数理解析研究所講究録
第 1562 巻 2007 年 117-126

Bayesian Communication under Rough Sets Information *

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Abstract. A communication model in the p-belief system is presented which leads to a Nash equilibrium of a strategic form game through robust messages. In the communication process each player predicts the other players' actions under his/her private information with conditional probability greater than p. The players communicate privately their conjectures through message according to the communication graph, where each recipient of the message learns and revises his/her conjecture. The emphasis is on that each player sends not exact information about his/her individual conjecture to the other player, but he/she sends robust information as the conditional probability about the other players' actions greater than his/her exact conjectures.
Keywords: Communication, , p-Belief system, Robust message, Nash equilibrium, Protocol, Conjecture, Non-corporative game.
AMS 2000 Mathematics Subject Classification: Primary 91A35, Secondary 03B45.
Journal of Economic Literature Classification: C62, C78.

1 Introduction

This article presents the communication model leading to a mixed strategy Nash equilibrium for a strategic form game as a learning process through robust messages in the p-belief system associated with a partitional information structure. We show that

Main theorem. *Suppose that the players in a strategic form game have the p-belief system with a common prior distribution. In a communication process of the game according to a protocol with revisions of their beliefs about the other players' actions, the profile of their future predictions converges to a mixed strategy Nash equilibrium of the game in the long run.*

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* This paper is a preliminary version, and the final form will be published elsewhere.
** 茨城県 ひたちなか市 中根８６６　Tel 029 272 5201 Fax 029 271 2813
*** Partially supported by the Grant-in-Aid for Scientific Research(C)(No.18540153) in the Japan Society for the Promotion of Sciences.

Recently, researchers in economics, AI, and computer science become entertained lively concerns about relationships between knowledge and actions. At what point does an economic agent sufficiently know to stop gathering information and make decisions? There are also concerns about the complexity of computing knowledge. The most interest to us is the emphasis on the considering the situation involving the knowledge of a group of agents rather than of just a single agent.

In game theoretical situations, the concept of mixed strategy Nash equilibrium (J.F. Nash [12]) has become central. Yet a little is known about the process by which players learn if they do. This article will give a communication protocol run by the mutual learning leading to a mixed strategy Nash equilibrium of a strategic form game from the point of distributed knowledge system.

Let us consider the following protocol: The players start with the same prior distribution on a state-space. In addition they have private information given by a partition of the state space. Beliefs of players are posterior probabilities: A player *p-believes* (simply, *believes*) an event with $0 < p \leq 1$ if the posterior probability of the event given his/her information is at least p. Each player predicts the other players' actions as his/her belief of the actions. He/she communicates privately their beliefs about the other players' actions through messages through robust messages, which message is approximate information about his/her individual conjecture on the others' actions greater than his/her exact conjectures as the conditional probability under his/her private information. The recipients update their belief according to the messages. Precisely, at every stage each player communicates privately not only his/her belief about the others' actions but also his/her rationality as messages according to a protocol,[1] and then the recipient updates their private information and revises her/his prediction. In addition, the players are assumed to be rational and maximizing their expected utility according their beliefs at every stage. When a player communicates with another, the other players are not informed about the contents of the message.

The main theorem says that the players' predictions regarding the future beliefs converge in the long run, which lead to a mixed strategy Nash equilibrium of a game. The emphasis is on the three points: First that each player sends not exact information about his/her individual conjecture but robust information about the actions greater than his/her exact conjectures as the conditional probability under his/her private information, secondly that each player's prediction is not required to be common-knowledge among all players, and finally that the communication graph is not assumed to be acyclic.

Many authors have studied the learning processes modeled by Bayesian updating. The papers by E. Kalai and E. Lehrer [5] and J. S. Jordan [4] (and references in therein) indicate increasing interest in the mutual learning processes in games that leads to equilibrium: Each player starts with initial erroneous belief regarding the actions of all the other players. They show the two strategies converges to an ε-mixed strategy Nash equilibrium of the repeated game.

[1] When a player communicates with another, the other players are not informed about the contents of the message.

