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京都大学
A subsolution for TU games

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1. Introduction and Preliminaries

In this paper we consider a subset of the imputation set for cooperative TU games, and examine its properties.

An $n$-person cooperative game with side payments (abbreviated as a game) is an ordered pair $(N, v)$, where $N = \{1, 2, \ldots, n\}$ is the set of players and $v$, called the characteristic function, is a real-valued function on the power set of $N$, satisfying $v(\emptyset) = 0$. For simplicity we express a game $(N, v)$ as $v$. A subset of $N$ is called a coalition. For any set $Z$, $|Z|$ denotes the cardinality of $Z$. For $S \subseteq N$ and $x \in \mathbb{R}^N$, we define $x(S) = \sum_{i \in S} x_i$ (if $S \neq \emptyset$) and $= 0$ (if $S = \emptyset$). A pre-imputation for a game $v$ is a vector $x \in \mathbb{R}^N$ that satisfies

$$x(N) = v(N).$$

We denote by $\mathcal{P}I \equiv \mathcal{P}I(v)$ the set of all pre-imputations for a game $v$. A pre-imputation $x \in \mathcal{P}I$ is said to be individually rational if

$$x_i \geq v(i), \quad \forall i \in N.$$  \hspace{1cm} (1.2)

An individually rational pre-imputation is called an imputation. We denote by $\mathcal{I} \equiv \mathcal{I}(v)$ the set of all imputations for a game $v$, which we call the imputation set. A pre-imputation $x \in \mathcal{P}I$ is said to be reasonable if

$$x_i \leq u_i, \quad \forall i \in N,$$  \hspace{1cm} (1.3)

where $u_i \equiv u_i(v) \equiv \max_{i \in S} \{v(S) - v(S \backslash \{i\})\}$ for all $i \in N$. We denote by $\mathcal{R} \equiv \mathcal{R}(v)$ the set of all reasonable pre-imputations for a game $v$. For $x, y \in \mathcal{I}$ and for a coalition $S \subseteq N$, we say that $x$ dominates $y$ via $S$, denoted by $x \succ sy$, if

$$\begin{cases} (i) \ x_i > y_i, \quad \forall i \in S, \\ (ii) \ x(S) \leq v(S). \end{cases}$$ \hspace{1cm} (1.4)

For $x, y \in \mathcal{I}$, we say that $x$ dominates $y$, denoted by $x \succ y$, if there is an $S$ such that $x \succ sy$.  

\footnote{For simplicity, we write $v(\{i\}), v(\{i, j\})$ as $v(i), v(ij)$.}
dominates $y$ via $S$. For $\mathcal{X} \subseteq \mathcal{I}$, we denote by $\text{Dom} \mathcal{X}$ the set of all imputations dominated by some element of $\mathcal{X}$. A set of imputations $\mathcal{X} \subseteq \mathcal{I}$ is called a stable set if it satisfies

$$
\begin{align*}
(\text{i}) \quad & \mathcal{X} \cap \text{Dom} \mathcal{X} = \emptyset \quad \text{(internal stability)}, \\
(\text{ii}) \quad & \mathcal{X} \cup \text{Dom} \mathcal{X} = \mathcal{I} \quad \text{(external stability)}.
\end{align*}
$$

(1.5)

The core of a game $v$, denoted by $\mathcal{C} \equiv \mathcal{C}(v)$, is defined by

$$
\mathcal{C} = \mathcal{I} \setminus \text{Dom} \mathcal{I}.
$$

(1.6)

2. A Subsolution

In this section we define a subset $\mathcal{Q}$ of the imputation set and examine properties of it. We assume that for every game in this section the imputation set is not empty, $\mathcal{I}(v) \neq \emptyset$, that is,

$$
v(N) \geq \sum_{i \in N} v(i).
$$

(2.1)

