Residuated mapping and CPS-translation
– Extended abstract –

Ken-etsu Fujita (藤田 憲悦)
Gunma University (群馬大学)
fujita@cs.gunma-u.ac.jp

Abstract
We provide a call-by-name CPS-translation from polymorphic \( \lambda \)-calculus \( \lambda^2 \) into existential \( \lambda \)-calculus \( \lambda^\exists \). Then we prove that the CPS-translation is a residuated mapping from the preordered set of \( \lambda^2 \)-terms to that of \( \lambda^\exists \)-terms. From the inductive proof, its residual (inverse translation) can be extracted, which constitutes the so-called Galois connection. It is also obtained that given the CPS-translation the existence of its inverse is unique.

1 Preliminaries

By a preordered set \( \langle A, \sqsubseteq \rangle \), we mean a set \( A \) on which there is defined a preorder, i.e., a reflexive and transitive relation \( \sqsubseteq \). If \( \langle A_1, \sqsubseteq_1 \rangle \) and \( \langle A_2, \sqsubseteq_2 \rangle \) are preordered sets, then we say that a mapping \( f : A_1 \to A_2 \) is monotone, if \( x \sqsubseteq_1 y \) implies \( f(x) \sqsubseteq_2 f(y) \) for any \( x, y \in A_1 \). A direct image under \( f \) is denoted by \( f[X] \) for every \( X \subseteq A_1 \), and an inverse image is denoted by \( f^{-1}[Y] \) for every \( Y \subseteq A_2 \). A subset \( B \subseteq A \) is a down-set of a preordered set \( \langle A, \sqsubseteq \rangle \), if \( y \sqsubseteq x \) together with \( y \in A \) and \( x \in B \) implies \( y \in B \). By a principal down-set, we mean a down-set of the form \( \{ y \in A \mid y \sqsubseteq x \} \), which is denoted by \( \downarrow x \).

Definition 1 (Residuated mapping) A mapping \( f : A \to B \) that satisfies the following condition is said to be residuated: The inverse image under \( f \) of every principal down-set of \( B \) is a principal down-set of \( A \).

2 Source calculus: \( \lambda^2 \)

We introduce our source calculus of 2nd order \( \lambda \)-calculus (Girard-Reynolds), denoted by \( \lambda^2 \). For simplicity, we adopt its domain-free style.

Definition 2 (Types)
\[ A ::= X \mid A \to A \mid \forall X.A \]

Definition 3 ((Pseudo)\( \lambda^2 \)-terms)
\[ \lambda^2 \exists M ::= x \mid \lambda x.M \mid MM \mid \lambda X.M \mid MA \]
4 (Reduction rules)  
(β)  $(\lambda x. M_1) M_2 \rightarrow M_1[x := M_2]$

(η)  $\lambda x. M x \rightarrow M$, if $x \not\in FV(M)$

($\beta_t$)  $(\lambda X. M) A \rightarrow M[X := A]$

($\eta_t$)  $\lambda X. M X \rightarrow M$, if $X \not\in FV(M)$

$FV(M)$ denotes a set of free variables in $M$.

We write $\rightarrow_{\lambda 2}$ for the compatible relation obtained from the reflexive and transitive closure of the one step reduction relation, and $\rightarrow^{+}_{\lambda 2}$ for that from the transitive closure. In particular, $\rightarrow_{R}$ denotes the subrelation of $\rightarrow$ restricted to the reduction rules $R \subseteq \{\beta, \eta, \beta_t, \eta_t\}$. We may write simply $(\beta)$ for either $(\beta)$ or $(\beta_t)$, and $(\eta)$ for either $(\eta)$ or $(\eta_t)$, if clear from the context. We employ the notation $\equiv$ to indicate the syntactic identity under renaming of bound variables.

3 Target calculus: $\lambda^3$

We next define our target calculus denoted by $\lambda^3$, which is logically a subsystem of minimal logic consisting of constant $\bot$, negation, conjunction and 2nd order existential quantification\footnote{For further introduction of the CPS target calculus $\lambda^3$ with let-expressions, see also [5].}.