As for as J.F. Nash's fundamental notion of strategic equilibrium is concerned, R.J. Aumann and A. Brandenburger [1] gives epistemic conditions for mixed strategy Nash equilibrium: They show that the common-knowledge of the predictions of the players having the partitional information (that is, equivalently, the **S5**-knowledge model) yields a Nash equilibrium of a game. However it is not clear just what learning process leads to the equilibrium.

To fill this gap from epistemic point of view, Matsuhisa ([6], [8], [9]) presents his communication system for a strategic game, which leads a mixed Nash equilibrium in several epistemic models. The articles [6], [8] [10] treats the communication model in the **S4**-knowledge model where each player communicates to other players by sending exact information about his/her conjecture on the others' action. In Matsuhisa and Strokan [10], the communication model in the p-belief system is introduced:[2]Each player sends exact information that he/she believes that the others play their actions with probability at least his/her conjecture as messages. Matsuhisa [9] extended the communication model to the case that the sending messages are non-exact information that he/she believes that the others play their actions with probability at least his/her conjecture. This article is in the line of [9]; each player sends his/her robust information about the actions greater than his/her exact conjectures as the conditional probability under his/her private information in the Bayesian communication model presented in Matsuhisa [9].

This paper organizes as follows. Section 2 recalls the p-belief system associated with a partition information structure, and we extend a game on p-belief system. The Bayesian belief communication process for the game is introduced where the players send robust messages about their conjectures about the other players' action. In Section 3 we give the formal statement of the main theorem (Theorem 1) and sketch the proof. In Section 4 we conclude with remarks. The illustrated example will be shown in the lecture presentation at AI*IA 2007.

2 The Model

Let Ω be a non-empty *finite* set called a *state-space*, N a set of finitely many *players* $\{1, 2, \ldots n\}$ at least two ($n \geq 2$), and let 2^{Ω} be the family of all subsets of Ω. Each member of 2^{Ω} is called an *event* and each element of Ω called a *state*. Let μ be a probability measure on Ω which is common for all players. For simplicity it is assumed that (Ω, μ) is a *finite* probability space with μ *full support*.[3]

2.1 p-Belief System[4]

[2] C.f.: Monderer and Samet [11] for the p-belief system.

[3] That is; $\mu(\omega) \neq 0$ for every $\omega \in \Omega$.

[4] Monderer and Samet [11].

Let p be a real number with $0 < p \leq 1$. The *p-belief system* associated with the partition information structure $(\Pi_i)_{i \in N}$ is the tuple

$$\langle N, \Omega, \mu, (\Pi_i)_{i \in N}, (B_i(*, p))_{i \in N} \rangle$$

consisting of the following structures and interpretations: (Ω, μ) is a finite probability space, and i's *p-belief operator* $B_i(*; p)$ is the operator on 2^{Ω} such that $B_i(E, p)$ is the set of states of Ω in which i *p-believes* that E has occurred with probability at least p ; that is,

$$B_i(E; p) := \{\omega \in \Omega \mid \mu(E \mid \Pi_i(\omega)) \geq p \}.$$

Remark 1. When $p = 1$ the 1-belief operator $B_i(*; 1)$ becomes the knowledge operator for **S5**-logic, i.e. the operator corresponding to the partition on a state space.

2.2 Game on p-Belief System[5]

By a *game G* we mean a *finite* strategic form game

$$\langle N, (A_i)_{i \in N}, (g_i)_{i \in N} \rangle$$

with the following structure and interpretations: N is a finite set of players $\{1, 2, \ldots, i, \ldots n\}$ with $n \geq 2$, A_i is a finite set of i's *actions* (or i's pure strategies) and g_i is an i's *payoff function* of A into $\mathbb{R}$, where A denotes the product $A_1 \times A_2 \times \cdots \times A_n$, A_{-i} the product $A_1 \times A_2 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n$. We denote by g the n-tuple $(g_1, g_2, \ldots g_n)$ and by a_{-i} the $(n-1)$-tuple $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ for a of A. Furthermore we denote $a_{-I} = (a_i)_{i \in N \setminus I}$ for each $I \subset N$.

