Relation between Equilibrium points for a Differential Inclusion and Solutions of a Variational Inequality

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We introduce a relation between equilibrium points for some differential inclusion and solutions of a variational inequality. At first we show convergence theorems of solutions for some differential inclusions.

1. Convergence Theorem for Differential Inclusions

Let $X$ be a Hilbert space with an inner product $(\cdot, \cdot)$, let $K \subset X$ be a non-empty closed convex subset of $X$, let $A$ be a set-valued mapping from $K$ to $2^X$ with convex compact set-value, and let $x(\cdot) : [0, \infty) \to X$. We consider a differential inclusion $DI(A,K)$ for $A$ and $K$ as follows:

$$DI(A,K) : \begin{cases} x(t) \in K \text{ for any } t \in [0, \infty), \\ \dot{x}(t) \in -A(x(t)) \text{ for almost any } t \in (0, \infty). \end{cases}$$

Next we give some definitions.

定義 (Def.) (1) $x(\cdot)$ is called a trajectory of $DI(A,K)$, if a mapping $x(\cdot) : [0, \infty) \to K$ is absolutely continuous and satisfies $DI(A,K)$.

(2) A point $x^* \in K$ is said to be an equilibrium point for $DI(A,K)$ if $0 \in -A(x^*)$.

Let $\varphi : X \to (-\infty, \infty]$ be a proper lower semi-continuous convex function. A subdifferential $\partial \varphi : X \to 2^X$ is defined by

$$\partial \varphi(x) = \{w \in X : \varphi(y) \geq \varphi(x) + (w,y-x) \text{ for any } y \in X\}.$$ 

Then it is well-known that $x^* \in K$ is an equilibrium point of $DI(\partial \varphi,K)$ if and only if $x^*$ is a minimum point of $\varphi$, i.e., $\varphi(x^*) = \min_{x \in X} \varphi(x)$. It is also known that $\partial \varphi$ has a property of demipositivity.

定義 (Def.) A set-valued mapping $A : X \to 2^X$ is said to be demipositional if (1), (2) and (3) hold:

(1) $(v,x-y) \geq 0$ for all $x \in X, y \in A^{-1}(0)$ and $v \in A(x)$.

(2) There exists $y_0 \in A^{-1}(0)$ such that $0 \in A(x)$ whenever $(v,x-y_0) = 0$ for all $v \in A(x)$. 

For the $y_0$ in (2), if $x_n \rightarrow x, v_n \in A(x_n)$, $\{v_n\}$ is bounded and
$\lim_{n \rightarrow \infty} (v_n, x_n - y_0) = 0$, then $0 \in A(x)$.

(Remark) If $A$ satisfies (1) and (2), $A$ is called firmly positive.

Bruck showed the convergence theorems with respect to a demi positive mapping.

**Theorem 1** ([1] Bruck, 1974)

Suppose $A : X \rightarrow 2^X$ is demi positive and that $x() : [0, \infty) \rightarrow X$ is an absolutely continuous mapping satisfying

\[
\begin{cases}
  x(t) \in D(A) \text{ for all } t \geq 0, \\
  \dot{x}(t) \in -A(x(t)) \text{ for almost all } t > 0, \\
  \|x(t)\| \in L^\infty(0, \infty).
\end{cases}
\]

Then there exists $x^* = w\lim_{t \rightarrow \infty} x(t)$ and $x^* \in A^{-1}(0)$.

**Theorem 2** ([1] Bruck, 1974) Let $\varphi : X \rightarrow (-\infty, \infty]$ be a proper lower semi-continuous convex even function with a minimum. Then there exists a unique solution $x() : [0, \infty) \rightarrow X$, which is absolutely continuous on $[\delta, \infty)$ for all $\delta > 0$, satisfying

\[
\begin{cases}
  x(t) \in D(\partial \varphi) \text{ for all } t > 0, \\
  \dot{x}(t) \in -\partial \varphi(x(t)) \text{ for almost all } t > 0,
\end{cases}
\]

and there exists $x^* = s\lim_{t \rightarrow \infty} x(t)$ such that $\varphi(x^*) = \min_{x \in X} \varphi(x)$.

We shall introduce the sufficient conditions of demipositivity.

**Theorem 3** ([1] Bruck, 1974) A set-valued mapping $A : X \rightarrow 2^X$ is demi positive when the following one of (a)-(e) holds:

(a) $A$ is a subdifferential $\partial \varphi$ of a proper lower semi-continuous convex function $\varphi : X \rightarrow [-\infty, \infty)$ with a minimum in $X$.

(b) $A$ is $I - T$, where $I$ is an identity function and $T$ is a non-expansive mapping with a fixed point.

(c) $A$ is maximal monotone, odd and firmly positive.

(d) $A$ is maximal monotone and $\text{int} A^{-1}(0) \neq \emptyset$.