Definition 5 (Types)

$$A ::= \bot | X | \neg A | A \land A | \exists X. A$$

Definition 6 ((Pseudo)$\lambda^3$-terms)

$$\Lambda^3 \ni M ::= x | \lambda x. M | MM | \langle M, M \rangle | \text{let } \langle x, x \rangle = M \text{ in } M$$

$$| \langle A, M \rangle | \text{let } \langle X, x \rangle = M \text{ in } M$$

Definition 7 (Reduction rules)  
(β)  $(\lambda x. M_1) M_2 \rightarrow M_1[x := M_2]$

(η)  $\lambda x. M x \rightarrow M$, if $x \not\in FV(M)$

(\text{let}_\lambda)  \text{let } \langle x_1, x_2 \rangle = \langle M_1, M_2 \rangle \text{ in } M \rightarrow M[x_1 := M_1, x_2 := M_2]$

(\text{let}_{\land})  \text{let } \langle x_1, x_2 \rangle = M_1 \text{ in } M[z := \langle x_1, x_2 \rangle] \rightarrow M[z := M_1], \quad \text{if } x_1, x_2 \not\in FV(M)$

(\text{let}_\exists)  \text{let } \langle X, x \rangle = \langle A, M_1 \rangle \text{ in } M \rightarrow M[X := A, x := M_1]$

(\text{let}_{\exists})  \text{let } \langle X, x \rangle = M_1 \text{ in } M[z := \langle X, x \rangle] \rightarrow M_2[z := M_1], \quad \text{if } X, x \not\in FV(M_2)$

We also write simply (\text{let}) for either (\text{let}_\lambda) or (\text{let}_\exists), and (\text{let}_\eta) for (\text{let}_{\land}) or (\text{let}_{\exists_\eta}). Similarly we write $\rightarrow_{\lambda 3}$ and $\rightarrow^{+}_{\lambda 3}$ as done for $\lambda 2$. 


4 CPS-translation * from $\Lambda^2$ into $\Lambda^3$

We define a translation, so-called modified CPS-translation * from pseudo $\lambda^2$-terms into pseudo $\lambda^3$-terms. In each case, a fresh and free variable $a$ is introduced, which is called a continuation variable.

**Definition 8**

1. $x^* = xa$
2. $(\lambda x.M)^* = \text{let} \langle x, a \rangle = a \text{ in } M^*$
3. $(M_1 M_2)^* = \begin{cases} M_1^*[a := \langle x, a \rangle] & \text{for } M_2 \equiv x \\ M_1^*[a := \langle \lambda a.M_2^*, a \rangle] & \text{otherwise} \end{cases}$
4. $(\lambda X.M)^* = \text{let} \langle X, a \rangle = a \text{ in } M^*$
5. $(MA)^* = M^*[a := \langle A^*, a \rangle]$
6. $X^* = X$; $(A_1 \Rightarrow A_2)^* = \neg A_1^* \land A_2^*$; $(\forall X.A)^* = \exists X.A^*$

Remarked that $M^*$ contains exactly one free occurrence of a continuation variable $a$, and $M^*$ has neither $\beta$-redex nor $\eta$-redex. Let $\lambda X.M$ have type $\forall X.A$. Then, under the translation, the parametric polymorphic function $\lambda X.M$ with respect to $X$ becomes an abstract data type $(\lambda X.M)^*$ for $X$, which is waiting for an implementation $a$ with type $\exists X.A^*$ together with an interface (a signature) with type $A^*$, i.e., $(\lambda X.M)^*$ is

\begin{align*}
\text{abstype } X \text{ with } a:A^* \text{ is } a \text{ in } M^*
\end{align*}
in a familiar notation.

**Lemma 1 (Monotone *)** If we have $M_1 \rightarrow_{\lambda^2} M_2$, then $M_1^* \rightarrow_{\lambda^3} M_2^*$ holds. In particular, if $M_1 \rightarrow_{\beta} M_2$, then $M_1^* \rightarrow_{\beta} M_2^*$. And if $M_1 \rightarrow_{\eta} M_2$, then $M_1^* \rightarrow_{\eta} M_2^*$.