A probability distribution ϕ_i on A_{-i} is said to be i's *overall conjecture* (or simply i's *conjecture*). For each player j other than i, this induces the marginal distribution on j's actions; we call it i's *individual conjecture* about j (or simply i's conjecture *about* j.) Functions on Ω are viewed like random variables in the probability space (Ω, μ). If $\mathbf{x}$ is a such function and x is a value of it, we denote by $[\mathbf{x} = x]$ (or simply by $[x]$) the set $\{\omega \in \Omega | \mathbf{x}(\omega) = x\}$.

The information structure (Π_i) with a common prior μ yields the distribution on $A \times \Omega$ defined by $\mathbf{q}_i(a, \omega) = \mu([\mathbf{a} = a] | \Pi_i(\omega))$; and the i's overall conjecture defined by the marginal distribution

$$\mathbf{q}_i(a_{-i}, \omega) = \mu([\mathbf{a}_{-i} = a_{-i}] | \Pi_i(\omega))$$

which is viewed as a random variable of ϕ_i. We denote by $[\mathbf{q}_i = \phi_i]$ the intersection $\bigcap_{a_{-i} \in A_{-i}} [\mathbf{q}_i(a_{-i}) = \phi_i(a_{-i})]$ and denote by $[\phi]$ the intersection $\bigcap_{i \in N} [\mathbf{q}_i = \phi_i]$. Let $\mathbf{g}_i$ be a random variable of i's payoff function g_i and $\mathbf{a}_i$ a random variable of an i's action a_i. Where we assume that $\Pi_i(\omega) \subseteq [a_i] := [\mathbf{a}_i = a_i]$ for all $\omega \in [a_i]$ and for every a_i of A_i. i's action a_i is said to be *actual* at a state ω if

[5] Aumann and Brandenburger [1]

$\omega \in [\mathbf{a}_i = a_i]$; and the profile a_I is said to be actually played at ω if $\omega \in [\mathbf{a}_I = a_I] := \bigcap_{i\in I}[\mathbf{a}_i = a_i]$ for $I \subset N$. The pay off functions $g = (g_1, g_2, \dots, g_n)$ is said to be *actually played* at a state ω if $\omega \in [\mathbf{g} = g] := \bigcap_{i\in N}[\mathbf{g}_i = g_i]$. Let **Exp** denote the expectation defined by

$$\mathbf{Exp}(g_i(b_i, \mathbf{a}_{-i}); \omega) := \sum_{a_{-i}\in A_{-i}} g_i(b_i, a_{-i})\ \mathbf{q}_i(a_{-i}, \omega).$$

A player i is said to be *rational* at ω if each i's actual action a_i maximizes the expectation of his actually played pay off function g_i at ω when the other players actions are distributed according to his conjecture $\mathbf{q}_i(\cdot\,;\omega)$. Formally, letting $g_i = \mathbf{g}_i(\omega)$ and $a_i = \mathbf{a}_i(\omega)$, $\mathbf{Exp}(g_i(a_i, \mathbf{a}_{-i}); \omega) \geq \mathbf{Exp}(g_i(b_i, \mathbf{a}_{-i}); \omega)$ for every b_i in A_i. Let R_i denote the set of all of the states at which i is rational.

2.3 Protocol [6]

We assume that the players communicate by sending *messages*. Let T be the time horizontal line $\{0, 1, 2, \cdots t, \cdots\}$. A *protocol* is a mapping $\Pr : T \to N \times N, t \mapsto (s(t), r(t))$ such that $s(t) \neq r(t)$. Here t stands for *time* and $s(t)$ and $r(t)$ are, respectively, the *sender* and the *receiver* of the communication which takes place at time t. We consider the protocol as the directed graph whose vertices are the set of all players N and such that there is an edge (or an arc) from i to j if and only if there are infinitely many t such that $s(t) = i$ and $r(t) = j$.

