数理解析研究所講究録
第 1562 巻 2007 年 19-24

On lattices with Elkan's formula

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Abstract

In this note we consider properties of lattices satisfying a special equation called Elkan's formula (E) : $(x \wedge y')' = y \vee (x' \wedge y')$. We show that, for any lattice L with a unary operator $'$ and a greatest element 1, if it satisfies (E) then $L^* = \{x'' \mid x \in L\}$ is a Boolean algebra.

1 Introduction

It is proved in [3] that each of de Morgan algebras and of orthomodular lattices with the formula $(x \wedge y')' = y \vee (x' \wedge y')$ is a Boolean algebra. Since the equation is presented in the theory of fuzzy logic by C.Elkan ([2]), we call it here the *Elkan's formula* and denote it by (E). The results obtained in [3] mean that Boolean algebras are characterized by de Morgan algebras and orthomodular lattices by use of this equation. It is also proved in [4] that every orthocomplemented lattice with (E) is a Boolean algebra. In this note we give a more general result than [4], that is, for any lattice with a greatest element 1, if it satisfies only the condition

$$\text{(E)} \quad (x \wedge y')' = y \vee (x' \wedge y'),$$

then $L^* = \{x'' \mid x \in L\}$ is a Boolean algebra. This implies a new characterization theorem of Boolean algebras.

2 Preliminaries

Let $\mathcal{L} = (L, \wedge, \vee, ', 0, 1)$ be a bounded lattice with a unary operator $'$. A unary operator $' : L \to L$ is called an *orthocomplementation* when it satisfies, for all $x, y \in L$,

1. $x \wedge x' = 0$;
2. $x \vee x' = 1$;
3. $x \leq y \Longrightarrow y' \leq x'$;
4. $x'' = x$.

By an orthcomplemented lattice, we mean the bounded lattices with an orthocomplementation. If an orthocomplemented lattice $\mathcal{L}$ satisfies the following conditions

(D1) $(x \wedge y)' = x' \vee y', \ (x \vee y)' = x' \wedge y'$
(D2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \ x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

then it is called a *de Morgan algebra.* An orthocomplemented lattice $\mathcal{L}$ is called an *orthomodular lattice* when it satisfies

(D1) $(x \wedge y)' = x' \vee y', \ (x \vee y)' = x' \wedge y'$
(OM) $x \leq y \Longrightarrow y = x \vee (x' \wedge y)$ (Orthomodular law)

It is proved in [3, 4] that

1. The only de Morgan algebras in which the law (E) $(x \wedge y')' = y \vee (x' \wedge y')$ holds are those that are Boolean algebras.
2. Every orthomodular lattice satisfying (E) is a Boolean algebra.
3. Every orthocomplemented lattice with (E) is a Boolean algebra.

3 Stronger result

We give a stronger result than [3, 4] in this section. Let $\mathcal{L} = (L; \wedge, \vee, 0, 1)$ be a bounded lattice with a unary operator $'$ satisfying (E) $(x \wedge y')' = y \vee (x' \wedge y')$. It is easy to prove the following results.

Proposition 1. *For all $x, y \in L$, we have*

1. $0' = 1$
2. $1'' = 1$
3. $x \vee x' = 1$
4. $x \leq x''$
5. $x \leq y'$ *if and only if* $y \leq x'$
6. *if* $x \leq y$ *then* $y' \leq x'$
7. $x' = x'''$
8. $(x \vee y)' = x' \wedge y'$

Theorem 1. *Let $\mathcal{L} = (L; \wedge, \vee, 0, 1)$ be a bounded lattice. If a unary operator $'$ satisfies only the following two conditions, then L is a Boolean algebra : For all $x, y \in L$,*

(c1) $1' = 0$
(E) $(x \wedge y')' = y \vee (x' \wedge y')$

Thus, the class of all bounded lattices with (c1) and (E) coincides with the class of Boolean algebras.

Proof. In order to prove our statement, it is enough to verify that the unary operator $'$ is an orthocomplementation. That is, we have to prove $x \wedge x' = 0$ and $x'' = x$. At first we show that $x'' = x$. In the equation (E), if we take $x = 1$ and $y = x$ simultaneously, then we have

$$(1 \wedge x')' = x \vee (1' \wedge x').$$

Since $1 \wedge x' = x'$ and $1' \wedge x' = 0 \wedge x' = 0$ by (c1), it follows that

$$x'' = (1 \wedge x')' = x \vee (1' \wedge x') = x \vee 0 = x.$$

We note from the above that

$$0' = 1'' = 1.$$

Since $1 = 0' = (0 \wedge x')' = x \vee (0' \wedge x') = x \vee x'$, if we take $y = x$ in (E), then $(x \wedge x')' = x \vee (x' \wedge x') = x \vee x' = 1$. It follows that

$$x \wedge x' = (x \wedge x')'' = 1' = 0.$$

□

As to the condition (c1) we can show the next result.

