数理解析研究所講究録
第 1561 巻 2007 年 128-143

Fixed Point Theorems and Nonlinear Ergodic Theorems for Nonexpansive Mappings

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1 Introduction

Let E be a Banach space and let C be a nonempty closed convex subset of E. Then, a mapping $T\colon C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The first fixed point theorem for a nonexpansive mapping whose domain C is not compact was established in 1965 by Browder [11]. He proved that if C is a bounded closed convex subset of a Hilbert space E and T is a nonexpansive mapping of C into itself, then T has a fixed point in C. Almost immediately, both Browder [12] and Göhde [21] proved that the same is true if E is a uniformly convex Banach space. Kirk [27] also proved the following theorem: Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has normal structure. Let T be a nonexpansive mapping of C into itself. Then the set $F(T)$ of fixed points of T is nonempty.

After Kirk's theorem, many fixed point theorems concerning nonexpansive mappings have been proved in a Hilbert space or a Banach space. In particular, Baillon and Schöneberg [9] introduced the concept of asymptotic normal structure and generalized Kirk's fixed point theorem as follows: Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has asymptotic normal structure. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is nonempty. A fixed point theorem for a family of nonexpansive mappings was first proved by DeMarr [18] by assuming that the family is commutative and C is compact. Later, Takahashi [56] extended this theorem to a noncommutative semigroup of nonexpansive mappings which is called amenable. Since then, many fixed point theorems for a nonexpansive mapping or a family of nonexpansive mappings have been established by many authors.

On the other hand, in 1975, Baillon [6] originally proved the first nonlinear ergodic theorem in the framework of Hilbert spaces: Let C be a closed and convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If $F(T)$ is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and Px is contained in the closure of convex hull of $\{T^n x : n = 1, 2, \dots\}$ for each $x \in C$. We call such a retraction *"an ergodic retraction"*. In 1981, Takahashi [58] proved the existence of ergodic retractions for amenable semigroups of nonexpansive mappings on Hilbert spaces. Rodé [49] also found a sequence of means on a semigroup, generalizing the Cesàro means, and extended Baillon's theorem. These results were extended to a uniformly convex Banach space whose norm is Fréchet differentiable in the case of commutative semigroups of nonexpansive mappings by Hirano, Kido and Takahashi [23]. In 1999, Lau, Shioji and Takahashi [33] extended Takahashi's result and Rodé's result to amenable semigroups of nonexpansive mappings in the Banach space.

In this article, we first discuss fixed point theorems for nonexpansive mappings or families of nonexpansive mappings in Banach spaces. In particular, we state a fixed point theorem for amenable semigroups of nonexpansive mappings in Banach spaces which generalizes Kirk's theorem and Takahashi's theorem, simultaneously. This theorem answers affirmatively a problem posed during the Conference on Fixed Point Theory and Applications held at CIRM, Marseille-Luminy, 1989 (see [29]). Then we show generalized nonlinear ergodic theorems for nonexpansive semigroups in Banach spaces. In particular, we discuss generalized nonlinear ergodic theorems for nonexpansive semigroups in uniformly convex Banach spaces or general Banach spaces. Using these results, we obtain some nonlinear ergodic theorems in cases of discrete and one-parameter semigroups of nonexpansive mappings. Finally, we discuss two iterative methods for approximation of fixed points of nonexpansive mappings which are different from the mean ergodic method.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the topological dual of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The *modulus of convexity* δ of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. Let ϵ and r be real numbers with $r > 0$ and $0 \leq \epsilon \leq 2r$. If E is uniformly convex, then δ satisfies that $\delta(\epsilon/r) > 0$ and

$$\left\|\frac{x+y}{2}\right\| \leq r\left(1 - \delta\left(\frac{\epsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \epsilon$. Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x-z\| \leq \|x-y\|$ for all $y \in C$. Putting $z = P_C(x)$, we call P_C the *metric projection* of E onto C. The *duality mapping* J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called *smooth*. The norm of E is said to be *uniformly Gâteaux differentiable* if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be *Fréchet differentiable* if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called *uniformly smooth* if the limit (2.1) is attained uniformly for $x, y \in U$. We know that if E is smooth, then the duality mapping J is single valued. Further, if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm to weak* continuous on each bounded subset of E. We know the following result: Let E be a uniformly convex Banach space with a Gâteaux differentiable norm. Let C be a nonempty closed convex subset of E and $x \in E$. Then, $x_0 = P_C(x)$ if and only if

$$\langle x_0 - y, J(x - x_0)\rangle \geq 0$$

for all $y \in C$, where J is the duality mapping of E; see [64, 65] for more details.

