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On scaling law and Tsallis entropy derived from a fundamental nonlinear differential equation

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Abstract

The so-called q-exponential function, one parameter generalization of the exponential function, is given by the solution of the fundamental nonlinear differential equation. Starting from the differential equation, it is shown that its solution has a scaling property and the corresponding information measure to the solution is uniquely determined to be Tsallis entropy. Moreover, the Shannon additivity, one of the axioms of Tsallis entropy, is found to be derived from q-multinomial coefficient and the Leibniz product rule of the q-derivative independently.

1 Scaling property derived from a fundamental nonlinear differential equation

The exponential function is often appeared in every scientific field. Among many properties of the exponential function, the linear differential function $dy/dx = y$ is the most important characterization of the exponential function. A slightly nonlinear generalization of this linear differential equation is given by

$$\frac{dy}{dx} = y^q \quad (q \in \mathbb{R}). \tag{1}$$

(See the equation (17) at page 5 of [1] and the equations (22)-(23) at page 8 of [2].) This nonlinear differential equation is equivalent to

$$\int \frac{1}{y^q} dy = \int dx. \tag{2}$$

Then we define the so-called *q-logarithm* $\ln_q x$.

$$\ln_q x := \frac{x^{1-q} - 1}{1-q} \tag{3}$$

as a generalization of $\ln x$. Applying the property:

$$\frac{d}{dx} \ln_q x = \frac{1}{x^q}, \tag{4}$$

to (2), we obtain

$$\ln_q y = x + C \tag{5}$$

where C is *any* constant [3]. Then we define the so-called *q-exponential* $\exp_q(x)$ as the inverse function of $\ln_q x$ as follows:

$$\exp_q(x) := \begin{cases} [1 + (1-q)x]^{\frac{1}{1-q}} & \text{if } 1 + (1-q)x > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Note that the q-logarithm and q-exponential recover the usual logarithm and exponential when $q \to 1$, respectively (See the pages 84-87 of [4] for the detail properties of these generalized functions $\ln_q x$ and $\exp_q(x)$). Thus, the general solution to the nonlinear differential equation (1) becomes

$$y = \exp_q(x+C) = \exp_q(C)\exp_q\left(\frac{x}{\left(\exp_q(C)\right)^{1-q}}\right) \tag{7}$$

where C is *any* constant satisfying $1+(1-q)C>0$. Dividing the both sides by $\exp_q(C)$ of the above solution, we obtain

$$\frac{y}{\exp_q(C)} = \exp_q\left(\frac{x}{\left(\exp_q(C)\right)^{1-q}}\right). \tag{8}$$

Under the following scaling:

$$y\prime := \frac{y}{\exp_q(C)}, \quad x\prime := \frac{x}{\left(\exp_q(C)\right)^{1-q}}, \tag{9}$$

we obtain

$$y\prime = \exp_q(x\prime). \tag{10}$$

This means that the solution of the nonlinear differential equation (1) obtained above is *"scale-invariant"* under the above scaling (9). Moreover, we can choose *any* constant C satisfying $1+(1-q)C>0$ because C is an integration constant of (2).

Note that the above scaling (9) with respect to both variables x and y can be observed only when $q \neq 1$ and $C \neq 0$. In fact, when $q=1$, i.e., $y=\exp(x+C)$, (9) reduces to the scaling with respect to only y, i.e., $x\prime = x$, and when $C=0$, both scalings in (9) disappears [3].

As similarly as the relation between the exponential function $\exp(x)$ and Shannon entropy, we expect the corresponding information measure to the q-exponential function $\exp_q(x)$. There exist some candidates such as Rényi entropy, Tsallis entropy and so on. But the algebra derived from the q-exponential function uniquely determines Tsallis entropy as the corresponding information measure. In the following sections of this paper, we present the two mathematical results to uniquely determine Tsallis entropy by means of the already established formulations such as the q-exponential law, the q-multinomial coefficient and q-Stirling's formula.