Definition. A set $\mathcal{Q} \equiv \mathcal{Q}(v) \subseteq \mathcal{I}$ is defined by

$$
\mathcal{Q} \equiv \{x \in \mathcal{I} : \forall y \in \mathcal{I} \text{ s.t. } y \succ x, \exists z \in \mathcal{I} \text{ s.t. } z \succ y \text{ and } z \not\simeq x\}.
$$

Remark. Let

$$
\mathcal{Q}' \equiv \{x \in \mathcal{I} : \forall y \in \mathcal{I} \text{ s.t. } y \succ x, \exists \approx \in \mathcal{I} \text{ s.t. } z \succ y\}.
$$

If $\mathcal{C} = \emptyset$ then $\mathcal{I} = \text{Dom} \mathcal{I}$. And so $\mathcal{Q}' = \mathcal{I}$.

Hereafter we fix a game $(N, v)$.

Proposition 2.1. For a game $v$, let $\mathcal{X}$ be a stable set. Then $\mathcal{X} \subseteq \mathcal{Q}$.

Proof: Let $x \in \mathcal{X}$ and suppose $y \succ x$ where $y \in \mathcal{I}$. By the internal stability, we have $y \notin \mathcal{X}$, and so by the external stability, there exists $z \in \mathcal{X}$ such that $z \succ y$. By the internal stability, $z \not\simeq x$. Hence $x \in \mathcal{Q}$. $\square$

Proposition 2.2. For a game $v$, it holds $\mathcal{Q} \subseteq \mathcal{R}$.

Proof: Let $x \in \mathcal{Q}$ and assume $x \notin \mathcal{R}$. There exists $i \in N$ such that $x_i > u_i$. This implies $x_i > v(N) - v(N \setminus \{i\})$, which implies $x(N \setminus \{i\}) < v(N \setminus \{i\})$. Hence we can take $y \in \mathcal{I}$ such that $y \succ x$ via $N \setminus \{i\}$ and $y_i \geq u_i$. Since $x \in \mathcal{Q}$, there exists $z \in \mathcal{I}$ such that $z \succ y$ via a coalition $S$ and $z \not\simeq x$. If $i \notin S$ then $z \succ x$ via $S$, which is a contradiction. If $z(S \setminus \{i\}) \leq v(S \setminus \{i\})$ then $z \succ x$ via $S \setminus \{i\}$, which is a contradiction. So we must have $i \in S$ and $z(S \setminus \{i\}) > v(S \setminus \{i\})$. Then $y_i < z_i = z(S) - z(S \setminus \{i\}) < v(S) - v(S \setminus \{i\}) \leq u_i$, which is a contradiction. $\square$
From this proposition we see that if \( v(S \cup \{i\}) = v(S) + v(i) \) for all \( S : i \notin S \) and \( x \in Q \) then it must hold \( x_i = v(i) \) since \( u_i = v(i) \).

**Proposition 2.3.** For a game \( v \), it holds \( C \subseteq Q \subseteq I \setminus \text{Dom}C \).

**Proof:** By Proposition 2.1, we have \( C \subseteq Q \). If \( x \in \text{Dom}C \), then there exists \( y \in C \) such that \( y \succ x \) and \( y \notin \text{Dom}I \). Hence \( x \in Q \).

**Proposition 2.4.** For a game \( v \), the core \( C \) is a stable set if and only if \( C = Q \).

**Proof:** Assume that \( C \) is a stable set. By Proposition 2.3, we have \( C \subseteq Q \). Let \( x \in Q \setminus C \). Since \( C \) is a stable set, by the external stability there exists \( y \in C \) such that \( y \succ x \). But there exists no imputation which dominates \( y \) because \( y \) is in the core. This is a contradiction. Hence \( Q \setminus C = \emptyset \).

Conversely assume \( C = Q \). Since \( C \subseteq X \) for any stable set \( X \), we have \( C \subseteq X \subseteq C \) from Proposition 2.1. Hence \( C \) is a unique stable set.