A Banach space E is said to satisfy *Opial's condition* [46] if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup y$ implies

$$\liminf_{n\to\infty} \|x_n - y\| < \liminf_{n\to\infty} \|x_n - z\|$$

for all $z \in E$ with $z \neq y$. A Hilbert space satisfies Opial's condition.

Let C be a closed convex subset of E. A mapping $T\colon C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote the set of all fixed points of T by $F(T)$. A closed convex subset C of a Banach space E is said to have *normal structure* if for each bounded closed convex subset of K of C which contains at least two points, there exists an element x of K which is not a diametral point of K, i.e.,

$$\sup\{\|x - y\| : y \in K\} < \delta(K),$$

where $\delta(K)$ is the diameter of K. It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure; see [64] for more details. Let D be a subset of C and let P be a mapping of C into D. Then P is said to be *sunny* if

$$P(Px + t(x - Px)) = Px$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A mapping P of C into C is said to be a *retraction* if $P^2 = P$. We denote by $\overline{D}$ and $\overline{co}D$ the closure of D and the convex hull of D, respectively.

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be *accretive* if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$. If A is accretive, then we have

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$$

for all $r > 0$. An accretive operator A is said to satisfy the *range condition* if $\overline{D(A)} \subset \bigcap_{r>0} R(I + rA)$. If A is accretive, then we can define, for each $r > 0$, a nonexpansive single valued mapping $J_r\colon R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$. It is called the *resolvent* of A. We also define the *Yosida approximation* A_r by $A_r = (I - J_r)/r$. We know that $A_r x \in AJ_r x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$. We also know that for an accretive operator A satisfying the range condition, $A^{-1}0 = F(J_r)$ for all $r > 0$. An accretive operator A is said to be *m-accretive* if $R(I + rA) = E$ for all $r > 0$. Reich

[48] proved the following result: Let E be a uniformly convex and uniformly smooth Banach space and let $A \subset E \times E$ be an m-accretive operator such that $A^{-1}0$ is nonempty. Then, for any $x \in E$, the strong limit $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}0$. In this case, putting $Px = \lim_{t\to\infty} J_t x$, we have that P is a sunny nonexpansive retraction of E onto $A^{-1}0$. We also know that $x_0 = Px$ if and only if

$$\langle x - x_0, J(x_0 - z)\rangle \geq 0$$

for all $z \in A^{-1}0$.

Let S be a semitopological semigroup, i.e., a semigroup with Hausdorff topology such that for each $s \in S$, the mappings $t \mapsto ts$ and $t \mapsto st$ of S into itself are continuous. Let $B(S)$ be the Banach space of all bounded real valued functions on S with supremum norm and let X be a subspace of $B(S)$ containing constants. Then, an element μ of X^* (the dual space of X) is called a *mean* on X if $\|\mu\| = \mu(1) = 1$. We know that $\mu \in X^*$ is a mean on X if and only if

$$\inf\{f(s) : s \in S\} \leq \mu(f) \leq \sup\{f(s) : s \in S\}$$

for every $f \in X$. A real valued function μ on X is called a *submean* on X if the following properties are satisfied:

(1) $\mu(f+g) \leq \mu(f) + \mu(g)$ for every $f, g \in X$;
(2) $\mu(\alpha f) = \alpha\mu(f)$ for every $f \in X$ and $\alpha \geq 0$;
(3) for $f, g \in X$, $f \leq g$ implies $\mu(f) \leq \mu(g)$;
(4) $\mu(c) = c$ for every constant function c.

Clearly every mean on X is a submean. The notion of submeans was first introduced by Mizoguchi and Takahashi [45]. For a submean μ on X and $f \in X$, sometimes we use $\mu_t(f(t))$ instead of $\mu(f)$. For each $s \in S$ and $f \in B(S)$, we define elements $\ell_s f$ and $r_s f$ of $B(S)$ given by $(\ell_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for all $t \in S$. Let X be a subspace of $B(S)$ containing constants which is invariant under ℓ_s, $s \in S$ (resp. r_s, $r \in S$). Then a mean μ on X is said to be *left invariant* (resp. *right invariant*) if $\mu(f) = \mu(\ell_s f)$ (resp. $\mu(f) = \mu(r_s f)$) for all $f \in X$ and $s \in S$. An *invariant mean* is a left and right invariant mean. A submean μ on X is said to be *left subinvariant* if $\mu(f) \leq \mu(\ell_s f)$ for all $f \in X$ and $s \in S$. Let S be a semitopological semigroup. Then S is called *left* (resp. *right*) *reversible* if any two closed right (resp. left) ideals of S have non-void intersection. If S is left reversible, $(S, \leq)$ is a directed system when the binary relation "$\leq$" on S is defined by $a \leq b$ if and only if $\{a\} \cup \overline{Sa} \supset \{b\} \cup \overline{Sb}$, $a, b \in S$. Similarly, we can define the binary relation "$\leq$" on a right reversible semitopological semigroup S.