2 q-exponential law

The exponential law plays an important role in mathematics, so this law is also expected to be generalized based on the q-exponential function $\exp_q(x)$. For this purpose, the new multiplication operation $\otimes_q$ is introduced in [5] and [6] for satisfying the following identities:

$$\ln_q(x \otimes_q y) = \ln_q x + \ln_q y, \tag{11}$$

$$\exp_q(x) \otimes_q \exp_q(y) = \exp_q(x+y). \tag{12}$$

The concrete form of the q-logarithm or q-exponential has been already given in the previous section, so that the above requirements as q-exponential law leads us to the definition of $\otimes_q$ between two positive numbers.

Definition 1 *For two positive numbers x and y, the q-product $\otimes_q$ is defined by*

$$x \otimes_q y := \begin{cases} \left[x^{1-q}+y^{1-q}-1\right]^{\frac{1}{1-q}}, & \textit{if } x>0,\ y>0,\ x^{1-q}+y^{1-q}-1>0, \\ 0, & \textit{otherwise.} \end{cases} \tag{13}$$

The *q-product* recovers the usual product such that $\lim_{q\to 1}(x\otimes_q y) = xy$. The fundamental properties of the q-product $\otimes_q$ are almost the same as the usual product, but

$$a\,(x\otimes_q y) \neq ax\otimes_q y \quad (a,x,y\in\mathbb{R}). \tag{14}$$

The other properties of the *q-product* are available in [5] and [6].

In order to see one of the validities of the q-product, we recall the well known expression of the exponential function $\exp(x)$ given by

$$\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n. \tag{15}$$

Replacing the power on the right side of (15) by the n times of the q-product $\otimes_q^n$:

$$x^{\otimes_q^n} := \underbrace{x\otimes_q\cdots\otimes_q x}_{n\ \text{times}}, \tag{16}$$

$\exp_q(x)$ is obtained. In other words, $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{\otimes_q^n}$ coincides with $\exp_q(x)$.

$$\exp_q(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{\otimes_q^n} \tag{17}$$

The proof of (17) is given in the appendix of [7]. This coincidence (17) indicates a validity of the q-product. In fact, the present results in the following sections reinforce it.

3 q-multinomial coefficient and q-Stirling's formula

We briefly review the q-multinomial coefficient and the q-Stirling's formula by means of the q-product $\otimes_q$. As similarly as for the q-product, *q-ratio* is introduced as follows:

Definition 2 *For two positive numbers x and y, the inverse operation to the q-product is defined by*

$$x\oslash_q y := \begin{cases} \left[x^{1-q}-y^{1-q}+1\right]^{\frac{1}{1-q}}, & \text{if } x>0,\ y>0,\ x^{1-q}-y^{1-q}+1>0,\\ 0, & \text{otherwise}\end{cases} \tag{18}$$

which is called q-ratio in [6].

$\oslash_q$ is also derived from the following satisfactions, similarly as for $\otimes_q$ [5][6].

$$\ln_q(x\oslash_q y) = \ln_q x - \ln_q y, \tag{19}$$

$$\exp_q(x)\oslash_q \exp_q(y) = \exp_q(x-y). \tag{20}$$

The q-product and q-ratio are applied to the definition of the q-multinomial coefficient [7].

Definition 3 *For $n=\sum_{i=1}^k n_i$ and $n_i\in\mathbb{N}$ $(i=1,\cdots,k)$, the q-multinomial coefficient is defined by*

$$\begin{bmatrix} & n & \\ n_1 & \cdots & n_k\end{bmatrix}_q := (n!_q)\oslash_q\left[(n_1!_q)\otimes_q\cdots\otimes_q(n_k!_q)\right]. \tag{21}$$

From the definition (21), it is clear that

$$\lim_{q\to 1}\begin{bmatrix} & n & \\ n_1 & \cdots & n_k\end{bmatrix}_q = \begin{bmatrix} & n & \\ n_1 & \cdots & n_k\end{bmatrix} = \frac{n!}{n_1!\cdots n_k!}. \tag{22}$$

In addition to the q-multinomial coefficient, the q-Stirling's formula is useful for many applications such as our main results. By means of the q-product (13), the q-factorial $n!_q$ is naturally defined as follows.