A protocol is said to be *fair* if the graph is strongly-connected; in words, every player in this protocol communicates directly or indirectly with every other player infinitely often. It is said to contain a *cycle* if there are players $i_1, i_2, \dots, i_k$ with $k \geq 3$ such that for all $m < k$, i_m communicates directly with i_{m+1}, and such that i_k communicates directly with i_1. The communications is assumed to proceed in *rounds*[7]

2.4 Communication on p-Belief System

Let ε be a real number with $0 \leq \varepsilon < 1$. A *Bayesian belief communication process* $\pi(G)$ with revisions of players' conjectures $(\phi_i^t)_{(i,t)\in N\times T}$ according to a protocol for a game G is a tuple

$$\pi(G) = \langle \Pr, (\Pi_i^t)_{i\in N}, (B_i^t)_{i\in N}, (\phi_i^t)_{(i,t)\in N\times T}\rangle$$

with the following structures: the players have a common prior μ on Ω, the protocol Pr among N, $\Pr(t) = (s(t), r(t))$, is fair and it satisfies the conditions that $r(t) = s(t+1)$ for every t and that the communications proceed in rounds. The revised information structure Π_i^t at time t is the mapping of Ω into 2^Ω for player i. If $i = s(t)$ is a sender at t, the message sent by i to $j = r(t)$ is M_i^t. An n-tuple $(\phi_i^t)_{i\in N}$ is a revision process of individual conjectures. These structures are inductively defined as follows:

[6] C.f.: Parikh and Krasucki [13]

[7] There exists a time m such that for all t, $\Pr(t) = \Pr(t+m)$. The *period* of the protocol is the minimal number of all m such that for every t, $\Pr(t+m) = \Pr(t)$.

– Set $\Pi_i^0(\omega) = \Pi_i(\omega)$.
– Assume that Π_i^t is defined. It yields the distribution

$$\mathbf{q}_i^t(a,\omega) = \mu([\mathbf{a} = a] | \Pi_i^t(\omega)).$$

Whence
- R_i^t denotes the set of all the state ω at which i is *rational* according to his conjecture $\mathbf{q}_i^t(\cdot\,;\omega)$; that is, each i's actual action a_i maximizes the expectation of his pay off function g_i being actually played at ω when the other players actions are distributed according to his conjecture $\mathbf{q}_i^t(\cdot\,;\omega)$ at time t.[8]
- The message $M_i^t : \Omega \to 2^\Omega$ sent by the sender i at time t is defined as a robust information:

$$M_i^t(\omega) = \bigcap_{a_{-i}\in A_{-i}} \{\xi \in \Omega \mid \mathbf{q}_i^t(a_{-i},\xi) \geq \mathbf{q}_i^t(a_{-i},\omega)\}.$$

Then:
– The revised partition Π_i^{t+1} at time $t+1$ is defined as follows:
  - $\Pi_i^{t+1}(\omega) = \Pi_i^t(\omega) \cap M_{s(t)}^t(\omega)$ if $i = r(t)$;
  - $\Pi_i^{t+1}(\omega) = \Pi_i^t(\omega)$ otherwise,
– The revision process $(\phi_i^t)_{(i,t)\in N\times T}$ of conjectures is inductively defined as follows:
  - Let $\omega_0 \in \Omega$, and set $\phi_{s(0)}^0(a_{-s(0)}) := \mathbf{q}_{s(0)}^0(a_{-s(0)},\omega_0)$
  - Take $\omega_1 \in M_{s(0)}^0(\omega_0) \cap B_{r(0)}([g_{s(0)}] \cap R_{s(0)}^0; p)$,[9] and set $\phi_{s(1)}^1(a_{-s(1)}) := \mathbf{q}_{s(1)}^1(a_{-s(1)},\omega_1)$
  - Take

$$\omega_{t+1} \in M_{s(t)}^t(\omega_t) \cap B_{r(t)}([g_{s(t)}] \cap R_{s(t)}^t; p),$$

  and set

$$\phi_{s(t+1)}^{t+1}(a_{-s(t+1)}) := \mathbf{q}_i^{t+1}(a_{-s(t+1)},\omega_{t+1}).$$

The specification is that a sender $s(t)$ at time t informs the receiver $r(t)$ his/her individual conjecture about the other players' actions with a probability greater than his/her belief. The receiver revises her/his information structure under the information. She/he predicts the other players action at the state where the player p-believes that the sender $s(t)$ is rational, and she/he informs her/his the predictions to the other player $r(t+1)$.