3 Fixed Point Theorems

In this section, we discuss fixed point theorems for a nonexpansive mapping or a family of nonexpansive mappings. The first fixed point theorem for nonexpansive mappings was established in 1965 by Browder [11]. He proved that if C is a bounded closed convex subset of a Hilbert space H and T is a nonexpansive mapping of C into itself, then T has a fixed point in C. Almost immediately, both Browder [12] and Göhde [21] proved that the same is true if E is a uniformly convex Banch space. Kirk [27] also proved the following theorem:

Theorem 3.1 ([27]). *Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has normal structure. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is nonempty.*

After Kirk's theorem, many fixed point theorems concerning nonexpansive mappings have been proved in a Hilbert space or a Banach space. In particular, Baillon and Schöneberg [9] introduced the concept of asymptotic normal structure and generalized Kirk's fixed point theorem as follows:

Theorem 3.2 ([9]). *Let E be a reflexive Banach space and let C be a non-empty bounded closed convex subset of E which has asymptotic normal structure. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is nonempty.*

On the other hand, DeMarr [18] proved the following fixed point theorem for a commutative family of nonexpansive mappings.

Theorem 3.3 ([18]). *Let C be a compact convex subset of a Banach space E and let $\mathcal{S}$ be a commutative family of nonexpansive mappings of C into itself. Then $\mathcal{S}$ has a common fixed point in C, i.e., there exists $z \in C$ such that $Tz = z$ for every $T \in \mathcal{S}$.*

Browder [12] proved the following fixed point theorem without compactness:

Theorem 3.4 ([12]). *Let C be a bounded closed convex subset of a uniformly convex Banach space E and let $\mathcal{S}$ be a commutative family of nonexpansive mappings of C into itself. Then $\mathcal{S}$ has a common fixed point in C.*

Further, let us consider to extend these theorems to a noncommutative semigroup of nonexpansive mappings. Let S be a semitopological semigroup and let C be a nonempty closed convex subset of a Banach space E. Then a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings of C into itself is called a *nonexpansive semigroup* on C if it satisfies the following:

(1) $T_{st}x = T_sT_tx$ for all $s, t \in S$ and $x \in C$;
(2) for each $x \in C$, the mapping $s \mapsto T_sx$ is continuous;
(3) for each $s \in S$, T_s is a nonexpansive mapping of C into itself.

For a nonexpansive semigroup $\mathcal{S} = \{T_s : s \in S\}$ on C, we denote by $F(\mathcal{S})$ the set of common fixed points of $T_s, s \in S$. Let S be a semitopological semigroup, let $C(S)$ be the Banach space of all bounded continuous functions on S and let $RUC(S)$ be the space of all bounded right uniformly continuous functions on S, i.e., all $f \in C(S)$ such that the mapping $s \mapsto r_sf$ is continuous. Then $RUC(S)$ is a closed subalgebra of $C(S)$ containing constants and invariant under ℓ_s and r_s, $s \in S$; see [40] for more details.

In 1969, Takahashi [56] proved the first fixed point theorem for a noncommutative semigroup of nonexpansive mappings which generalizes DeMarr's fixed point theorem, that is, he proved that any discrete left amenable semigroup has a common fixed point. Mitchell [41] generalized Takahashi's result by showing that any discrete left reversible semigroup has a common fixed point. Lau proved the following theorem in [28]:

Theorem 3.5 ([28]). *Let S be a semitopological semigroup and let $A(S)$ be the space of all $f \in C(S)$ such that $\{\ell_sf : s \in S\}$ is relatively compact in the norm topology of $C(S)$. Let $\mathcal{S} = \{T_s : s \in S\}$ be a nonexpansive semigroup on a compact convex subset C of a Banach space E. Then $A(S)$ has a left invariant mean if and only if $\mathcal{S}$ has a common fixed point in C.*

Lim [38] generalized Kirk's result [27], Browder's result [12] and Mitchell's result [41] by showing the following theorem:

Theorem 3.6 ([38]). *Let S be a left reversible semitopological semigroup. Let C be a weakly compact convex subset of a Banach space E which has normal structure and let $\mathcal{S} = \{T_s : s \in S\}$ be a nonexpansive semigroup on C. Then $\mathcal{S}$ has a common fixed point in C.*

Takahashi and Jeong [66] also generalized Browder's result [12] by using the concept of submeans.