Definition 4 *For a natural number $n \in \mathbb{N}$ and $q \in \mathbb{R}^+$, the q-factorial $n!_q$ is defined by*

$$n!_q := 1 \otimes_q \cdots \otimes_q n. \tag{23}$$

Using the definition of the q-product (13), $\ln_q(n!_q)$ is explicitly expressed by

$$\ln_q(n!_q) = \frac{\sum\limits_{k=1}^{n} k^{1-q} - n}{1-q}. \tag{24}$$

If an approximation of $\ln_q(n!_q)$ is not needed, this explicit form should be directly used for its computation. However, in order to clarify the correspondence between the studies $q = 1$ and $q \neq 1$, the approximation of $\ln_q(n!_q)$ is useful. In fact, using the following q-Stirling's formula, we obtain the unique generalized entropy corresponding to the q-exponential function $\exp_q(x)$, shown in the following sections.

Theorem 5 *Let $n!_q$ be the q-factorial defined by (23). The rough q-Stirling's formula $\ln_q(n!_q)$ is computed as follows:*

$$\ln_q(n!_q) = \begin{cases} \dfrac{n}{2-q}\ln_q n - \dfrac{n}{2-q} + O(\ln_q n) & \textit{if} \quad q \neq 2, \\ n - \ln n + O(1) & \textit{if} \quad q = 2. \end{cases} \tag{25}$$

The proof of the above formulas (25) is given in [7].

4 Tsallis entropy uniquely derived from the q-multinomial coefficient and q-Stirling's formula

In this section we show that Tsallis entropy is uniquely and naturally derived from the fundamental formulations presented in the previous section. In order to avoid separate discussions on the positivity of the argument in (21), we consider the q-logarithm of the q-multinomial coefficient to be given by

$$\ln_q \begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_q = \ln_q(n!_q) - \ln_q(n_1!_q) \cdots - \ln_q(n_k!_q). \tag{26}$$

The unique generalized entropy corresponding to the q-exponential is derived from the q-multinomial coefficient using the q-Stirling's formula as follows [7].

Theorem 6 *When n is sufficiently large, the q-logarithm of the q-multinomial coefficient coincides with Tsallis entropy (28) as follows:*

$$\ln_q \begin{bmatrix} & n & \\ n_1 & \cdots & n_k \end{bmatrix}_q \simeq \begin{cases} \dfrac{n^{2-q}}{2-q} \cdot S_{2-q}\left(\dfrac{n_1}{n}, \cdots, \dfrac{n_k}{n}\right) & \textit{if} \quad q > 0,\ q \neq 2 \\ -S_1(n) + \sum\limits_{i=1}^{k} S_1(n_i) & \textit{if} \quad q = 2 \end{cases} \tag{27}$$

where S_q is Tsallis entropy [8]:

$$S_q := \frac{1 - \sum\limits_{i=1}^{n} p_i^q}{q-1} \tag{28}$$

and $S_1(n)$ is given by $S_1(n) := \ln n$.

The proof of this theorem is given in [7].

Note that the above relation (27) reveals a surprising *symmetry*: (27) is equivalent to

$$\ln_{1-(1-q)}\left[\begin{array}{ccc} & n & \\ n_1 & \cdots & n_k \end{array}\right]_{1-(1-q)} \simeq \frac{n^{1+(1-q)}}{1+(1-q)}\cdot S_{1+(1-q)}\left(\frac{n_1}{n},\cdots,\frac{n_k}{n}\right) \tag{29}$$

for $q>0$ and $q\neq 2$. This expression represents that there exists a *symmetry* with a factor $1-q$ around $q=1$ in the algebra of the q-product. Substitution of some concrete values of q into (27) or (29) helps us understand the meaning of this symmetry.