Proposition 2. *Let L be a bounded lattice with (E). Then the following statements are equivalent:*

1. $1' = 0$
2. $x = x''$ *for all* $x \in L$
3. $x \wedge x' = 0$ *for all* $x \in L$
4. *if* $x' \leq y$, *then* $y' \leq x$ *for all* $x, y \in L$

Proof. 1. $\Longrightarrow$ 2. : Suppose that $1' = 0$. For every element $x \in L$, we have

$$x'' = (1 \wedge x')' = x \vee (1' \wedge x') = x \vee (0 \wedge x') = x \vee 0 = x.$$

2. $\Longrightarrow$ 3. : We suppose that $x = x''$ for every $x \in L$. We note that $0' = 1$ by (E). Indeed, if we take $x = 0$ and $y = x$ in the equation (E) then for all $y \in L$ we have

$$0' = (0 \wedge y')' = y \vee (0' \wedge y') \geq y.$$

This means that $0' = 1$ and $1' = 0'' = 0$. It follows from the above that $1 = 0' = (0 \wedge x')' = x \vee (0' \wedge x') = x \vee (1 \wedge x') = x \vee x'$. Thus we have

$$x \wedge x' = (x \wedge x')'' = (x \vee (x' \wedge x'))' = (x \vee x')' = 1' = 0.$$

3. $\Longrightarrow$ 4. : We assume that $x \wedge x' = 0$ for every $x \in L$ and $x' \leq y$. It follows from the assumption that $1' = 0$. Then for all $x \in L$, since $x'' = (1 \wedge x')' = x \vee (1' \wedge x') = x \vee (0 \wedge x') = x \vee 0 = x$ and $x' \wedge y' = 0$ by $x' \leq y$, we have

$$x \wedge y' = (x \wedge y')'' = (y \vee (x' \wedge y'))' = (y \vee 0)' = y'.$$

This means that $y' \leq x$.

4. $\Longrightarrow$ 1. : Assume that if $x' \leq y$ then $y' \leq x$ for all $x, y \in L$. Since $x' \leq 1$ for every $x \in L$, we have $1' \leq x$ by the assumption. Thus we have $1' = 0$.

□

4 Substructure L^*

For any lattice $\mathcal{L} = (L; \wedge, \vee)$ with a unary operator $'$ and the Elkan's formula (E), we put $L^* = \{x'' \,|\, x \in L\}$. Since L^* is a subset of L, it has a partial order $\leq^*$ which is a restriction of $\leq$ on L^*. Thus, for all $x, y \in L^*$ we have

$$x \leq^* y \iff x \leq y \text{ in } L$$

As to the order $\leq^*$, we have a following result.

Proposition 3. *$\mathcal{L}^* = (L^*, \leq^*)$ is also a lattice with a unary operator $'$, that is, for all $x, y \in L^*$*

$$\inf_{L^*}\{x, y\} = x \wedge y$$
$$\sup_{L^*}\{x, y\} = (x \vee y)''$$

Proof. We only show the case of "sup". It is obvious that $x, y \leq x \vee y \leq (x \vee y)''$. For every $u \in L^*$ such that $x, y \leq^* u$, since $x, y \leq u$ and $x \vee y \leq u$, we have $(x \vee y)'' \leq^* u'' = u$. This means that $\sup_{L^*}\{x, y\} = (x \vee y)''$. □

We use symbols $\sqcap$ and $\sqcup$ for "inf" and "sup" in L^* respectively to avoid confusions: For $x, y \in L^*$,

$$x \sqcap y = \inf_{L^*}\{x, y\}$$
$$x \sqcup y = \sup_{L^*}\{x, y\}$$

We can show that the lattice L^* satisfies the *orthomodular law* : if $a \leq^* b$ then $b = a \sqcup (a' \sqcap b)$.

Proposition 4. *The lattice L^* satisfies the orthomodular law : if $a \leq^* b$ then $b = a \sqcup (a' \sqcap b)$.*

Proof. Suppose that $a \leq^* b$ for $a, b \in L^*$. Since $b = b''$ and $b' \leq^* a'$, we have

$$\begin{aligned} b = b'' &= (b' \wedge a')' \\ &= \{(b' \wedge a')'\}'' = \{a \vee (b'' \wedge a')\}'' \\ &= \{a \vee (b \wedge a')\}'' = a \sqcup (b \wedge a') \\ &= a \sqcup (a' \sqcap b). \end{aligned}$$

Thus, the lattice L^* satisfies the orthomodular law. □

Next we consider a case of L having a greatest element 1. We see that $1'$ is the least element in L^*. Moreover we have the following result.