Theorem 3.7 **([66]).** *Let S be a semitopological semigroup. Let $\mathcal{S} = \{T_s : s \in S\}$ be a nonexpansive semigroup on a bounded closed convex subset C of a uniformly convex Banach space E. Suppose that $RUC(S)$ has a left subinvariant submean. Then $\mathcal{S}$ has a common fixed point in C.*

To prove Theorem 3.7, we need the following lemma [72]:

Lemma 3.8 **([72]).** *Let $p > 1$ and $b > 0$ be two fixed numbers. Then a Banach space E is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function (depending on p and b) $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x-y\|)$$

for all $x, y \in B_b$ and $0 \leq \lambda \leq 1$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$ and B_b is the closed ball with radius b and centered at the origin.

We may comment on the relationship between "$RUC(S)$ has an invariant mean" and "S is left reversible". As well known, they do not imply each other in general. But if $RUC(S)$ has sufficiently many functions to separate closed sets, then "$RUC(S)$ has an invariant mean" would imply "S is left and right reversible". Lau and Takahashi [36] generalized Lim's result [38] and Takahashi and Jeong's result [66].

Theorem 3.9 **([36]).** *Let S be a semitopological semigroup, let C be a nonempty weakly compact convex subset of a Banach space E which has normal structure and let $\mathcal{S} = \{T_s : s \in S\}$ be a nonexpansive semigroup on C. Suppose $RUC(S)$ has a left subinvariant submean. Then $\mathcal{S}$ has a common fixed point in C.*

To prove Theorem 3.9, we need two lemmas.

Lemma 3.10 **([37]).** *A closed convex subset C of a Banach space has normal structure if and only if it does not contain a sequence $\{x_n\}$ such that for some $c > 0$,*

$$\|x_n - x_m\| \leq c \text{ and } \|x_{n+1} - \overline{x_n}\| \geq c - \frac{1}{n^2}$$

for all $n \geq 1$ and $m \geq 1$, where $\overline{x_n} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

Lemma 3.11 **([20]).** *Let X be a compact convex subset of a separated topological vector space E, let $f_1, f_2, \ldots, f_n$ be a finite family of lower semicontinuous convex functions from X into R and let $c \in R$, where R denotes the set of real numbers. Then the following conditions (1) and (2) are equivalent:*

(1) There exists $x_0 \in X$ such that $f_i(x_0) \leq c$ for all $i = 1, 2, \ldots, n$;
(2) for any finite non-negative real numbers $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ with $\sum_{i=1}^{n} \alpha_i = 1$, there exists $y \in X$ such that $\sum_{i=1}^{n} \alpha_i f_i(y) \leq c$.

Theorem 3.9 answers affirmatively a problem posed during the Conference on Fixed Point Theory and Applications held at CIRM, Marseille-Luminy, 1989 (see [29]), whether Lim's result and Takahashi and Jeong's result can be fully extended to such Banach spaces for amenable semigroups.

Problem. Would "normal structure "in Theorem 3.9 be replaced by "asymptotic normal structure"?

4 Nonlinear Ergodic Theorems

In this section, we discuss nonlinear ergodic theorems. The first nonlinear ergodic theorem for nonexpansive mappings was established in 1975 by Baillon [6] in the framework of a Hilbert space.

Theorem 4.1 ([6]). *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro means*

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$.

This theorem was extended to a uniformly convex Banach space whose norm is Fréchet differentiable by Bruck [15].

Theorem 4.2 ([15]). *Let C be a closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm. If $T : C \to C$ is a nonexpansive mapping with a fixed point, then the Cesàro means of $\{T^n x\}$ converge weakly to a fixed point of T.*

In their theorems, putting $y = Px$ for each $x \in C$, we have that P is a nonexpansive retraction of C onto $F(T)$ such that $PT^n = T^n P = P$ for all $n = 1, 2, \ldots$ and $Px \in \overline{co}\{T^n x : n = 0, 1, 2, \ldots\}$ for each $x \in C$, where $\overline{co}A$ is the closure of the convex hull of A. Takahashi [58] called such a retraction "ergodic retraction". In general, let S be a semitopologcal semigroup, let C be a closed convex subset of E, and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C. Then, a mapping P of C onto $F(\mathcal{S})$ is called a *nonexpansive ergodic retraction* if it satisfies the following conditions:

(1) $\|Px - Py\| \leq \|x - y\|$ for all $x, y \in C$;
(2) $P^2 = P$;
(3) $PT_t = T_t P = P$ for all $t \in S$;
(4) $Px \in \overline{co}\{T_t x : t \in S\}$ for all $x \in C$.