We denote by ∞ a sufficient large τ such that for all $\omega \in \Omega$, $\mathbf{q}_i^\tau(\cdot\,;\omega) = \mathbf{q}_i^{\tau+1}(\cdot\,;\omega) = \mathbf{q}_i^{\tau+2}(\cdot\,;\omega) = \cdots$. Hence we can write $\mathbf{q}_i^\tau$ by $\mathbf{q}_i^\infty$ and ϕ_i^τ by ϕ_i^∞.

Remark 2. This communication model is a variation of the model introduced by Matsuhisa [6].

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[8] Formally, letting $g_i = \mathbf{g}_i(\omega)$, $a_i = \mathbf{a}_i(\omega)$, the expectation at time t, $\mathbf{Exp}^t$, is defined by

$$\mathbf{Exp}^t(g_i(a_i,\mathbf{a}_{-i});\omega) := \sum_{a_{-i}\in A_{-i}} g_i(a_i,a_{-i})\ \mathbf{q}_i^t(a_{-i},\omega).$$

An player i is said to be rational according to his conjecture $\mathbf{q}_i^t(\cdot,\omega)$ at ω if for all b_i in A_i, $\mathbf{Exp}^t(g_i(a_i,\mathbf{a}_{-i});\omega) \geq \mathbf{Exp}^t(g_i(b_i,\mathbf{a}_{-i});\omega)$.

[9] We denote $[g_i] := [\mathbf{g}_i = g_i]$

3 The Result

We can now state the main theorem :

Theorem 1. *Suppose that the players in a strategic form game G have the knowledge structure with μ a common prior. In the Bayesian belief communication process $\pi(G)$ according to a protocol* Pr *among all players in the game, the n-tuple of their conjectures $(\phi_i^t)_{(i,t)\in N\times T}$ converges to a mixed strategy Nash equilibrium of the game in finitely many steps.*

The proof is based on the below proposition:

Proposition 1. *Notation and assumptions are the same in Theorem 1. For any players $i, j \in N$, their conjectures $\mathbf{q}_i^\infty$ and $\mathbf{q}_j^\infty$ on $A \times \Omega$ must coincide; that is, $\mathbf{q}_i^\infty(a;\omega) = \mathbf{q}_j^\infty(a;\omega)$ for every $a \in A$ and $\omega \in \Omega$.*

Proof. On noting that Pr is fair, it suffices to verify that $\mathbf{q}_i^\infty(a;\omega) = \mathbf{q}_j^\infty(a;\omega)$ for $(i,j) = (s(\infty), r(\infty))$. Since $\Pi_i(\omega) \subseteq [a_i]$ for all $\omega \in [a_i]$, we can observe that $\mathbf{q}_i^\infty(a_{-i};\omega) = \mathbf{q}_i^\infty(a;\omega)$, and we let define the partitions of Ω, $\{W_i^\infty(\omega) \mid \omega \in \Omega\}$ and $\{Q_j^\infty(\omega) \mid \omega \in \Omega\}$, as follows:

$$W_i^\infty(\omega) := \bigcap_{a_{-i}\in A_{-i}} [\mathbf{q}_i^\infty(a_{-i},*) = \mathbf{q}_i^\infty(a_{-i},\omega)] = \bigcap_{a\in A} [\mathbf{q}_i^\infty(a,*) = \mathbf{q}_i^\infty(a,\omega)],$$
$$Q_j^\infty(\omega) := \Pi_j^\infty(\omega) \cap W_i^\infty(\omega).$$