**Proof.** By induction on the derivation. \hfill $\square$

In order to give an inverse translation, first we provide the mutual inductive definitions, respectively for denotations $\text{Univ}$ and continuations $C$, as follows. Both $\text{Univ}$ and $C$ are down-sets in the above sense.

\[
\begin{array}{c}
a \in C \\
\frac{C \in C}{\langle x, C \rangle \in C}
\end{array}
\]
\[
\begin{array}{c}
C \in C \quad P \in \text{Univ} \\
\frac{\langle \lambda a.P, C \rangle \in C}{C \in C}
\end{array}
\]
\[
\begin{array}{c}
C \in C \quad xC \in \text{Univ} \\
\frac{x \in C}{C \in xC}
\end{array}
\]
\[
\begin{array}{c}
C \in C \\
\frac{P \in \text{Univ}}{\text{let} \langle x, a \rangle = C \text{ in } P \in \text{Univ}}
\end{array}
\]
\[
\begin{array}{c}
C \in C \quad P \in \text{Univ} \\
\frac{\langle \lambda a.P \rangle C \in \text{Univ}}{C \in \text{Univ}}
\end{array}
\]
\[
\begin{array}{c}
C \in C \\
\frac{\langle X, a \rangle = C \text{ in } P \in \text{Univ}}{\text{let} \langle X, a \rangle = C \text{ in } P \in \text{Univ}}
\end{array}
\]
We write \( \langle R_1, R_2, \ldots, R_n \rangle \) for \( \langle R_1, \langle R_2, \ldots, R_n \rangle \rangle \) with \( n > 1 \), and \( \langle R_1 \rangle \) for \( R_1 \) with \( n = 1 \). \( C \in \mathcal{C} \) is in the form of \( \langle R_1, \ldots, R_n, a \rangle \) where \( R_i \) (1 \( \leq i \leq n \)) is \( x, \lambda a. P \), or \( A^* \) with \( n \geq 0 \). We explicitly mention that \( C \in \mathcal{C} \) has exactly one occurrence of free variable \( a \) such that \( C \equiv \langle R_1, \ldots, R_n, a \rangle \) with \( n \geq 0 \). \( P \in \text{Univ} \) also has exactly one occurrence of free variable \( a \) such as \( C \) as a proper subterm of \( P \).

The inductively defined sets \( \text{Univ}, \mathcal{C} \subseteq \Lambda^3 \) are down-sets with respect to \( \rightarrow_{\lambda^3} \).

**Lemma 2**

1. If \( P_1 \in \text{Univ} \) and \( P_1 \rightarrow_{\lambda^3} P_2 \), then \( P_2 \in \text{Univ} \).

2. If \( C_1 \in \mathcal{C} \) and \( C_1 \rightarrow_{\lambda^3} C_2 \), then \( C_2 \in \mathcal{C} \).

**Proof.**

Let \( P, P_1 \in \text{Univ} \) and \( C, C_1 \in \mathcal{C} \). Then \( P[a := C_1], P[x := \lambda a. P_1], P[X := A^*] \in \text{Univ} \), and \( C[a := C_1], C[x := \lambda a. P_1], C[X := A^*] \in \mathcal{C} \).

**Proposition 1**

1. \( \text{Univ} \) is strongly normalizing with respect to \( \rightarrow_{\beta\eta} \), i.e., for any \( P \in \text{Univ} \), there is no infinite reduction sequence of \( \rightarrow_{\beta\eta} \) starting with \( P \).

2. \( \text{Univ} \) is Church-Rosser with respect to \( \rightarrow_{\beta\eta} \), i.e., for any \( P, P_1, P_2 \in \text{Univ} \), if we have \( P \rightarrow_{\beta\eta} P_1 \) and \( P \rightarrow_{\beta\eta} P_2 \), then there exists some \( P_3 \in \text{Univ} \) such that \( P_1 \rightarrow_{\beta\eta} P_3 \) and \( P_2 \rightarrow_{\beta\eta} P_3 \).

**Proof.**

1. Since every \( \lambda \)-abstraction \( \lambda a. P \in \text{Univ} \) is linear, for any \( P_1 \rightarrow_{\beta\eta} P_2 \), the contractum \( P_2 \) has less length than that of \( P_1 \).

2. \( \text{Univ} \) is weak Church-Rosser with respect to \( \rightarrow_{\beta\eta} \), and hence the property of Church-Rosser holds from Newman’s Lemma.