Let $\{\mu_\alpha : \alpha \in A\}$ be a net of means on $RUC(S)$. Then $\{\mu_\alpha \in A\}$ is said to be *asymptotically invariant* if for each $f \in RUC(S)$ and $s \in S$,

$$\mu_\alpha(f) - \mu_\alpha(\ell_s f) \to 0 \quad \text{and } \mu_\alpha(f) - \mu_\alpha(r_s f) \to 0.$$

Let us give an example of asymptotically invariant nets. Let $S = \{0, 1, 2, \ldots\}$ and let N be the set of positive integers. Then for $f = (x_0, x_1, \ldots) \in B(S)$ and $n \in N$, the real valued function μ_n defined by

$$\mu_n(f) = \frac{1}{n}\sum_{k=0}^{n-1} x_k$$

is a mean. Further since for $f = (x_0, x_1, \ldots) \in B(S)$ and $m \in N$

$$\begin{aligned} |\mu_n(f) - \mu_n(r_m f)| &= \left|\frac{1}{n}\sum_{k=0}^{n-1} x_k - \frac{1}{n}\sum_{k=0}^{n-1} x_{k+m}\right| \\ &\leq \frac{1}{n} \cdot 2m\|f\| \to 0, \end{aligned}$$

as $n \to \infty$, $\{\mu_n\}$ is an asymptotically invariant net of means.

If C is a nonempty closed convex subset of a Hilbert space H and $\mathcal{S} = \{T_s : s \in S\}$ is a nonexpansive semigroup on C such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$, then we know that for each $u \in C$ and $v \in H$, the functions $f(t) = \|T_t u - v\|^2$ and $g(t) = (T_t u, v)$ are in $RUC(S)$. Let μ be a mean on $RUC(S)$. Then since for each $x \in C$ and $y \in H$, the real valued function $t \mapsto (T_t x, y)$ is in $RUC(S)$, we can define the value $\mu_t(T_t x, y)$ of μ at this function. By linearity of μ and of the inner product, this is linear in y; moreover, since

$$|\mu_t(T_t x, y)| \leq \|\mu\| \cdot \sup_t |(T_t x, y)| \leq (\sup_t \|T_t x\|) \cdot \|y\|,$$

it is continuous in y. So, by the Riesz theorem, there exists an $x_0 \in H$ such that

$$\mu_t(T_t x, y) = (x_0, y)$$

for every $y \in H$. We write such an x_0 by $T_\mu x$; see [58,64] for more details.

In 1981, Takahashi [58] proved the first nonlinear ergodic theorem for noncommutative semigroups of nonexpansive mappings in a Hilbert space.

Theorem 4.3 ([58]). *Let C be a nonempty closed convex subset of a Hilbert space and let S be a semitopological semigroup such that $RUC(S)$ has an invariant mean. Let $\mathcal{S} = \{T_t : t \in S\}$ be a one-parameter nonexpansive semigroup on C such that $\{T_t x : t \in S\}$ is bounded for some $x \in C$. Then, there exists a unique nonlinear ergodic retraction P from C onto $F(\mathcal{S})$ such that $PT_t = T_t P = P$ for each $t \in S$ and $Px \in \overline{co}\{T_t x : t \in S\}$ for each $x \in C$.*

Further, Rodé [49] prove the following theorem.

Theorem 4.4 ([49]). *Let C be a nonempty closed convex subset of a Hilbert space H and let S be a semitopological semigroup such that $RUC(S)$ has an invariant mean. Let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $\{T_t x : t \in S\}$ is bounded for some $x \in C$. Then, for an asymptotically invariant net $\{\mu_\alpha : \alpha \in A\}$ of means on $RUC(S)$, the net $\{T_{\mu_\alpha} x : \alpha \in A\}$ converges weakly to an element $x_0 \in F(\mathcal{S})$.*

Using Theorem 4.4, we have Theorem 4.1. By the same method, we can prove the following nonlinear ergodic theorems:

Theorem 4.5. *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. If $F(T)$ is nonempty, then for each $x \in C$,*

$$S_r(x) = (1-r)\sum_{k=0}^{\infty} r^k T^k x,$$

as $r \uparrow 1$, converges weakly to an element $y \in F(T)$.

Theorem 4.6. *Let C be a closed convex subset of a Hilbert space H and let $\mathcal{S} = \{S(t) : t \in [0,\infty)\}$ be a one-parameter nonexpansive semigroup on C. If $F(\mathcal{S})$ is nonempty, then for each $x \in C$,*

$$S_\lambda(x) = \frac{1}{\lambda}\int_0^\lambda S(t)x dt,$$

as $\lambda \to \infty$, converges weakly to an element $y \in F(\mathcal{S})$.