Remark that the above correspondence (27) and the symmetry (29) reveals that the q-exponential function (6) derived from (1) is consistent with Tsallis entropy only as information measure.

5 The generalized Shannon additivity derived from the q-multinomial coefficient

This section shows another way to uniquely determine the generalized entropy. More precisely, the identity derived from the q-multinomial coefficient coincides with the generalized Shannon additivity which is the most important axiom for Tsallis entropy.

Consider a partition of a given natural number n into k groups such as $n=\sum_{i=1}^{k} n_i$. In addition, each natural number n_i $(i=1,\cdots,k)$ is divided into m_i groups such as $n_i=\sum_{j=1}^{m_i} n_{ij}$ where $n_{ij}\in\mathbf{N}$.

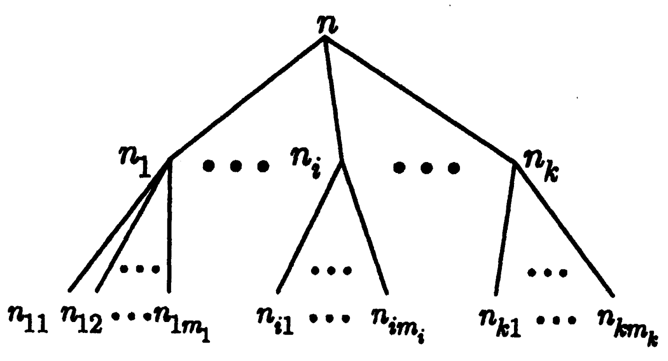

図 1: partition of a natural number n

Then, the following identity holds for the q-multinomial coefficient.

$$\left[\begin{array}{ccc} & n & \\ n_{11} & \cdots & n_{km_k} \end{array}\right]_q = \left[\begin{array}{ccc} & n & \\ n_1 & \cdots & n_k \end{array}\right]_q \otimes_q \left[\begin{array}{ccc} & n_1 & \\ n_{11} & \cdots & n_{1m_1} \end{array}\right]_q \otimes_q\cdots\otimes_q \left[\begin{array}{ccc} & n_k & \\ n_{k1} & \cdots & n_{km_k} \end{array}\right]_q \tag{30}$$

It is very easy to prove the above relation (30) by taking the q-logarithm of the both sides and using (26).

On the other hand, the above identity (30) is reformed to the generalized Shannon additivity in the following way. Taking the q-logarithm of the both sides of the above relation (30), we have

$$\ln_q\left[\begin{array}{ccc} & n & \\ n_{11} & \cdots & n_{km_k} \end{array}\right]_q = \ln_q\left[\begin{array}{ccc} & n & \\ n_1 & \cdots & n_k \end{array}\right]_q + \sum_{i=1}^{k}\ln_q\left[\begin{array}{ccc} & n_i & \\ n_{i1} & \cdots & n_{im_i} \end{array}\right]_q. \tag{31}$$

From the relation (27), we obtain

$$S_{2-q}\left(\frac{n_{11}}{n},\cdots,\frac{n_{km_k}}{n}\right)=S_{2-q}\left(\frac{n_1}{n},\cdots,\frac{n_k}{n}\right)+\sum_{i=1}^{k}\left(\frac{n_i}{n}\right)^{2-q}S_{2-q}\left(\frac{n_{i1}}{n_i},\cdots,\frac{n_{im_i}}{n_i}\right). \tag{32}$$