**Proposition 2.5.** Suppose \((N, v)\) is symmetric, that is, \( v \) depends on only the number of members in a coalition. For every \( S \subseteq N \), let \( v(s) = v(S) \) where \( s = |S| \). Assume \( v(1) = 0 \). Then

\[
x^* = \left( \frac{v(n)}{n}, \ldots, \frac{v(1)}{n} \right) \in Q.
\]

**Proof:** Suppose \( y \succ x^* \) via \( S \) and \( y \npreceq x^* \) via every \( T \) such that \( T \subset S \). Then

\[
y_j > \frac{v(1)}{n}, \forall j \in S, \quad y(S) \leq v(S), \quad \text{and} \quad y(T) > v(T), \forall T \subset S, T \neq S.
\]

Then

\[
v(n) = y(N) = y(N \setminus S) + y(S) > y(N \setminus S) + \frac{|S|}{n}v(n).
\]

This implies \( \frac{n-|S|}{n}v(n) > y(N \setminus S) \), and so there exists \( i \in N \setminus S \) such that \( y_i < \frac{v(n)}{n} \). For some \( j_0 \in S \), let \( S^0 = (S \setminus \{j_0\}) \cup \{i\} \). Define \( z \in P_I \) by

\[
z_j = \begin{cases} y_j + \epsilon, & j \in S^0 = (S \setminus \{j_0\}) \cup \{i\}; \\ \frac{v(n)}{n} - \delta, & j \in N \setminus S^0. \end{cases}
\]

Then \( y(S^0) < y(S) \leq v(S) = v(S^0) \), and so for sufficiently small \( \epsilon > 0 \), we have

\[
z(S^0) = y(S^0) + \epsilon|S^0| \leq v(S^0), \quad z_j > y_j, \forall j \in S^0.
\]
Hence \( z \succ y \) via \( S^0 \), and \( z \not\succ x^\ast \) since \( z_i = y_i + \epsilon \leq \frac{v(n)}{n} \) and \( z(T) = y(T) + \epsilon|T| \succ y(T) \succ v(T) \) for every \( T \subset S^0 \setminus \{j_0\} \). It remains to see that it is possible to find \( \epsilon > 0 \) and \( \delta > 0 \) such that 
\[
\forall T \subset S^0 \setminus \{j_0\}, \quad z(S^0) \leq v(S^0) \quad \text{and} \quad z(N) = v(n) \quad \text{if and only if}
\]
\[
y(S^0 \setminus \{i\}) - \frac{|S^0| - 1}{n} v(n) + \epsilon|S^0| = \frac{v(n)}{n} - y_i - \delta(n - |S^0|).
\]
(2.2)

0 < \delta \leq \frac{v(n)}{n} if and only if 
\[
\frac{v(n)}{n} - \frac{y(S^0)}{|S^0|} < \epsilon \leq \frac{v(n) - y(S^0)}{|S^0|}. 
\]
(2.3)

\( z(S^0) \leq v(S^0) \) if and only if 
\[
\epsilon \leq \frac{v(S^0) - y(S^0)}{|S^0|}. 
\]
(2.4)

Since \( x^\ast(S) < y(S) \leq v(S^0) \), we have \( \frac{v(n)}{n} < \frac{v(S^0)}{|S^0|} \). Hence there exist \( \epsilon \) and \( \delta \) which satisfy (2.2) - (2.4). \( \square \)

Proposition 2.5 implies that \( Q \neq \emptyset \) when \( v \) is symmetric.