Proposition 5. *Let L be a lattice with a unary operator $'$ satisfying the Elkan's formula (E). If there is a greatest element 1 in L, then we have for all $x \in L$*

(1) $1 = 1''$;
(2) $x \vee x' = 1$;
(3) $x' \wedge x'' = 1'$.

Proof. Suppose that L is a lattice with a unary operator $'$ satisfying (E) and a greatest element 1.

(1) If we take $x = y = 1$ in (E), then we have

$$1'' = (1 \wedge 1')' = 1 \vee (1' \wedge 1') \geq 1.$$

(2) Take $x = 1', y = x$. We get that

$$1 = 1'' = (1' \wedge x')' = x \vee (1'' \wedge x') = x \vee (1 \wedge x') = x \vee x'.$$

(3) It is trivial that $x' \wedge x'' = (x \vee x')' = 1'$. □

For any lattice L with meeting the orthomodular law, an element $x \in L$ is said to be *commutative* with y (denoted by xCy) if $x = (x \wedge y) \vee (x \wedge y')$. As to the commutativity, it is obvious that (1) if $x \leq y$ then xCy and (2) if xCy then xCy'.

A set $\{x, y, z\}$ of elements of L is called *distributive triple* if xCy and xCz. It is well-known ([1]) that if $\{x, y, z\}$ is distributive triple then the sublattice generated by $\{x, y, z\}$ is a ditributive lattice (Foulis-Holland theorem), thus the following holds :

$$\begin{aligned}
x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \\
y \wedge (z \vee x) &= (y \wedge z) \vee (y \wedge x) \\
z \wedge (x \vee y) &= (z \wedge x) \vee (z \wedge y) \\
(x \wedge y) \vee z &= (x \vee z) \wedge (y \vee z) \\
(y \wedge z) \vee x &= (y \vee x) \wedge (z \vee x) \\
(z \wedge x) \vee y &= (z \vee y) \wedge (x \vee y).
\end{aligned}$$

From the fact above, we can conclude that

Lemma 1. *For any $a, b \in L^*$, we have aCb, that is,*

$$a = (a \sqcap b) \sqcup (a \sqcap b').$$

Hence L^* is a distributive lattice.

Proof. Since the orthomodular law holds in the lattice L^* and $a \sqcap b' \leq^* a$, we have

$$\begin{aligned} a &= (a \sqcap b') \sqcup \{(a \sqcap b')' \sqcup a\} \\ &= (a \sqcap b') \sqcup [\{b \sqcup (a' \sqcap b')\} \sqcup a]. \end{aligned}$$

The fact that $a' \sqcap b' \leq^* a', b'$ implies that $(a' \sqcap b')Ca'$ and $(a' \sqcap b')Cb'$. Hence we get $(a' \sqcap b')Ca$ and $(a' \sqcap b')Cb$ by $a = a''$ and $b'' = b$ for $a, b \in L^*$. This means that a set $\{a,\ b,\ a' \sqcap b'\}$ is distributive triple. From the Foulis-Holland theorem, it follows that

$$\begin{aligned} a \sqcap \{b \sqcup (a' \sqcap b')\} &= (a \sqcap b) \sqcup (a \sqcap a' \sqcap b') \\ &= (a \sqcap b) \sqcup (1' \sqcap b) \\ &= (a \sqcap b) \sqcup 1' \\ &= a \sqcap b, \end{aligned}$$

because $1'$ is the least element in L^*. This implies that

$$a = (a \sqcap b') \sqcup (a \sqcap b).$$

That is, for all elements $a, b \in L^*$, we have aCb. Thus L^* is a distributive lattice. □

Moreover, we see that each element $a \in L^*$ has a complement $a' \in L^*$ from the proposition above.

Theorem 2. *If a lattice L has a greatest element* 1 *and satisfies the Elkan's formula, then the lattice $\mathcal{L}^* = (L^*, \sqcap, \sqcup, ', 1', 1)$ is a maximal Boolean algebra included in L.*

Proof. We only show its maximality. If $L^* \subset B$ for some Boolean algebra included in L, then there is an element $x \in B$ such that $x \notin L^*$. Since B is the Boolean algebra, we have $x = x''$ and thus $x \in L^*$. This is a contradiction. Hence L^* is the maximal Boolean algebra included in L. □

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