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Author(s)	Nakanishi, Kenji
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Scattering theory for the Gross-Pitaevskii equation

京都大学 大学院理学研究科 中西賢次
Graduate School of Science, Kyoto University Kenji Nakanishi

ABSTRACT

The Gross-Pitaevskii equation is algebraically equivalent to the defocusing cubic nonlinear Schrödinger equation, but the natural solutions should approach non-zero equillibria at the spatial infinity. We study large-time behavior of such solutions in the simplest case, i.e., for small perturbations of space-independent solutions. In three or higher dimensions, we see that we need only a linear modification for the free Schrödinger equation to approximate the asymptotic behavior, whereas in two dimensions, we need some quadratic modifications also. This article is based on the joint work with Stephen Gustafson and Tai-Peng Tsai [8, 9].

1. INTRODUCTION

There has been a large amount of study on long-time behavior of solutions for the nonlinear Schrödinger equation (NLS) and similar ones in terms of the scattering theory. The typical statement is that each solution under some conditions can be approximated at the time infinity by a sum of bound states solving nonlinear elliptic equations and a dispersive component evolving by the linear equation. Such a description relies crucially on the fact that the nonlinear interaction becomes weaker for the dispersive component both with itself and with the bound states for large time. To derive time decay of those interactions, the spatial decay of each component has played dominant roles.

However, it is not always natural in the physical context to assume spatial decay of the solutions. A typical example is the so-called Gross-Pitaevskii equation (GP) modelling the Bose-Einstein condensation, or superfluidity

$$i\psi_t + \Delta\psi = (|\psi|^2 - 1)\psi, \quad \psi(t,x) : \mathbb{R}^{1+d} \to \mathbb{C}. \tag{1.1}$$

This is equivalent to the defocusing cubic NLS by the change of variable $\psi \mapsto e^{it}\psi$. What makes it different from the usual NLS is the boundary condition given by

$$|\psi(t,x)| \to 1 \quad (|x| \to \infty). \tag{1.2}$$

Hence those scattering results in L^2 or any Sobolev space H^s for the NLS do not apply in this context. In fact, the long-time behavior of solutions is generally quite different between them; it is well known [1, 4] that there exist finite energy traveling waves for (GP) of the form

$$\psi(t,x) = \varphi(x - ct), \quad \lim_{|x|\to\infty} \varphi(x) = 1, \tag{1.3}$$

whereas every H^1 solution of the same NLS disperses and approach a free solution, at least in three or higher dimensions [6]. Heuristically the dynamics of (GP) is more complicated and difficult to analyse, because the interaction with the non-zero back ground does not decay at the spatial infinity. A consequence of it appears in the decay of fininte energy traveling waves [7]:

$$|\varphi(x)| \gtrsim |x|^{1-d} \quad (|x| \gg 1), \tag{1.4}$$

which is in a striking contrast with the exponential decay of solitary waves for the focusing NLS. We will see a similar phenomenon for the dispersive component of (GP) in the two dimensional case.

Before going to the scattering problem, it is necessary to recall the global existence for (GP). It was shown in [1] that the equation (1.1) is globally wellposed in the class $\psi \in 1 + H^1_x$ for $d \le 3$. The H^1 norm is related to the conserved quantities

$$(1.5) \qquad E(\psi) = \int_{\mathbb{R}^d} |\nabla\psi|^2 + \frac{(|\psi|^2-1)^2}{2} dx, \quad Q(\psi) = \int_{\mathbb{R}^d} (|\psi|^2 - 1) dx,$$

which however do not control the L^2 norm. Actually the spatial asymptotic (1.4) implies that finite energy traveling waves do not belong to $L^2(\mathbb{R}^2)$. Thus [5] extended the global wellposedness to the natural class of finite energy defined by

$$(1.6) \qquad \{\varphi \in \dot H^1 \cap L^2_{loc} \mid |\varphi|^2 - 1 \in L^2.\},$$

which is equivalent to $1 + H^1$ for $d = 3, 4$, but not for $d = 2$. In the recent paper [3], the above result was further extened to include the stationary vortex solutions

$$(1.7) \qquad \psi(t, x_1, x_2) = \varphi(r)e^{im\theta}, \quad x_1 + ix_2 = re^{i\theta}, \quad m \in \mathbb{Z} \setminus \{0\},$$

which have infinite energy due to the phase gradient. These results use conservation laws to extend the solutions globally, without specifying the asymptotic behavior at the time infinity, on which our knowledge is very limited so far.