Next, let us state a nonlinear ergodic theorem for nonexpansive semigroups in a Banach space. Before stating it, we give a definition. A net $\{\mu_\alpha\}$ of continuous linear functionals on $RUC(S)$ is called *strongly regular* if it satisfies the following conditions:

(1) $\sup_{\alpha} \|\mu_\alpha\| < +\infty$;

(2) $\lim_{\alpha} \mu_\alpha(1) = 1$;

(3) $\lim_{\alpha} \|\mu_\alpha - r_s^* \mu_\alpha\| = 0$ for every $s \in S$.

Theorem 4.7 ([23]). *Let S be a commutative semitopological semigroup and let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let C be a nonempty closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S})$ is nonempty. Then there exists a unique nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for every $x \in C$. Further, if $\{\mu_\alpha\}$ is a strongly regular net of continuous linear functionals on $RUC(S)$, then for each $x \in C$, $T_{\mu_\alpha}T_tx$ converges weakly to Px uniformly in $t \in S$.*

We have not known whether Theorem 4.7 would hold in the case when S is noncommutative (cf. [62]). Lau, Shioji and Takahashi [33] solved the problem as follows:

Theorem 4.8 ([33]). *Let C be a closed convex subset of a uniformly convex Banach space E, let S be a semitopological semigroup which $RUC(S)$ has an invariant mean, and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C with $F(\mathcal{S}) \neq \phi$. Then there exists a nonexpansive ergodic retraction P from C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for each $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for each $x \in C$.*

This is a generalization of Takahashi's result [58] for an amenable semigroup of nonexpansive mappings on a Hilbert space. Further they extended Rodé's result [49] to an amenable semigroup of nonexpansive mappings on a uniformly convex Banach space whose norm is Fréchet differentiable.

Theorem 4.9 ([33]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let S be a semitopological semigroup. Let C be a closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C with $F(\mathcal{S}) \neq \phi$. Suppose that $RUC(S)$ has an invariant mean. Then there exists a unique nonexpansive retraction P from C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for each $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for each $x \in C$. Further, if $\{\mu_\alpha\}$ is an asymptotically invariant net of means on X, then for each $x \in C$, $\{T_{\mu_\alpha}x\}$ converges weakly to Px.*

To prove Theorem 4.9, they used Theorem 4.8 and the following lemma which has been proved in Lau, Nishiura and Takahashi [31].

Lemma 4.10 ([31]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let S be a semitopological semigroup. Let C be a closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C with $F(\mathcal{S}) \neq \phi$. Then, for each $x \in C$, $F(\mathcal{S}) \cap \bigcap_{s\in S} \overline{co}\{T_{ts}x : t \in S\}$ consists of at most one point.*

The following theorem has been proved in Takahashi [60] and Lau, Nishiura and Takahashi [31] when E is a Hilbert space.

Theorem 4.11 ([33]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let S be a semitopological semigroup. Let C be a closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C with $F(\mathcal{S}) \neq \phi$. Suppose that for each $x \in C$, $F(\mathcal{S}) \cap \bigcap_{s\in S} \overline{co}\{T_{ts}x : t \in S\}$ is nonempty. Then there exists a nonexpansive retraction P from C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for each $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for each $x \in C$.*

Recently, Miyake and Takahashi [44] proved nonlinear ergodic theorems for nonexpansive mappings with compact domains in general Banach spaces.

Theorem 4.12 ([44]). *Let C be a compact and convex subset of a Banach space E, let S be a semigroup, let $\mathcal{S} = \{T_s : s \in S\}$ be a nonexpansive semigroup on C into itself, let X be a subspace of $B(S)$ containing 1 such that $\ell_s X \subset X$ for each $s \in S$ and the functions $s \mapsto \langle T_s x, x^* \rangle$ and $s \mapsto \|T_s x - y\|$ are contained in X for each $x, y \in C$ and $x^* \in E^*$ and let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on X. Then, for each $x \in C$, $T_{r_h^* \mu_\alpha} x$ converges uniformly in $h \in S$.*

Next, applying Theorem 4.12, we obtain a nonlinear ergodic theorem for nonexpansive semigroups on a compact and convex subset of a strictly convex Banach space.