It follows that

$$Q_j^\infty(\xi) \subseteq W_i^\infty(\omega) \quad \text{for all } \xi \in W_i^\infty(\omega),$$

and hence $W_i^\infty(\omega)$ can be decomposed into a disjoint union of components $Q_j^\infty(\xi)$ for $\xi \in W_i^\infty(\omega)$;

$$W_i^\infty(\omega) = \bigcup_{k=1,2,\ldots,m} Q_j^\infty(\xi_k) \quad \text{for} \quad \xi_k \in W_i^\infty(\omega).$$

It can be observed that

$$\mu([\mathbf{a}=a] \mid W_i^\infty(\omega)) = \sum_{k=1}^{m} \lambda_k \mu([\mathbf{a}=a] \mid Q_j^\infty(\xi_k)) \tag{1}$$

for some $\lambda_k > 0$ with $\sum_{k=1}^m \lambda_k = 1$.[10]

On noting that $W_j^\infty(\omega)$ is decomposed into a disjoint union of components $\Pi_j^\infty(\xi)$ for $\xi \in W_j^\infty(\omega)$, it can be observed that

$$\mathbf{q}_j^\infty(a;\omega) = \mu([\mathbf{a}=a] \mid W_j^\infty(\omega)) = \mu([\mathbf{a}=a] \mid \Pi_j^\infty(\xi_k)) \tag{2}$$

[10] This property is called the *convexity* for the conditional probability $\mu(X|*)$ in Parikh and Krasucki [13].

for any $\xi_k \in W_i^\infty(\omega)$. Furthermore we can verify that for every $\omega \in \Omega$,

$$\mu([\mathbf{a} = a] \| W_j^\infty(\omega)) = \mu([\mathbf{a} = a] \| Q_j^\infty(\omega)). \tag{3}$$

In fact, we first note that $W_j^\infty(\omega)$ can also be decomposed into a disjoint union of components $Q_j^\infty(\xi)$ for $\xi \in W_j^\infty(\omega)$. We shall show that for every $\xi \in W_j^\infty(\omega)$, $\mu([\mathbf{a} = a] \| W_j^\infty(\omega)) = \mu([\mathbf{a} = a] \| Q_j^\infty(\xi))$. For: Suppose not, the disjoint union G of all the components $Q_j(\xi)$ such that $\mu([\mathbf{a} = a] \| W_j^\infty(\omega)) = \mu([\mathbf{a} = a] \| Q_j^\infty(\xi))$ is a proper subset of $W_j^\infty(\omega)$. It can be shown that for some $\omega_0 \in W_j^\infty(\omega) \setminus G$ such that $Q_j(\omega_0) = W_j^\infty(\omega) \setminus G$. On noting that $\mu([\mathbf{a} = a] \| G) = \mu([\mathbf{a} = a] \| W_j^\infty(\omega))$ it follows immediately that $\mu([\mathbf{a} = a] \| Q_j^\infty(\omega_0)) = \mu([\mathbf{a} = a] \| W_j^\infty(\omega))$, in contradiction. Now suppose that for every $\omega_0 \in W_j^\infty(\omega) \setminus G$, $Q_j(\omega_0) \neq W_j^\infty(\omega) \setminus G$. The we can take an infinite sequence of states $\{\omega_k \in W_j^\infty(\omega) \mid k = 0, 1, 2, 3, \dots\}$ with $\omega_{k+1} \in W_j^\infty(\omega) \setminus (G \cup Q_j^\infty(\omega_0) \cup Q_j^\infty(\omega_1) \cup Q_j^\infty(\omega_2) \cup \cdots \cup Q_j^\infty(\omega_k))$ in contradiction also, because Ω is finite.