(e) $A$ is maximal monotone, firmly positive and weakly closed.

(Remark) (1) $A : X \rightarrow 2^X$ is called monotone if $(u - v, x - y) \geq 0$ for any $x, y \in D(A)$ and $u \in A(x), v \in A(y)$.

(2) $A$ is called maximal monotone if it is not properly contained in any other monotone subset of $X$.

(3) $A$ is said to be weakly closed if $x_n \rightarrow x, v_n \rightarrow v, v_n \in A(x_n)$ and then $v \in A(x)$.

There are many results of approximations of equilibrium points for Maximal operators ([2], [3], etc.)

2. Solutions of Variational Inequality and Equilibrium points of Differential
Variational Inequality

Let $F : K \to 2^X$ be a upper semi-continuous set-valued mapping such that $F(x)$ is a non-empty convex compact subset of $X$ for any $x \in X$, where $F$ is said to be upper semi-continuous if for any open set $U$ containing $F(x_0)$ there exists a neighborhood $V$ of $x_0$ such that $F(V) \subset U$, where $F(V) = \bigcup_{x \in V} F(x)$. We give some definitions for solutions of variational inequalities for $F$ and $K$.

定義 (Def.) [4] (1) $x^* \in K$ is called a solution of Stampacchia variational inequality $SVI(F,K)$ if there exists $\xi^* \in F(x^*)$ such that $(\xi^*, y - x^*) \geq 0$ for all $y \in K$.

定義 (Def.) (2) $x^* \in K$ is called a solution of Strong Minty variational inequality $SMVI(F,K)$ if for all $y \in K$, $(\eta, y - x^*) \geq 0$ for all $\eta \in F(y)$.

定義 (Def.) (3) $x^* \in K$ is called a solution of Weak Minty variational inequality $WMVI(F,K)$ if for any $y \in K$ there exists $\eta_0 \in F(y)$ such that $(\eta_0, y - x^*) \geq 0$.

$F$ is said to be pseudomonotone if for all $x, y \in K$ there exists $u \in F(x)$ such that $(u, y - x) \geq 0$ then $(v, y - x) \geq 0$ for all $v \in F(y)$. If $F$ is pseudomonotone, the set of solutions of $SVI(F,K)$ coincides with the set of solutions of $SMVI(F,K), WMVI(F,K)$. Results with respect to the convergence theorems of variational inequalities are shown in many approaches ([5], [6], [7]).

Let $T_K(x) = \{v \in X : x + \alpha_n v_n \in K, \alpha_n > 0, \alpha_n \to 0, v_n \to v(n \to \infty)\}$ and let $N_K(x) = \{v \in X : (v, v) \leq 0 \text{ for any } v \in T_K(x)\}$. $T_K(x)$ is called a tangent cone and $N_K(x)$ is called a normal cone. The following differential inclusion is said to be a differential variational inequality $DVI(F,K)$.

$$DVI(F,K) : \begin{cases}
x(t) \in K \text{ for all } t \in [0, \infty), \\
\dot{x}(t) \in -(F + N_K)(x(t)) \text{ for a.e. } t \in [0, \infty).
\end{cases}$$

And we call the following differential inclusion a projected differential inclusion $PDI(F,K)$.

$$PDI(F,K) : \begin{cases}
x(t) \in K \text{ for all } t \in [0, \infty), \\
\dot{x}(t) \in P_{T_K(x)}(-F)(x(t)) \text{ for a.e. } t \in [0, \infty),
\end{cases}$$

where $P_{T_K(x)}$ is a projection onto $T_K(x(t))$. It is shown that $x(t)$ is a solution of $DVI(F,K)$ if and only if $x(t)$ is a solution of $PDI(F,K)$.

G.P.Crespi and M.Rocca showed the following theorems ([8]).

定理 (Theorem (G.P.Crespi and M.Rocca,2004)) Let $x^* \in K$ be an equilibrium point of $DVI(F,K)$ and assume that $F$ is pseudomonotone. Then every solution $x(t)$ of $DVI(F,K)$ satisfies that

$$\|x(t) - x^*\| \leq \|x(s) - x^*\| \text{ for } t \geq s.$$
定理 (Theorem) Let $K \subset X$ be a closed convex subset, and let $F : X \to 2^X$ be an upper semi-continuous mapping with non-empty convex and compact values. Assume $F$ is pseudomonotone. Then, the following (a) and (b) are equivalent:

(a) $x^* \in K$ is an equilibrium point of $DVI(F,K)$. 
(b) $x^* \in K$ is a solution of $SVI(F,K)$.

There are many results of convergence theorems to solutions of $SVI(F,K)$ by using iterative schemes and also given many results of approximating solutions. We try to study the approximation theory and the iterative methods in order to find an equilibrium point of $DVI(F,K)$ with respect to the fixed point theory with good compositions of operator $F$ of a large class of set-valued mappings.([9])

References