Any (pseudo) term \( P \in \text{Univ} \) is Church-Rosser and strongly normalizing with respect to \( \beta\eta \)-reductions, and the unique \( \beta\eta \)-normal form is denoted by \( \downarrow_{\beta\eta} P \). The same property naturally holds for \( \mathcal{C} \) as well. A normalization function \( \downarrow_{\beta\eta} \) can be inductively defined as follows:

**Definition 9** \((\downarrow_{\beta\eta})\)

1. For \( P \in \text{Univ} \):

   \begin{enumerate}
   \item \( \downarrow_{\beta\eta} (xC) = x(\downarrow_{\beta\eta} C) \)
   \item \( \downarrow_{\beta\eta} (\lambda a. P) C = \downarrow_{\beta\eta} (P[a := C]) \)
   \item \( \downarrow_{\beta\eta} \text{let} \langle \chi, a \rangle = C \text{ in } P = \text{let} \langle \chi, a \rangle = \downarrow_{\beta\eta} C \text{ in } \downarrow_{\beta\eta} P \)
   \end{enumerate}

2. For \( C \equiv \langle R_1, \ldots, R_n, a \rangle \in \mathcal{C} \) with \( n \geq 0 \), where \( R_i \equiv x, \lambda a. P \), or \( A^* \):

   \begin{enumerate}
   \item \( R \equiv x \):
      \( \downarrow_{\beta\eta} x = x \)
   \item \( R \equiv \lambda a. P \):
      \begin{enumerate}
      \item \( \downarrow_{\beta\eta} (\lambda a.xa) = x \), if \( P \equiv xa \);
      \item \( \downarrow_{\beta\eta} (\lambda a. P) = \lambda a.(\downarrow_{\beta\eta} P) \), otherwise;
      \end{enumerate}
   \item \( R \equiv A^* \):
      \( \downarrow_{\beta\eta} A^* = A^* \)
   \end{enumerate}
5 Residuated CPS-translation

Proposition 2 The following conditions are equivalent.

1. $f : A \rightarrow B$ is a residuated mapping.

2. $f : A \rightarrow B$ is monotone and there exists a monotone mapping $g : B \rightarrow A$ such that $A \ni a \subseteq g(f(a))$ and $f(g(b)) \subseteq b \in B$.

Proof. A residuated mapping is monotone in general. On the other hand, from the condition 1, for any $b \in B$ there exists $a \in A$ such that $f^\downarrow b = \downarrow a$ which cannot be empty, whence one has a choice function $g : B \rightarrow A$ by $g(b) = a$. Hence $g(b) \in \downarrow g(b) = f^\downarrow b$ holds true, so that we have $f(g(b)) \subseteq b$. We also have $a \in f^\downarrow f(a) = \downarrow g(f(a))$ by the definition, and hence we have $a \subseteq g(f(a))$.

From the condition 2, we have that $f(a) \subseteq b$ if and only if $a \subseteq g(b)$. Hence, we have $f^\downarrow b = \downarrow g(b)$ for every $b \in B$. \hfill \Box

We write $M \subseteq N$ for $N \leadsto M$, i.e., the contextual and reflexive-transitive closure of one-step reduction $\rightarrow$.

Lemma 3 For any $P \in \text{Univ}$, there uniquely exists $M \in \Lambda 2$ such that $\downarrow_{\beta\eta} P \equiv M^*$. 

Proof. By induction on $P \in \text{Univ}$.

1. Case of $P \equiv xC \equiv x(R_1, \ldots, R_n, a)$ with $n \geq 0$

   (a) If $R_i \equiv x_i$, then we take $N_i \equiv x_i$, whence $\downarrow_{\beta\eta} R_i \equiv x_i \equiv N_i^*$. 

   (b) Case of $R_i \equiv \lambda a.P_i$

      If $P_i \equiv x_i a$, then we take $N_i \equiv x_i$, and whence $\downarrow_{\beta\eta} R_i \equiv x_i \equiv N_i^*$.

      Otherwise, from the induction hypothesis for $P_i$, there uniquely exists $N_i$ such that $\downarrow_{\beta\eta} P_i \equiv N_i^*$. Now we have $\downarrow_{\beta\eta} R_i = \lambda a.(\downarrow_{\beta\eta} P_i) \equiv \lambda a.N_i^*$.

