1(L_T^2)}$$

$$+ C\|\langle D_x \rangle^{s'-3/2} K_{\nu'} (B_{\nu'}(u_{\nu'}) - B_{\nu'}(\varphi)) \eta_\nu * \partial_x^2 \bar{u}_{\nu}\|_{L_x^1(L_T^2)}$$

Note that the estimates of integral kernels give

$$\|\langle D_x \rangle^{s'-3/2} (K_{\nu} - K_{\nu'}) \tilde{f}\|_{L_x^p(L_T^2)} \leq C\nu^{\beta} \|\langle D_x \rangle^{s-3/2} \tilde{f}\|_{L_x^p(L_T^2)}$$

Then, we have

$$\|\langle D_x \rangle^{s'-3/2} \partial_x (\bar{w}_{\nu} - \bar{w}_{\nu'})\|_{L_x^p(L_T^2)} \leq C\nu^{\beta} \|\bar{f}\|_{L_x^p(L_T^2)}$$

By Lemma 3.1 (3.3), it is also possible to derive

$$\|\bar{w}_{\nu} - \bar{w}_{\nu'}\|_{L_x^p(H_{x}^{s'-1})} \leq C\nu^{\beta} \|\bar{f}\|_{L_x^p(H_{x}^{s'-1})}$$

Thus, Lemma 5.2 gives

$$\|u_{\nu} - u_{\nu'}\|_{L_x^p(H_{x}^s)} + \|\langle D_x \rangle^{s'-3/2} \partial_x^2 (u_{\nu} - u_{\nu'})\|_{L_x^p(L_T^2)} \leq C\nu^{\beta} \|\bar{f}\|_{L_x^p(H_{x}^{s'-1})}$$
Applying Lemma 3.2 to the integral equation (5.2), we can show that
\[
\max_{j=0,1} \| (D_x)\nu' \partial_x^j (u_{\nu} - u_{\nu'}) \|_{L_x^2(L_T^\infty)} \leq CT^\beta (4C_0\delta_0)^2 \| u_{\nu} - u_{\nu'} \|_{Z_T} \\
+ C(\nu^\beta + \nu'^\beta)(4C_0\delta_0)^3.
\]
(5.24)

Then, (5.23), (5.24) and Lemma 5.3 yield (5.4). □

We now prove our main theorem.

**Proof of Theorem 1.1.** By Proposition 5.1(5.3), we can take a convergent subsequence of \( \{u_{\nu}\}_{\nu \in (0,1]} \) such that
\[
\lim_{\nu \downarrow 0} u_{\nu'} = u \quad \text{weakly-* in } L_T^\infty(H_x^s),
\]
\[
\lim_{\nu \downarrow 0} \langle D_x \rangle^s \partial_x^2 u_{\nu'} = \langle D_x \rangle^s \partial_x^2 u \quad \text{weakly-* in } L_x^2(L_T^\infty),
\]
\[
\lim_{\nu \downarrow 0} \langle D_x \rangle^\mu \partial_x \theta_x u_{\nu'} = \langle D_x \rangle^\mu \theta_x u \quad \text{weakly-* in } L_x^2(L_T^\infty),
\]
where we identify \( L_T^\infty(H_x^s) \) (resp. \( L_x^2(L_T^\infty) \) and \( L_x^2(L_T^\infty) \)) with \( (L_T^1(H_x^{-s}))^* \) (resp. \( (L_T^1(L_x^2))^* \) and \( (L_T^1(L_x^2))^* \)). From Proposition 5.1(5.4), it follows that \( \mathcal{N}(u_{\nu'}, \eta_{\nu'} \partial_x u_{\nu'}) \) tends to \( \mathcal{N}(u, \partial_x u) \) in \( L_T^\infty(L_x^2) \) and so \( u \) satisfies the integral equation:
\[
u = U(t)u_0 - iGN(u, \partial_x u) \quad \text{in } L_T^\infty(L_x^2). 
\]
(5.25)

We next show the continuity in time of \( u \) as an \( H_x^s \) valued function. In (5.25), it is easy to see that \( U(t)u_0 \in C([0,T];H_x^s) \). As for \( GN(u, \partial_x u) \equiv GN(t) \), we observe that
\[
GN(t+h) - GN(t) = U(t+h) \int_t^{t+h} U(-\tau) \mathcal{N}(\tau) d\tau \\
+ (U(t+h) - U(t)) \int_0^t U(-\tau) \mathcal{N}(\tau) d\tau \\
= G_1(h) + G_2(h).
\]
(5.26)

Let \( I = [t, t+h] \) if \( h > 0 \) and \( I = [t+h, t] \) if \( h < 0 \). Note that, by the dual estimate of \( \| D_x^{1/2} U(t) \phi \|_{L_x^\infty L_x^2} \leq C \| \phi \|_{L_x^2} \), we have \( \| D_x^{1/2} \int_0^t U(-\tau) \mathcal{N}(\tau) d\tau \|_{L_x^2} \leq C \| \mathcal{N} \|_{L_x^1(L_T^2)}. \) Then, Lebesgue’s convergence theorem yields
\[
\| D_x^{-1} \partial_x G_1(h) \|_{L_x^2} \leq C \| D_x^{-3/2} \partial_x \mathcal{N} \|_{L_x^1(L_T^2)} \\
\rightarrow 0 \quad \text{as } h \rightarrow 0.
\]

Since \( \| D_x^{-1} \partial_x \int_0^t U(-\tau) \mathcal{N}(\tau) d\tau \|_{L_x^2} < \infty \) by Lemma 3.1 (3.3), the strong continuity of the Schrödinger group yields \( \lim_{h \rightarrow 0} D_x^{-1} \partial_x G_2(h) = 0 \) in \( L_x^2 \). Hence, \( u \in C([0,T]; H_x^s) \). The uniqueness and Lipschitz’ dependence on the initial data follow from the routine work. □

**References**