Then, by means of the following probabilities defined by

$$p_{ij}:=\frac{n_{ij}}{n}\quad (i=1,\cdots,k,\quad j=1,\cdots,m_k), \tag{33}$$

$$p_i:=\sum_{j=1}^{m_i}p_{ij}=\sum_{j=1}^{m_i}\frac{n_{ij}}{n}=\frac{n_i}{n}\quad \left(\because n_i=\sum_{j=1}^{m_i}n_{ij}\right), \tag{34}$$

the identity (32) becomes

$$S_q\left(p_{11},\cdots,p_{km_k}\right)=S_q\left(p_1,\cdots,p_k\right)+\sum_{i=1}^{k}p_i^qS_q\left(\frac{p_{i1}}{p_i},\cdots,\frac{p_{im_i}}{p_i}\right). \tag{35}$$

The formula (35) obtained from the q-multinomial coefficient is exactly same as the generalized Shannon additivity (See [GSK3] given below) which is the most important axiom for Tsallis entropy [9].

In fact, the generalized Shannon-Khinchin axioms and the uniqueness theorem for the nonextensive entropy are already given and rigorously proved in [9]. The present result (35) and the already established axiom [GSK3] perfectly coincide with each other.

Theorem 7 *Let Δ_n be defined by the n-dimensional simplex:*

$$\Delta_n:=\left\{\left(p_1,\ldots,p_n\right)\,\middle|\, p_i\geq 0,\ \sum_{i=1}^{n}p_i=1\right\}. \tag{36}$$

The following axioms [GSK1]~[GSK4] determine the function $S_q:\Delta_n\to\mathbb{R}^+$ such that

$$S_q\left(p_1,\ldots,p_n\right)=\frac{1-\sum_{i=1}^{n}p_i^q}{\phi(q)}, \tag{37}$$

where $\phi(q)$ satisfies properties (i) ~ (iv):

(i) $\phi(q)$ is continuous and has the same sign as $q-1$,i.e.,

$$\phi(q)(q-1)>0; \tag{38}$$

(ii)

$$\lim_{q\to 1}\phi(q)=\phi(1)=0,\quad \phi(q)\neq 0\ (q\neq 1); \tag{39}$$

(iii) there exists an interval $(a,b)\subset\mathbb{R}^+$ such that $a<1<b$ and $\phi(q)$ is differentiable on the interval

$$(a,1)\cup(1,b); \tag{40}$$

and

(iv) there exists a constant $k>0$ such that

$$\lim_{q\to 1}\frac{d\phi(q)}{dq}=\frac{1}{k}. \tag{41}$$

[GSK1] *continuity*: S_q is continuous in Δ_n and $q \in \mathbb{R}^+$,

[GSK2] *maximality*: for any $q \in \mathbb{R}^+$, any $n \in \mathbb{N}$ and any $(p_1, \cdots, p_n) \in \Delta_n$,

$$S_q(p_1, \cdots, p_n) \le S_q\left(\frac{1}{n}, \ldots, \frac{1}{n}\right), \tag{42}$$

[GSK3] *generalized Shannon additivity*: if

$$p_{ij} \ge 0, \ p_i = \sum_{j=1}^{m_i} p_{ij} \ \ \forall i = 1, \cdots, n, \forall j = 1, \cdots, m_i, \tag{43}$$

then the following equality holds:

$$S_q(p_{11}, \cdots, p_{nm_k}) = S_q(p_1, \cdots, p_n) + \sum_{i=1}^{n} p_i^q S_q\left(\frac{p_{i1}}{p_i}, \cdots, \frac{p_{im_i}}{p_i}\right), \tag{44}$$

[GSK4] *expansibility*:

$$S_1(p_1, \ldots, p_n, 0) = S_1(p_1, \ldots, p_n). \tag{45}$$

Note that, in order to uniquely determine the Tsallis entropy (28) in the above set of the axioms, "$\lim_{q\to 1}$" should be removed from (41), that is, $\frac{d\phi(q)}{dq} = \frac{1}{k}$ (i.e., $\phi(q) = \frac{1}{k}(q-1)$) should be used instead of (41). The general form $\phi(q)$ perfectly corresponds to Tsallis' original introduction of the so-called Tsallis entropy in 1988 [8]. See his original characterization shown in page 9 of [1] for the detail ($\phi(q)$ corresponds to "a" in his notation. His simplest choice of "a" coincides with the simplest form of $\phi(q)$ i.e., $\frac{d\phi(q)}{dq} = \frac{1}{k}$.).