   (c) If $R_i \equiv A_i^*$, then we take $N_i \equiv A_i$.

Hence, we take $M \equiv xN_1 \ldots N_n$, and then there uniquely exists $M \in \Lambda 2$ such that $\downarrow_{\beta\eta} P$

$= x(\downarrow_{\beta\eta} R_1, \ldots, \downarrow_{\beta\eta} R_n, a)$

$\equiv x(N_1^{*'}, \ldots, N_n^{*'}, a)$

$= M^*$,

where $N_i^{*'} = \lambda a.N_i^*$ if $R_i \equiv \lambda a.P_i$ with no outmost $\eta$-redex; otherwise $N_i^{*'} = N_i^*$.

2. Case of $P \equiv (\lambda a.P')C$

   Since $a$ is a linear variable, by the induction hypothesis for $P'[a := C]$, there uniquely exists $M \in \Lambda 2$ such that $\downarrow_{\beta\eta} (P'[a := C]) \equiv M^*$. Therefore, we have a unique $M \in \Lambda 2$ such that $\downarrow_{\beta\eta} P \equiv M^*$.

3. Case of $P \equiv \mathsf{let} (x, a) = C$ in $P_1$ with $C \equiv (R_1, \ldots, R_n, a)$ and $n \geq 0$
(a) From the induction hypothesis for $P_1$, there uniquely exists $M_1 \in \Lambda 2$ such that \[ \Downarrow_{\beta\eta}P_1 \equiv M_1^* \).

(b) If $R_i \equiv x_i$, then we take $N_i \equiv x_i$, whence $\Downarrow_{\beta\eta}R_i \equiv x_i \equiv N_i^*$. 

(c) Case of $R_i \equiv \lambda a.P_i$

\[ \text{If } P_i \equiv x_i a, \text{ then we take } N_i \equiv x_i, \text{ and } \Downarrow_{\beta\eta}R_i \equiv x_i \equiv N_i^*. \]

Otherwise, from the induction hypothesis for $P_i$, there uniquely exists $N_i$ such that $\Downarrow_{\beta\eta}P_i \equiv N_i^*$. Now we have $\Downarrow_{\beta\eta}R_i = \lambda a.(\Downarrow_{\beta\eta}P_i) \equiv \lambda a.N_i^*$. 

(d) If $R_i \equiv A_i^*$, then we take $N_i \equiv A_i$. 

Hence, we take $M \equiv xN_1 \ldots N_n$, and then there uniquely exists $M \in \Lambda 2$ such that
\[ \Downarrow_{\beta\eta}P = \text{let } \langle x, a \rangle = \langle \Downarrow_{\beta\eta}R_1, \ldots, \Downarrow_{\beta\eta}R_n, a \rangle \text{ in } (\Downarrow_{\beta\eta}P_1) \equiv \text{let } \langle x, a \rangle = (N_i^*, \ldots, N_i^*, a) \text{ in } M_1^* \equiv M^*, \]

where $N_i^* = \lambda a.N_i^*$ if $R_i \equiv \lambda a.P_i$ with no outmost $\eta$-redex; otherwise $N_i^* = N_i^*$. 

4. Case of $P \equiv \text{let } \langle X, a \rangle = C \text{ in } P'$ is handled similarly. \hfill \square

From the inductive proof of Lemma 3 above, an extracted function giving a witness is written down here.

1. $x^\# = x; (\lambda a.P)^\# = P^\#; (A^*)^\# = A$
2. $(x(R_1, \ldots, R_n, a))^\# = xR_1^\# \ldots R_n^\#$
3. $((\lambda a.P)C)^\# = (P[a := C])^\#$
4. $(\text{let } \langle x, a \rangle = \langle R_1, \ldots, R_n, a \rangle \text{ in } P)^\# = (\lambda x.P^\#)R_1^\# \ldots R_n^\#$
5. $(\text{let } \langle X, a \rangle = \langle R_1, \ldots, R_n, a \rangle \text{ in } P)^\# = (\lambda X.P^\#)R_1^\# \ldots R_n^\#$

where the clause 1 is for $R_i$ appeared in $\langle R_1, \ldots, R_n, a \rangle \in C$, and the clause 2 through 5 are for $P \in \text{Univ}$. 

