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<tbody>
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Abstract. A subring $R$ of a division ring $D$ is said to be an invariant valuation ring if, for any non-zero element $d$ of $D$, we have $d \in R$ or $d^{-1} \in R$, and $dRd^{-1} = R$. An $R$-submodule $N$ of a left $R$-module $M$ is said to be relatively divisible (an RD-module for short) if $aN = N \cap aM$ for any $a \in M$. Every finitely generated left $R$-module $M$ has an RD-composition series with non-decreasing sequence of annihilators. Any two RD-composition series of $M$ is isomorphic and the length of RD-composition series of $M$ is equal to the number of minimal generators of $M$.  

1 Non-commutative valuation rings

Finitely generated modules over commutative valuation rings have been greatly investigated from 1980's (see [FS1], [SZ], [Z]). In this note, we report some results about finitely generated modules over non-commutative valuation rings.

At first, we introduce some non-commutative valuation rings. We refer to [MMU] for details about non-commutative valuation rings.

Let $Q$ be a simple Artinian ring and let $R$ be an order in $Q$, that is, $R$ is a subring of $Q$ which satisfies the following conditions;

1. any non zero-divisor of $R$ has its inverse in $Q$, and
2. for any element $q$ of $Q$, there exist $a, b, c, d \in R$ with $b, d$ non zero-divisor, such that $q = ab^{-1} = d^{-1}c$.

An order $R$ in a simple Artinian ring $Q$ is called a Dubrovin valuation ring if $R$ is a local Bezout order, that is, if every finitely generated one-sided ideal of $R$ is principal and $R/J(R)$ is simple Artinian, where $J(R)$ is the Jacobson radical of $R$. There is some characterization of Dubrovin valuation rings (see [MMU, Theorem 5.11]).

1This is an abstract and the paper will appear elsewhere.
A total valuation ring is an order $R$ in a division ring $D$ which satisfies the following condition;

(T) for any non-zero element $d \in D$, we have $d \in R$ or $d^{-1} \in R$.

If an order $R$ satisfies the condition (T) and the following condition (I), $R$ is called an invariant valuation ring;

(I) for any non-zero element $d$, $dRd^{-1} = R$.

It is clear that an invariant valuation ring is a total valuation ring, and a total valuation ring is a Dubrovin valuation ring (see [MMU, Theorem 5.11]).

Conversely, if a total valuation ring $R$ is integral over its center, then $R$ is an invariant valuation ring (see [MMU, Corollary 8.6]), and a Dubrovin valuation ring $R$ is a total valuation ring if $R/J(R)$ is a division ring (see [MMU, Lemma 8.13]).

## 2 Modules over non-commutative valuation rings

Throughout this section, let $R$ be an invariant valuation ring in a division ring $D$, and we consider finitely generated modules over $R$.

Let $M$ be a left $R$-module. An $R$-submodule $N$ of $M$ is said to be relatively divisible (RD-submodule for short) if, for any element $a \in R$, we have $aN = N \cap aM$.

Then we have following theorem:

**Theorem 2.1** Let $R$ be an invariant valuation ring and let $M$ be a finitely generated left $R$-module. Then there exists a sequence

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

of $R$-submodules of $M$ such that

1. each $M_i$ is an RD-submodule of $M$, and
2. $M_i/M_{i-1}$ is cyclic ($i = 1, 2, \cdots, n$).

The sequence in Theorem 2.1 is called an RD-composition series of $M$. Two RD-composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ and $0 = N_0 \subset N_1 \subset \cdots \subset N_k = M$ of $M$ are said to be isomorphic if $n = k$ and there is some permutation $\sigma$ of the number $0, 1, \cdots, n - 1$ such that $M_i/M_{i-1} \cong N_{\sigma(i)}/N_{\sigma(i)-1}$ ($i = 1, 2, \cdots, n$).
For an RD-composition series \( 0 = M_0 \subset M_1 \subset \cdots \subset M_n = M \) of \( M \), we set \( A_i \) to be the annihilator of \( M_i/M_{i-1} \), that is,

\[
A_i = \text{Ann}_R(M_i/M_{i-1}) = \{a \in R \mid a(M_i/M_{i-1}) = 0\}.
\]

If \( A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \), then we say that the annihilator sequence \( A_1, A_2, \cdots, A_n \) is non-decreasing. Then

**Theorem 2.2** For any RD-composition series of a finitely generated left \( R \)-module \( M \), there exists an isomorphic RD-composition series of \( M \) with non-decreasing annihilator sequence.

In some particular case, \( M \) is a direct sum of cyclic modules:

**Theorem 2.3** Let \( 0 = M_0 \subset M_1 \subset \cdots \subset M_n = M \) be an RD-composition series. If there is some \( k \) (\( \leq n \)) such that

\[
\text{Ann}_R(M_1) = \text{Ann}_R(M_2/M_1) = \cdots = \text{Ann}_R(M_k/M_{k-1}),
\]

then \( M_k \) is a direct sum of cyclic \( R \)-modules. In particular, if all annihilators are equal, then \( M \) is a direct sum of cyclic \( R \)-modules.

Concerning the length of RD-composition series, we have the following:

**Theorem 2.4** The length \( l(M) \) of an RD-composition series of \( M \) is equal to the number of minimal generators of \( M \).

We don't know about the relation between the length \( l(M) \) of a RD-composition series of \( M \) and the Goldie dimension \( g(M) \) of \( M \). But, in commutative case, it is proved that \( g(M) \leq l(M) \) in general, and that \( l(M) = g(M) \) if \( M \) is a direct sum of cyclic modules (see [SZ]).

We note that, about modules over total valuation rings or Dubrovin valuation rings, nothing is known yet.
References