In viewing (1), (2) and (3) it follows that

$$\mathbf{q}_i^\infty(a;\omega) = \sum_{k=1}^{m} \lambda_k \mathbf{q}_j^\infty(a;\xi_k) \tag{4}$$

for some $\xi_k \in W_i^\infty(\omega)$. Let ξ_ω be the state in $\{\xi_k\}_{k=1}^m$ attains the maximal value of all $\mathbf{q}_j^\infty(a;\xi_k)$ for $k = 1, 2, 3, \cdots, m$, and let $\zeta_\omega \in \{\xi_k\}_{k=1}^m$ be the state that attains the minimal value. By (4) we obtain that $\mathbf{q}_j^\infty(a;\zeta_\omega) \leq \mathbf{q}_i^\infty(a;\omega) \leq \mathbf{q}_j^\infty(a;\xi_\omega)$ for $(i,j) = (s(\infty), t(\infty))$.

On continuing this process according to the *fair* protocol Pr, it can be plainly verified: For each $\omega \in \Omega$ and for any $t \geq 1$,

$$\mathbf{q}_i^\infty(a;\zeta'_\omega) \leq \cdots \leq \mathbf{q}_j^\infty(a;\zeta_\omega) \leq \mathbf{q}_i^\infty(a;\omega) \leq \mathbf{q}_j^\infty(a;\xi_\omega) \leq \cdots \leq \mathbf{q}_i^\infty(a;\xi'_\omega)$$

for some $\zeta'_\omega, \cdots, \zeta_\omega, \xi_\omega, \cdots \xi'_\omega \in \Omega$, and thus $\mathbf{q}_i^\infty(a;\omega) = \mathbf{q}_j^\infty(a;\omega)$ because $\mathbf{q}_j^\infty(a;\zeta_\omega) \leq \mathbf{q}_j^\infty(a;\omega) \leq \mathbf{q}_j^\infty(a;\xi_\omega)$ and $\mathbf{q}_i^\infty(a;\zeta) = \mathbf{q}_j^\infty(a;\xi)$ for every $\zeta, \xi \in \Omega$. in completing the proof.

Proof of Theorem 1: We denote by $\Gamma(i)$ the set of all the players who directly receive the message from i on N; i.e., $\Gamma(i) = \{\, j \in N \mid (i,j) = \mathrm{Pr}(t) \text{ for some } t \in T\}$. Let F_i denote $[\phi_i^\infty] := \bigcap_{a_{-i} \in A_i} [\mathbf{q}_i^\infty(a_{-i};*) = \phi_i^\infty(a_{-i})]$. It is noted that $F_i \cap F_j \neq \emptyset$ for each $i \in N$, $j \in \Gamma(i)$.

We observe the first point that for each $i \in N$, $j \in \Gamma(i)$ and for every $a \in A$, $\mu([\mathbf{a}_{-j} = a_{-j}] \mid F_i \cap F_j) = \phi_j^\infty(a_{-j})$. Then summing over a_{-i}, we can observe that $\mu([\mathbf{a}_i = a_i] \mid F_i \cap F_j) = \phi_j^\infty(a_i)$ for any $a \in A$. In view of Proposition 1 it can be observed that $\phi_j^\infty(a_i) = \phi_k^\infty(a_i)$ for each $j, k, \neq i$; i.e., $\phi_j^\infty(a_i)$ is independent of the choices of every $j \in N$ other than i. We set the probability distribution σ_i on A_i by $\sigma_i(a_i) := \phi_j^\infty(a_i)$, and set the profile $\sigma = (\sigma_i)$.

We observe the second point that for every $a \in \prod_{i \in N} \mathrm{Supp}(\sigma_i)$, $\phi_i^\infty(a_{-i}) = \sigma_1(a_1) \cdots \sigma_{i-1}(a_{i-1})\sigma_{i+1}(a_{i+1}) \cdots \sigma_n(a_n)$: In fact, viewing the definition of σ_i we shall show that $\phi_i^\infty(a_{-i}) = \prod_{k \in N \setminus \{i\}} \phi_i^\infty(a_k)$. To verify this it suffices to show that for every $k = 1, 2, \cdots, n$, $\phi_i^\infty(a_{-i}) = \phi_i^\infty(a_{-I_k}) \prod_{k \in I_k \setminus \{i\}} \phi_i^\infty(a_k)$: We prove it by induction on k. For $k = 1$ the result is immediate. Suppose it is true for $k \geq 1$. On noting the protocol is fair, we can take the sequence of sets of players $\{I_k\}_{1 \leq k \leq n}$ with the following properties:

(a) $I_1 = \{i\} \subset I_2 \subset \cdots \subset I_k \subset I_{k+1} \subset \cdots \subset I_m = N$:

(b) For every $k \in N$ there is a player $i_{k+1} \in \bigcup_{j \in I_k} \Gamma(j)$ with $I_{k+1} \setminus I_k = \{i_{k+1}\}$.