Corollary 1 (Composition of $*$ and $\|$)

1. For any $P \in \text{Univ}$, we have $P \rightarrow_{\beta\eta} (P^\#)^*$. 

2. For any $M \in \Lambda 2$, we have $(M^*)^\# \equiv M$. 

Proof.

1. From Lemma 3, we have $\Downarrow_{\beta\eta}P \equiv (P^\#)^*$ and $P \rightarrow_{\beta\eta} \Downarrow_{\beta\eta}P$. Therefore, $P \rightarrow_{\beta\eta} (P^\#)^*$ holds for any $P \in \text{Univ}$. 

2. From the definition of $*$, $M^*$ has neither $\beta$- nor $\eta$-redex. Hence, $\Downarrow_{\beta\eta} (M^*) \equiv M^*$ holds, and then $(M^*)^\# \equiv M$ for any $M \in \Lambda 2$. \hfill \square

Lemma 4 (Monotone $\|$)
The above mapping $\| : \text{Univ} \rightarrow \Lambda 2$ is monotone.
Proof. By the definition of $\hat{\#}$. In particular, let $P_1, P_2 \in \text{Univ}$, then the following holds.

1. If $P_1 \implies_{\beta\eta} P_2$, then $P_1^\# \equiv P_2^\#$.
2. If $P_1 \implies_{1\text{et}} P_2$, then $P_1^\# \rightarrow_{\beta} P_2^\#$.
3. If $P_1 \implies_{1\cdot t_{\eta}} P_2$, then $P_1^\# \rightarrow_{\eta} P_2^\#$.

$\square$

6 Residuated CPS-translation

As expected from the previous results, the CPS-translation forms a residuated mapping from $\lambda 2$ to $\text{Univ}$.

Theorem 1 (Residuated CPS-trans.) The CPS-translation $*$ is a residuated mapping from $\lambda 2$ to $\text{Univ}$.

Proof. From Proposition 2, Lemmata 1 and 4, and Corollary 1, the translation $*$ is a residuated mapping. In other words, for any $P \in \text{Univ}$, we have

$$\{M \in \lambda 2 \mid M^* \subseteq P\} = \downarrow P^\#.$$ 

In fact, from Lemma 1 and Corollary 1, we have $\downarrow P^\# \subseteq \{M \in \lambda 2 \mid M^* \subseteq P\}$. On the other hand, from Lemma 4 and Corollary 1, the inverse direction $\{M \in \lambda 2 \mid M^* \subseteq P\} \subseteq \downarrow P^\#$ holds true.

We summarize results induced from the discussion above.

Corollary 2

1. $\lambda 2$ is strongly normalizing if and only if $\text{Univ}$ is strongly normalizing.
2. $\lambda 2$ is weakly normalizing if and only if $\text{Univ}$ is weakly normalizing.
3. $\lambda 2$ is Church-Rosser if and only if $\text{Univ}$ is Church-Rosser.

We remark that $\lambda^3$ itself is not Church-Rosser.

4. Let $\downarrow P$ be $\{Q \mid P \rightarrow_{\lambda^3} Q\}$ for $P \in \text{Univ}$. Then the inverse image under $*$ of $\downarrow P$ is a principal down-set generated by $P^\# \in \lambda 2$.

5. Given the CPS-translation $*$. Then an existence of its residual (inverse translation) is unique.

6. Define $P_1 \sim_{\rho n} P_2$ by $\downarrow_{\rho n} P_1 \equiv \downarrow_{\rho n} P_2$ for $P_1, P_2 \in \text{Univ}$. There exists a bijection $*$ between $\lambda 2$ and $\text{Univ}/ \sim_{\rho n}$. In particular, there exists a one-to-one correspondence between $\lambda 2$-normal forms and $\text{Univ}$-normal forms.

7. Let $\downarrow_{\lambda^3}[\lambda 2]^*$ be the down-set generated by $[\lambda 2]^*$, i.e., $\{P \mid M^* \rightarrow_{\lambda^3} P \text{ for some } M \in \lambda 2\}$. Let $\uparrow_{\rho n}[\lambda 2]^*$ be the up-set generated by $[\lambda 2]^*$, i.e., $\{P \in \text{Univ} \mid P \rightarrow_{\rho n} M^* \text{ for some } M \in \lambda 2\}$.