**Definition.** (Roth (1976)) A set \( \mathcal{Y} \subseteq \mathcal{I} \) is called a *subsolution* if

\[
\begin{align*}
(i) & \quad \mathcal{Y} \subseteq \mathcal{I} \setminus \text{Dom}\mathcal{Y}, & \text{(internal stability)} \\
(ii) & \quad \mathcal{Y} = \mathcal{I} \setminus \text{Dom}(\mathcal{I} \setminus \text{Dom}\mathcal{Y}).
\end{align*}
\]

**Proposition 2.6.** Let \( \mathcal{Y} \) be a subsolution. Then \( \mathcal{Y} \subseteq Q \).

**Proof:** Let \( \mathcal{Y} \) be a subsolution and suppose \( x \in \mathcal{Y} \). For any \( y \in \mathcal{I} \) such that \( y \succ x \), it holds \( y \notin \mathcal{Y} \) since \( \mathcal{Y} \) is internally stable. So \( y \notin \mathcal{I} \setminus \text{Dom}(\mathcal{I} \setminus \text{Dom}\mathcal{Y}) \) by the definition of subsolution. Hence \( y \in \text{Dom}(\mathcal{I} \setminus \text{Dom}\mathcal{Y}) \). This implies that there exists \( z \in \mathcal{I} \setminus \text{Dom}\mathcal{Y} \) such that \( z \succ y \). Since \( x \notin \text{Dom}(\mathcal{I} \setminus \text{Dom}\mathcal{Y}) \), it holds that \( z \not\succ x \). Hence \( x \in Q \). \( \square \)

The next example says that the set \( Q \) is different from the union of all stable sets. A remaining problem is whether the set \( Q \) coincides or not with the union of all stable sets when there exists a stable set.

**Example 2.1.** The 10-Person Game (Lucas (1969)). Let us consider the 10-person game:

\[
\begin{align*}
v(N) &= 5, v(13579) = 4, v(3579) = v(1579) = v(1379) = 3, \\
v(1479) &= v(3679) = v(2579) = 2, v(357) = v(157) = v(137) = 2, \\
v(359) &= v(159) = v(139) = 2, v(12) = v(34) = v(56) = v(78) = v(90) = 1, \\
v(i) &= 0 \quad \forall i \in N
\end{align*}
\]
and, for other $S, v(S) = 0$. Let
\[ B = \{ x \in I : x(12) = x(34) = x(56) = x(78) = x(90) = 1, \ x_i \geq 0, \forall i \in N \}. \]

It is easy to check that the core $C$ of this game is:
\[ C = \{ x \in B : x(13579) \geq 4 \}. \]

Define the following subsets of $B$:
\[
\begin{align*}
\mathcal{E}_1 &= \{ x \in B : x_3 = x_5 = 1, x_1 < 1, x(79) < 1 \}, \\
\mathcal{F}_{35} &= \{ x \in B : x(35) = 1, x_1 < 1, x(79) \geq 1 \} \setminus C, \\
\mathcal{F}_{51} &= \{ x \in B : x(15) = 1, x_3 < 1, x(79) \geq 1 \} \setminus C, \\
\mathcal{F}_{13} &= \{ x \in B : x(13) = 1, x_5 < 1, x(79) \geq 1 \} \setminus C, \\
\mathcal{F}_7 &= \{ x \in B : x_7 = 1, x_9 < 1, x(359) \geq 2, x(159) \geq 2, x(139) \geq 2 \} \setminus C, \\
\mathcal{F}_9 &= \{ x \in B : x_9 = 1, x_7 < 1, x(357) \geq 2, x(157) \geq 2, x(137) \geq 2 \} \setminus C, \\
\mathcal{F}_{79} &= \{ x \in B : x_7 = x_9 = 1 \} \setminus C. 
\end{align*}
\]