Theorem 4.13 ([44]). *Let C be a compact and convex subset of a strictly convex Banach space E, let S be a semigroup, let $\mathcal{S} = \{T_s : s \in S\}$ be a nonexpansive semigroup on C into itself, let X be a subspace of $B(S)$ containing 1 such that $\ell_s X \subset X$ for each $s \in S$ and the functions $s \mapsto \langle T_s x, x^* \rangle$ and $s \mapsto \|T_s x - y\|$ are contained in X for each $x, y \in C$ and $x^* \in E^*$ and let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on X. Then, for each $x \in C$, $T_{r_h^* \mu_\alpha} x$ converges strongly to a common fixed point of $\mathcal{S}$ uniformly in $h \in S$.*

Using Theorem 4.12 and Theorem 4.13, we obtain some nonlinear ergodic theorems in cases of discrete and one-parameter semigroups of nonexpansive mappings.

Theorem 4.14. *Let C be a compact and convex subset of a Banach space E and let T be a nonexpansive mapping of C into itself. Then, for each $x \in C$,*

$$\frac{1}{n}\sum_{i=0}^{n-1} T^{i+h}x$$

converges uniformly in $h \in N$.

Theorem 4.15. *Let C be a compact and convex subset of a strictly convex Banach space E and let T be a nonexpansive mapping of C into itself. Then, for each $x \in C$,*

$$\frac{1}{n}\sum_{i=0}^{n-1} T^{i+h}x$$

converges to a fixed point of T uniformly in $h \in N$.

Theorem 4.16. *Let C be a compact and convex subset of a Banach space E and let $\mathcal{S} = \{T(t) : t \in R\}$ be a one-parameter nonexpansive semigroup on C. Then, for each $x \in C$,*

$$\frac{1}{t}\int_0^t T(s+h)x \, ds$$

converges uniformly in $h \in R$.

Theorem 4.17. *Let C be a compact and convex subset of a strictly convex Banach space E and let $\mathcal{S} = \{T(t) : t \in R\}$ be a one-parameter nonexpansive semigroup on C. Then, for each $x \in C$,*

$$\frac{1}{t}\int_0^t T(s+h)x \, ds$$

converges to a common fixed point of $\mathcal{S}$ uniformly in $h \in R$.

5 Approximation of fixed points

There are two iterative methods for approximation of fixed points of nonexpansive mappings in a Hilbert space which are different from the Cesàro means.

Mann [39] introduced the following iterative scheme for finding a fixed point of a nonexpansive mapping. For the proof, see Takahashi [65].

Theorem 5.1 ([39]). *Let C be a closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let P be the metric projection of H onto $F(T)$. Let $x \in C$ and let $\{x_n\}$ be a sequence defined by $x_1 = x$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n = 1, 2, \dots,$$

where $\{x_n\} \subset [0,1]$ satisfies

$$0 \leq \alpha_n < 1 \text{ and } \sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty.$$

Then, $\{x_n\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} Px_n$.

Wittmann [71] dealt with the following iterative scheme to approximate a fixed point of a nonexpansive mapping in a Hilbert space; see originally Halpern [22]. For the proof, see Takahashi [65].

Theorem 5.2 ([71]). *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let P be the metric prjection of H onto $F(T)$. Let $x \in C$ and let $\{x_n\}$ be a sequence defined by $x_1 = x$ and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)Tx_n, \quad n = 1, 2, \dots,$$

where $\{\alpha_n\} \subset [0,1]$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then, $\{x_n\}$ converges strongly to $Px \in F(T)$.

Shimizu and Takahashi [50] introduced the first iterative schemes for finding common fixed points of families of nonexpansive mappings and proved strong convergence theorems for discrete and one-parameter nonexpansive semigroups in Hilbert spaces. Atsushiba, Shioji and Takahashi [2] established a weak convergence theorem of Mann's type for a nonexpansive semigroup in a Banach space.

Theorem 5.3 ([2]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let C be a nonempty closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \phi$. Let $\{\mu_n\}$ be a sequence of means on $C(S)$ such that $\|\mu_n - \ell_s^*\mu_n\| = 0$ for every $s \in S$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n, \quad n = 1, 2, \dots,$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, then $\{x_n\}$ converges weakly to an element $x_0 \in F(\mathcal{S})$.

Using Theorem 5.3, we can prove a weak convergence theorem of Mann's type for a one-parameter nonexpansive semigroup.

Theorem 5.4. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a closed convex subset of E. Let $\mathcal{S} = \{S(t) : t \in [0, \infty)\}$ be a one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \phi$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt, \quad n = 1, 2, \dots,$$

where $s_n \to \infty$ as $n \to \infty$ and $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0, a]$ for some a with $0 < a < 1$, then $\{x_n\}$ converges weakly to a common fixed point $z \in F(\mathcal{S})$.