A natural step toward understanding the asymptotic behavior is to investigate the dispersive property of small solutions, namely the case where $|\psi - 1|$ is small enough with decay at the spatial infinity. In terms of the standard NLS, this is equivalent to investigating small perturbation of non-zero plane wave solutions

$$(1.8) \qquad i\dot u + \Delta u = |u|^2 u, \quad u = \sqrt{\omega - |\xi|^2} e^{i\xi x} e^{-i\omega t} + \text{"small"},$$

which by itself seems to be interesting. The main issue is how to control the lower order interactions with the non-zero constant amplitude. We will see in the two dimensional case that the quadratic interaction has nontrivial long-time effect on the dispersive component, besides from the obvious linear interaction.

Now let us formulate the equation for the dispersive component. Putting $\psi = 1+u$ in (1.1), we get

$$(1.9) \qquad iu_t + \Delta u + 2\Re u = u^2 + 2|u|^2 + |u|^2 u,$$

where $2\Re$ is the linear interaction with the background 1. We can linearize (in the complex sense) the left hand side by change of variable $u \mapsto v$ defined by the Fourier multiplier $U := \sqrt{-\Delta(2-\Delta)^{-1}}$:

$$(1.10) \qquad u = u_1 + iu_2 = Uv_1 + iv_2, \quad v = v_1 + iv_2,$$

where $u = u_1 + iu_2$ denotes the decomposition into the real and imaginary parts. Then the equation for v is given

$$(1.11) \qquad iv_t - Hv = 3u_1^2 + u_2^2 + |u|^2 u_1 + iU^{-1}(2u_1u_2 + |u|^2 u_2),$$

where $H := \sqrt{-\Delta(2-\Delta)}$. Then we may ask if the solution v can be approximated by the unitary evolution group e^{-itH} for large time:

$$(1.12) \qquad \|v - e^{-itH}\varphi\|_{H^s} \to 0 \quad (t \to \infty)$$

for some final state φ and some Sobolev space H^s. This means that the background interaction remains effective only for the linear order. We will see that it is the case for all small solutions in four or higher dimensions, and for a class of solutions in three dimensions, but not completely correct in two dimensions.

An apparent obstruction in deriving such results is the singularity of U^{-1} at the Fourier origin in the nonlinearity (1.11), since singularity in the Fourier space corresponds to slow decay in the physical space. The more essential difficulty is estimating those quadratic terms especially in two dimensions.

For a comparison, let us mention the known results for the quadratic NLS:

$$iu_t + \Delta u = B(u, \overline{u}), \quad u : \mathbb{R}^{1+2} \to \mathbb{C}. \tag{1.13}$$

If $B = \lambda_1 u^2 + \lambda_2 \overline{u}^2$, then it is known [14, 12] that for every small and rapidly decaying final state φ with vanishing moments, there exists a nonlinear solution u which approach the free solution $e^{it\Delta}\varphi$ (i.e., the wave operator can be defined for such final data). If $B = \overline{\lambda}[\Re(\lambda u)]^2$, then there exists a solution u with the modified asymptotic profile [11]:

$$u \sim u^0 + \frac{i\overline{\lambda}}{2} \int_\infty^t |\lambda u^0(s)|^2 ds, \quad u^0 := e^{it\Delta}\varphi. \tag{1.14}$$

But the general case including $|u|^2$ remains open (see [15] for nonexistence of the wave operator). The difficulty is that the quadratic terms have the critical time decay if approximated by the free solution:

$$\|u(t)^2\|_{L^2_x(\mathbb{R}^2)} \gtrsim 1/t \notin L^1(1, \infty), \tag{1.15}$$

and therefore the modification is very sensitive to the form of nonlinearity.