It is well-known that
\[ I \setminus B, \ B \setminus (C \cup \mathcal{E} \cup \mathcal{F}), \ C, \ \mathcal{E}, \ \mathcal{F} \]
constitute a partition of $I$. It is known that
\[ I \setminus (C \cup \mathcal{E} \cup \mathcal{F}) \subseteq \text{Dom} \ C, \]
from which and from Proposition 2.3, we have $Q \subseteq C \cup \mathcal{E} \cup \mathcal{F}$. It is known that the set $C \cup \mathcal{F}$ is a subsolution, and so $C \cup \mathcal{F} \subseteq Q$. Let's see $\mathcal{E} \subset Q$. Assume $x \in \mathcal{E}_1$. If $y \succ x$ then $y \notin C \cup \mathcal{F}$ since $\mathcal{E} \cap \text{Dom}(C \cup \mathcal{F}) = \emptyset$. So $y \notin \mathcal{E} \cup \text{Dom} \ C$. Suppose $y \in \mathcal{E} \cup \text{Dom} \ C$. Then there exists $z \in C$ such that $z \succ y$, but $z \not\succ x$ since $\mathcal{E} \cap \text{Dom} \ C = \emptyset$. Hence $x \in Q$. Suppose $y \in \mathcal{E}$. Then $y \in \mathcal{E}_3$. There exists $z \in \mathcal{E}_3$ such that $z \succ y$, but $z \not\succ x$ since $\mathcal{E}_3 \cap \text{Dom}(\mathcal{E}_1 \cup \mathcal{E}_3) = \emptyset$. Hence $x \in Q$. So $\mathcal{E}_1 \subset Q$. By permutation, we see $\mathcal{E}_3 \cup \mathcal{E}_5 \subset Q$. Consequently we have that $Q = C \cup \mathcal{E} \cup \mathcal{F}$. Note that the set $C \cup \mathcal{F}$ is a subsolution and it is the supercore$^2$.

The next example says that the set $\mathcal{Q}$ is not always a convex set.

**Example 2.2.** (Lucas 1969) Let $n = 8$ and

$$v(N) = 4, v(1467) = 2, v(12) = v(34) = v(56) = v(78) = 1$$

and $v(S) = 0$ for all other $S$. Let

$$B = \{ x \in \mathcal{I} : x(12) = x(34) = x(56) = x(78) = 1 \}.$$  

For $i = 1, 4, 6, 7$, let

$$\mathcal{F}_i = B \cap \{ x \in \mathcal{I} : x_i = 1 \}. $$

The core is

$$\mathcal{C} = \{ x \in B : x(1467) \geq 2 \}. $$

It is known that

$$\mathcal{K} = \mathcal{C} \cup \mathcal{F}_1 \cup \mathcal{F}_4 \cup \mathcal{F}_6 \cup \mathcal{F}_7 $$

is a unique solution which is nonconvex. Let's see $\mathcal{Q} = \mathcal{K}$. It is known that $\mathcal{I} \setminus B \subseteq \text{Dom} \mathcal{C}$, which implies $\mathcal{Q} \subseteq B$. Let $x \in B \setminus \mathcal{K}$. Then $x(1467) < 2$ and $x_i < 1$ for $i = 1, 4, 6, 7$. Define $y \in B$ by

$$x_i < y_i < 1, \text{ for } i = 1, 4, 6, 7, \text{ and } y(1467) = 2, \text{ and } y(i, i + 1) = 1, \text{ for } i = 1, 2, 3, 4.$$ 

Then $y \succ x$ via $\{1, 4, 6, 7\}$ and $y \in \mathcal{C}$. So $x \in \text{DomC}$ and $x \notin \mathcal{Q}$. Hence $\mathcal{Q} = \mathcal{K}$.

3. **A Subsolution and the Nucleolus**

In this section we examine an inclusion relation between the nucleolus and the $\mathcal{Q}$.

Let $v$ be a game. For $x \in \mathcal{I}(v)$ let $\theta(x)$ be the $2^n$-vector whose components are the numbers $e(S, x), S \subseteq N$, arranged in nonincreasing order, i.e., $\theta(x)_i \geq \theta(x)_j$ whenever $1 \leq i \leq j \leq 2^n$. We say that $\theta(x)$ is lexicographically smaller than $\theta(y)$, denoted $\theta(x) <_L \theta(y)$, if and only if there is an index $k$ such that $\theta(x)_i = \theta(y)_i$ for all $i < k$, and $\theta(x)_k < \theta(y)_k$. We write $\theta(x) \leq_L \theta(y)$ for not $\theta(y) <_L \theta(x)$. The *nucleolus* for $v$ is the set $\mathcal{N}$ of vectors in $\mathcal{I}$ that minimizes $\theta$ in the lexicographic ordering, i.e.,

$$\mathcal{N} = \{ x \in \mathcal{I} : \theta(x) \leq_L \theta(y) \text{ for all } y \in \mathcal{I} \}. $$