Shioji and Takahashi [53] also established the following strong convergence theorem for a nonexpansive semigroup of Halpern's type in a Banach space.

Theorem 5.5 ([53]). *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \phi$. Let $\{\mu_n\}$ be a sequence of means on $C(S)$ such that $\|\mu_n - \ell_s^* \mu_n\| = 0$ for every $s \in S$. Suppose that $x, y_1 \in C$ and $\{y_n\}$ is given by*

$$y_{n+1} = \beta_n x + (1 - \beta_n) T_{\mu_n} y_n, \quad n = 1, 2, \dots,$$

where $\{\beta_n\}$ is in $[0,1]$. If $\{\beta_n\}$ is chosen so that $\lim_{n\to\infty} \beta_n = 0$ and $\Sigma_{n=1}^{\infty} \beta_n = \infty$, then $\{y_n\}$ converges strongly to an element of $F(\mathcal{S})$.

Suzuki and Takahashi [55] established a strong convergence theorem of Mann's type for a one-parameter nonexpansive semigroup in a Banach space without strict convexity.

Theorem 5.6 ([55]). *Let C be a compact convex subset of a Banach space E and let $\mathcal{S} = \{S(t) : t \in \mathbb{R}_+\}$ be a one-parameter nonexpansive semigroup on C. Let $x_1 \in C$ and define a sequence in C by*

$$x_{n+1} = \frac{\alpha_n}{t_n}\int_0^{t_n} S(s)x_n \, ds + (1 - \alpha_n)x_n$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ and $\{t_n\} \subset (0, \infty)$ satisfy the following conditions:

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1, \quad \lim_{n\to\infty} t_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.$$

Then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$.

Recently, Miyake and Takahashi [42] extended Suzuki and Takahashi's result to a general commutative nonexpansive semigroup in a Banach space.

Theorem 5.7 ([42]). *Let C be a compact convex subset of a Banach space E and let S be a commutative semigroup with identity 0. Let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C. Let X be a subspace of $B(S)$ containing 1 such that $\ell_s X \subset X$ for each $s \in S$ and the functions $s \mapsto \langle T_s x, x^* \rangle$ and $s \mapsto \|T_s x - y\|$ are contained in X for each $x, y \in C$ and $x^* \in E^*$ and let $\{\mu_n\}$ be an asymptotically invariant sequence of means on X such that $\lim_{n\to\infty} \|\mu_n - \mu_{n+1}\| = 0$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that*

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1.$$

Let $x_1 \in C$ and let $\{x_n\}$ be the sequence defined by

$$x_{n+1} = \alpha_n T_{\mu_n} x_n + (1 - \alpha_n) x_n$$

for every $n = 1, 2, \ldots$. Then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$.

Miyake and Takahashi [43] also obtained a strong convergence theorem of Halpern's type for a general commutative nonexpansive semigroup in a Banach space. See also Lau, Miyake and Takahashi [30] for amenable semigroups.

Theorem 5.8 (**[43]**). *Let C be a compact convex subset of a smooth and strictly convex Banach space E, let S be a commutative semigroup with identity 0. Let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C, let X be a subspace of $B(S)$ containing 1 such that $\ell_s X \subset X$ for each $s \in S$ and the functions $s \mapsto \langle T_s x, x^* \rangle$ and $s \mapsto \|T_s x - y\|$ are contained in X for each $x, y \in C$ and $x^* \in E^*$ and let $\{\mu_n\}$ be a strongly regular sequence of means on X. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$. Let $x \in C$ and let $\{x_n\}$ be the sequence defined by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T_{\mu_n} x_n$$

for every $n = 1, 2, 3, \ldots$. Then $\{x_n\}$ converges strongly to Px, where P is a unique sunny nonexpansive retraction of C onto $F(\mathcal{S})$.

Using Theorem 5.8, we can obtain the following strong convergence theorem for a one-parameter nonexpansive semigroup.

Theorem 5.9. *Let C be a compact convex subset of a smooth and strictly convex Banach space E and let $\mathcal{S} = \{S(t) : t \in \mathbb{R}_+\}$ be a one-parameter nonexpansive semigroup on C. Let $x_1 = x \in C$ and let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} S(s) x_n \, ds$$

for every $n = 1, 2, 3, \ldots$, where $\{\alpha_n\}$ is a sequnece in $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$ and $\{t_n\}$ is an increasing sequence in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = 1$. Then $\{x_n\}$ converges strongly to Px, where P is a unique sunny nonexpansive retracton of C onto $F(\mathcal{S})$.

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