Now we state the main results. In four or higher dimensions, we have [8]

Theorem 1.1. *Let $d \geq 4$ and $s \geq d/2 - 1$. There exists $\delta > 0$ such that for any $\varphi \in H^s(\mathbb{R}^d)$ satisfying $\|\varphi\|_{H^s} \leq \delta$, the unique global solution ψ of* (1.1) *with $\psi(0) = 1 + \varphi$ satisfies*

$$\psi = 1 + Uv_1 + iv_2, \quad \|v(t) - e^{-iHt}\varphi_+\|_{H^s} \to 0 \ (t \to \infty), \tag{1.16}$$

for some $\varphi_+ \in H^s(\mathbb{R}^d)$. Conversely for any $\varphi_+ \in H^s(\mathbb{R}^d)$, there is a global solution ψ satisfying the above asymptotic behavior. Moreover, the correspondence $\varphi \mapsto \varphi_+$ defines a local homeomorphism around 0 *in $H^s(\mathbb{R}^d)$.*

The regularity $s = d/2 - 1$ is the scaling critical exponent for the cubic NLS, while the L^2 is that for the quadratic NLS in $d = 4$, where the scaling critical means that the space $\dot{H}^s$ is invariant under the scaling $\varphi(x) \mapsto \lambda^\alpha \varphi(\lambda x)$ which leaves the equation invariant. Therefore the above seems to be optimal as a scattering result in H^s by the current technology of perturbative arguments. However the existence of traveling waves does not exclude the possibility of this kind of result for $d = 2, 3$, because in three dimensions there seems to be a lower bound on the energy of traveling waves, and in two dimensions they are not in L^2.

In three dimensions, we can define a wave operator in smaller spaces [9]

Theorem 1.2. *Let $d = 3$ and $q < 3/2$. Then for any $\varphi \in H^1 \cap W^{1,q}(\mathbb{R}^3)$, there exists a unique global solution ψ of (1.1) satisfying*

$$\begin{aligned} \psi = 1 + Uv_1 + iv_2, \quad & \|v(t) - e^{-iHt}\varphi\|_{(L_t^\infty H_x^1 \cap L^2 W_x^{1,6})(T,\infty)} = o(T^{-1/4}), \\ & \|v(t)\|_{W^{1,3}} = O(t^{-1/2}). \end{aligned} \tag{1.17}$$

The same result holds true for small $\varphi \in H^1 \cap W^{1,3/2}$. The spaces $W^{1,3/2}$ and $W^{1,3}$ are related to the L^p-L^q decay of the linearized operator:

$$\|e^{-itH}\varphi\|_{B^0_{3,2}} \lesssim |t|^{-1/2}\|\varphi\|_{B^0_{3/2,2}}, \tag{1.18}$$

where $B^0_{p,2}$ denotes the Besov space with the L^p_x norm on each dyadic frequency and the ℓ^2 on the dyadic parameter. The criticality of this estimate for quadratic terms can be observed by applying it to the Duhamel formula for the nonlinear term:

$$t^{1/2}\|\int_\infty^t e^{-i(t-s)H}u(s)^2 ds\|_{L^3_x} \lesssim \int_t^\infty (s/t-1)^{-1/2}\frac{ds}{s}\ \|t^{1/2}u(t)\|^2_{L_t^\infty L_x^3}. \tag{1.19}$$

In the hardest case $d = 2$, we have the following modified asymptotics [9]. We denote by $\mathcal{F}$ the Fourier transform on $\mathbb{R}^d$.

Theorem 1.3. *Let $d = 2$. There exists $\delta > 0$ such that for any $\varphi \in H^1$ satisfying $\|\varphi\|_{\dot{B}^1_{1,1}} \le \delta$ and $\langle\xi\rangle^{-1/2}|\xi|^{|\alpha|}\partial^\alpha \mathcal{F}\varphi(\xi) \in L^\infty_\xi \cap L^2_\xi$ for $|\alpha| \le 2$, there exists a unique global solution ψ of (1.1) satisfying*

$$\begin{aligned} &\psi = 1 + Uv_1 + iv_2, \quad \|v + \nu - z^0 - z^1\|_{H^1} \lesssim t^{-1+\varepsilon}, \\ &\nu := H^{-1}|\psi - 1|^2, \quad z^0 := e^{-iHt}\varphi, \quad z^1 := i\int_\infty^t e^{-iH(t-s)}|Uz^0(s)|^2 ds, \end{aligned} \tag{1.20}$$

for any $\varepsilon > 0$.