It is known that the nucleolus is included in the core whenever the core is non-empty. So the nucleolus is included in the set $\mathcal{Q}$ by Proposition 2.3 whenever the core is non-empty. Since the nucleolus satisfies the symmetry, Proposition 2.5 implies that the nucleolus is included in the set $\mathcal{Q}$ when the game is symmetric.
Proposition 3.1. Assume $v(S) = 0$ for $S$ such that $|S| \leq n - 2$. The nucleolus $\mathcal{N}$ is included in the set $Q$.

Proof: If $C \neq \emptyset$ it holds $\mathcal{N} \subseteq Q$ by Proposition 2.3 since it is known that $\mathcal{N} \subseteq C$. Assume that $C = \emptyset$. Let $\mathcal{N} = \{x^*\}$. Without loss of generality assume

$$e(N \setminus \{1\}, x^*) \geq \ldots \geq e(N \setminus \{n\}, x^*).$$

Since $x^* \notin C$ there exists $y \in \mathcal{I}$ such that $y \succ x^*$ via $N \setminus \{i\}$ for $i \in N$. Then

$$\begin{cases} e(N \setminus \{j\}, y) > e(N \setminus \{1\}, x^*), & \forall j \neq i; \\
e(N \setminus \{i\}, x^*) > e(N \setminus \{i\}, y) \geq 0. \end{cases}$$

Assume $i \geq 2$. Define $z \in \mathcal{I}$ by

$$e(N \setminus \{j\}, z) = \begin{cases} e(N \setminus \{j\}, y) + \epsilon, & j \neq 1; \\
e(N \setminus \{j\}, x^*) - (n-1)\epsilon, & j = 1, \end{cases}$$

so that $e(N \setminus \{1\}, x^*) - (n-1)\epsilon \geq 0$ and $e(N \setminus \{i\}, y) + \epsilon \leq e(N \setminus \{i\}, x^*)$. That is,

$$0 < \epsilon \leq \min\{\frac{e(N \setminus \{1\}, x^*)}{n-1}, e(N \setminus \{i\}, x^*) - e(N \setminus \{i\}, y)\}.$$

We have $z \succ y$ via $N \setminus \{1\}$. In order for $z$ to dominate $x^*$, it must dominate only via $N \setminus \{1\}$. This is impossible because $e(N \setminus \{i\}, z) \leq e(N \setminus \{i\}, x^*)$. So $z \neq x^*$.

Next assume $i = 1$. Assume $e(N \setminus \{1\}, x^*) > e(N \setminus \{2\}, x^*)$. Since the nucleolus satisfies, what is called, Property $P^3$, we must have $x^*_1 = v(1) = 0$. Then $e(N \setminus \{1\}, x^*) > e(N \setminus \{1\}, y) \geq 0$, which implies $y_1 < 0$ contradicting $y \in \mathcal{I}$. Hence we have $e(N \setminus \{1\}, x^*) = e(N \setminus \{2\}, x^*)$. Exchange $e(N \setminus \{2\}, x^*)$ with $e(N \setminus \{1\}, x^*)$. Then it reduces to the case $i = 2$. \hfill $\square$

4. Remarks

For 3-person games, by Proposition 3.1, the nucleolus is included in the set $Q$ and also the reader could see that the set $Q$ coincides with the union of all stable sets.

It is interesting to examine whether the nucleolus is included in the set $Q$ or not for broader classes of games.

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3See, for example, pp.328-332 of Owen (1995).
References