Then we have $\downarrow_{\lambda^3}[\lambda 2]^* \subseteq \text{Univ} = \uparrow_{\rho n}[\lambda 2]^*$. We remark that $\subseteq$ is strict. For instance, $xa \in \downarrow_{\lambda^3}[\lambda 2]^*$ and $(\lambda a.xa)a \in \text{Univ}$, but $(\lambda a.xa)a \not\in \downarrow_{\lambda^3}[\lambda 2]^*$. 
Proof.

1. If $M_1 \rightarrow_{\lambda_2} M_2$, then we have $M_1^* \rightarrow_{\lambda_3}^{+} M_2^*$ by induction on the derivation. Therefore, strong normalization of Univ implies that of $\lambda_2$.

On the other hand, $\rightarrow_{\rho\eta}$ in Univ is strongly normalizing. If Univ has an infinite reduction path of $\rightarrow_{\lambda_3}$, then the path should contain an infinite reduction path consisting of $\rightarrow_{\lambda_1 \rightarrow_{\lambda_1} ... \rightarrow_{\lambda_1}}$. Now, from Lemma 4, $\lambda_2$ has an infinite reduction path of $\rightarrow_{\rho\eta}$. Hence, strong normalization of $\lambda_2$ implies that of Univ.

2. From the monotone translations between $\Lambda 2$ and Univ, and the one-to-one correspondence between $\lambda_2$-normal forms and Univ-normal forms.

3. $\Lambda 2$ and Univ form the so-called Galois connection under $\star$ and $\sharp$.

4. The CPS-translation $\star$ forms a residuated mapping.

5. Suppose we had two inverse translations $\|_1$ and $\|_2$, then $P^{\|_1} \equiv P^{\|_2}$ for any $P \in$ Univ.

Because we have $P \rightarrow_{\rho\eta} P^{\|}$ for any $P \in$ Univ from Corollary 1 (1). Hence, we have $P^{\sharp} \equiv (P^{\|})^\# \equiv P^{\|\#}$ from Lemma 4 (1).

6. Since $\sim_{\beta\eta}$ is an equivalence relation over Univ, we take

$[P]_{\sim_{\beta\eta}} = \{ P' \in$ Univ $| P \sim_{\beta\eta} P' \}$ for $P \in$ Univ.

Then we define $\star(M) = [M^*]_{\sim_{\beta\eta}}$. In other words,

$\star(M) = \uparrow_{\beta\eta}(M^*) = \{ P \in$ Univ $| P \rightarrow_{\beta\eta} M^* \}$.

Then $\star: \Lambda 2 \rightarrow$ Univ$/\sim_{\beta\eta}$ is a bijection. In fact, for any $[P] \in$ Univ$/\sim_{\beta\eta}$, there exists $M \in \Lambda 2$ such that $\star(M) = [P]$. Because we take $M \equiv P^{\#}$, whence $P \rightarrow_{\beta\eta} (P^{\|})^* \star$ and $\star(P^{\#}) = [P]$. On the other hand, suppose $M_1 \neq M_2$. Then $\star(M_1) \neq \star(M_2)$, since $M_1^*$ and $M_2^*$ are distinct $\beta\eta$-normal forms.

7. For any $M \in \Lambda 2$, we have $M^* \in$ Univ, and Univ is a down-set with respect to $\rightarrow_{\lambda_3}$.

Then we have $\downarrow_{\lambda_3}[\Lambda 2]^* \subseteq$ Univ.

For any $P \in$ Univ, we have $P^{\sharp} \equiv \Lambda 2$ and $P \rightarrow_{\beta\eta} P^{\|}$ from Lemma 1. Hence, $P \equiv \uparrow_{\beta\eta}[\Lambda 2]^*$ holds true. The inverse direction is clear, and therefore we have Univ $= \uparrow_{\beta\eta}[\Lambda 2]^*$.

It is remarked that instead of pseudo-terms, when we take account of well-typed terms, the binary relations $\rightarrow_{\lambda_2}$ and $\rightarrow_{\lambda_3}$ form partial orders on $\lambda$-terms.

References