We let take $j \in I_k$ such that $i_{k+1} \in \Gamma(j)$. Set $H_{i_{k+1}} := [\mathbf{a}_{i_{k+1}} = a_{i_{k+1}}] \cap F_j \cap F_{i_{k+1}}$. It can be verified that $\mu([\mathbf{a}_{-j-i_{k+1}} = a_{-j-i_{k+1}}] \mid H_{i_{k+1}}) = \phi_{-j-i_{k+1}}^\infty(a_{-j})$. Dividing $\mu(F_j \cap F_{i_{k+1}})$ yields that

$$\mu([\mathbf{a}_{-j} = a_{-j}] \mid F_j \cap F_{i_{k+1}}) = \phi_{i_{k+1}}^\infty(a_{-j}) \mu([\mathbf{a}_{i_{k+1}} = a_{i_{k+1}}] \mid F_j \cap F_{i_{k+1}}).$$

Thus $\phi_j^\infty(a_{-j}) = \phi_{i_{k+1}}^\infty(a_{-j-i_{k+1}}) \phi_j^t(a_{i_{k+1}})$; then summing over a_{I_k} we obtain $\phi_j^\infty(a_{-I_k}) = \phi_{i_{k+1}}^\infty(a_{-I_k - i_{k+1}}) \phi_j^\infty(a_{i_{k+1}})$. It immediately follows from Proposition 1 that $\phi_i^\infty(a_{-I_k}) = \phi_i^\infty(a_{-I_k - i_{k+1}}) \phi_i^\infty(a_{i_{k+1}})$, as required.

Furthermore we can observe that all the other players i than j agree on the same conjecture $\sigma_j(a_j) = \phi_i^\infty(a_j)$ about j. We conclude that each action a_i appearing with positive probability in σ_i maximizes g_i against the product of the distributions σ_l with $l \neq i$. This implies that the profile $\sigma = (\sigma_i)_{i \in N}$ is a mixed strategy Nash equilibrium of G, in completing the proof. □

4 Concluding remarks

We have observed that in a communication process with revisions of players' beliefs about the other actions, their predictions induces a mixed strategy Nash equilibrium of the game in the long run. It is well to end some remarks on related literatures. The **S5**-knowledge model is an operator model equivalent to the Kripke semantics for the modal logic **S5** (= KT45), which is the binary relation on a state-space satisfying reflectivity, transitivity and symmetry. The **S4**-knowledge model is equivalent to the Kripke semantics for the modal logic **S4** (= KT4), which is the binary relation on a state-space satisfying reflectivity, transitivity.

Matsuhisa [6] and [8] established the same assertion in the **S4**-knowledge model. Furthermore Matsuhisa [7] showed a similar result for ε-mixed strategy Nash equilibrium of a strategic form game in the **S4**-knowledge model, which gives an epistemic aspect in Theorem of E. Kalai and E. Lehrer [5]. This article highlights a communication among the players in a game through sending rough information, and shows that the convergence to an exact Nash equilibrium is guaranteed even in such communication on approximate information after long run.

The main theorem in this article is an extension in the Bayesian communication for the **S5**-knowledge model.[11] There is an agenda to further research;

[11] See Remark 1.

first, to extend our main theorem to **S4**-knowledge model, which gives another generalization of the theorem for the **S5**-knowledge model, because it coincides with the theorems in Matsuhisa [6] and [8], and secondly, to unify all the communication models in the preceding papers ([6], [8], [10], [9]) including the result presented in this article.

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