When one of the authors (H.S.) submitted the paper [9] in 2002, nobody presented the idea of the q-product. However, as shown above, the identity on the q-multinomial coefficient [7] which was formulated based on the q-product [5][6] coincides with one of the axioms ([GSK3]: generalized Shannon additivity) in [9]. This means that the whole theory based on the q-product is self-consistent. Moreover, other fundamental applications of the q-product, such as law of error [10] and the derivation of the unique non self-referential q-canonical distribution [11][12], are also based on the q-product.

6 The generalized Shannon additivity derived from the Leibniz product rule of the q-derivative

Jackson's q-derivative often appeared in the studies of quantum group is applied to another characterization of Tsallis entropy.

Definition 8 *For a function $f : \mathbb{R} \to \mathbb{R}$, q-derivative of the function f is defined by*

$$D_q f := \frac{f(qx) - f(x)}{(q-1)x}. \tag{46}$$

Then, Tsallis entropy is characterized as the following form [13].

Theorem 9 *Let $\{p_i\}$ be a probability distribution. For the function f:*

$$f(x) := \sum_i p_i^x, \tag{47}$$

Tsallis entropy is given by

$$S_q = -D_q f(x)\big|_{x=1}. \tag{48}$$

Straightforward computation proves the above result.

For the q-derivative, we have the following Leibniz product rule [14].

Theorem 10 *For functions $f, g : \mathbb{R} \to \mathbb{R}$,*

$$D_q (f (x) g (x)) = D_q (f (x)) M_q (g (x)) + M_q (f (x)) D_q (g (x)) \tag{49}$$

where D_q is the q-derivative (46) and M_q is the average operator defined by

$$M_q f (x) := \frac{f (qx) + f (x)}{2}. \tag{50}$$

Then we derive the generalized Shannon additivity (35) as an application of the Leibniz product rule (49).

Theorem 11 *For a joint probability distribution $\{p_{ij}\}$, the two functions are defined by*

$$f (x) := p_i^x, \quad g (x) := p_{j|i}^x := \frac{p_{ij}}{p_i} = \frac{p_{ij}}{\sum_j p_{ij}}. \tag{51}$$

The functions $f (x)$, $g (x)$ and $f (x) g (x) = p_{ij}^x$ are applied to the above Leibniz product rule (49) with summing up with respect to i and j and taking a limit $x \to 1$, so that we obtain the generalized Shannon additivity (35).

This is also easily confirmed by a straightforward computation.

The discussions in this section is generalized to the two-parameter entropies by using Chakrabarti and Jagannathan (CJ) difference operator [15] instead of the q-derivative (46). See [14] for the detail.

7 Conclusion

Starting from a fundamental nonlinear equation $dy/dx = y^q$, we present the scaling property and the algebraic structure of its solution. Moreover, we prove that the algebra determined by its solutions is mathematically consistent with Tsallis entropy only as the corresponding unique information measure based on the following 2 mathematical reasons: (1) derivation of Tsallis entropy from the q-multinomial coefficient and q-Stirling's formula, (2) coincidence of the identity derived from the q-multinomial coefficient with the generalized Shannon additivity which is the most important axiom for Tsallis entropy. Moreover, we show that the generalized Shannon additivity is also derived from the Leibniz product rule of the q-derivative. As shown in this paper, the generalized Shannon additivity plays an important role in mathematical structure in Tsallis statistics. Very recently, using the generalized Shannon additivity one of the authors (H.S.) shows that a lower bound of average description length for the q-generalized D-ary code tree is given by Tsallis entropy [16].

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