Those quadratic modifiers have decay

$$\|\nu\|_{\dot{H}^1 \cap \dot{H}^2} + \|z^1\|_{\dot{H}^1} \lesssim t^{-1+\varepsilon}, \quad \|\nu\|_{\dot{H}^\varepsilon} + \|z^1\|_{\dot{H}^\varepsilon} \lesssim t^{-\varepsilon/2}, \tag{1.21}$$

for $0 < \varepsilon < 1$. However, they do not belong to L^2 in general. It is easy to see that $\nu \notin L^2_x$ (unless $\psi = 1$) due to the singularity of H^{-1} at $\xi = 0$. Moreover, we have the following asymptotic of z^1 in the Fourier space. Let $\xi_1 + i\xi_2 = re^{i\theta}$. Then we have

$$\lim_{r \to +0} r\mathcal{F}[e^{iHt}z^1(t)](\xi) = i\int_0^\infty \int_{\mathbb{R}^2} e^{i(\sqrt{2} - \nabla H(\eta)\cdot\theta)t}|\mathcal{F}\varphi(\eta)|^2 d\eta ds. \tag{1.22}$$

Since $1/|\xi| \notin L^2(\mathbb{R}^2)$, we deduce that $u_2(t) \notin L^2_x$ unless the right hand side vanishes for all $\theta \in \mathbb{R}$. The modifier z^1 is essentially the same as that in (1.14), but the latter was simplified by using $e^{it\Delta} \sim 1 + it\Delta$, which is not useful in the case of e^{-itH}. Both the results exploit special structure of the nonlinearity. The argument in [11] crucially depends on the fact that the modifier is completely killed in the nonlinearity because of the special choice of coefficients. In our argument, we exploit the fact that the modifier has singularity only at $\xi = 0$, which is compensated by U in the nonlinearity after a certain change of variable, which we will detail below.

The key ingredient of our proof is the following nonlinear transform of the solution, which resolves both the difficulties, the U^{-1} singularity and the slow decay of the quadratic terms in two dimensions. Let $z = v + H^{-1}|u|^2$. Then we have

$$\begin{aligned} &iz_t - Hz = 2u_1^2 - 4iH^{-1}\nabla \cdot (u_1 \nabla u_2) + |u|^2 u_1 + iU[|u|^2 u_2], \\ &u_1 + (2-\Delta)^{-1}|u|^2 = Uz_1, \quad u_2 = z_2. \end{aligned} \tag{1.23}$$

Hence the U^{-1} singularity has disappeared and moreover the quadratic terms are roughly of the form $(Uz)^2$ and the cubic terms are like $z^2 Uz$. Thus the above transform can be regarded as a "partial" normal form removing the most singular part around $\xi = 0$; similar arguments have been successfully used for the NLS, see for example [10].

For the linear evolution e^{-iHt}, we have the following L^p decay estimate by the stationary phase argument [8]:

$$\|e^{-iHt}\varphi\|_{\dot{B}^0_{p,2}} \lesssim |t|^{-d(1/2-1/p)} \|U^{(d-2)(1/2-1/p)}\varphi\|_{\dot{B}^0_{p',2}}, \tag{1.24}$$

for $2 \le p \le \infty$ and $1/p + 1/p' = 1$. Thus we gain some power of U if $d \ge 3$, compared with the free Schrödinger evolution. Then the Strichartz estimate with some gain at $\xi = 0$ follows from the above one by a standard argument, and the above results in three or higher dimensions are obtained by using those linear estimates together with the Hölder and the Sobolev inequalities on the nonlinear terms.

In the two dimensional case, the Hölder with the linear decay estimate is not sufficient and we have to exploit the oscillatory property of the quadratic terms for dispersive solutions. The key ingredient is the following decay estimate on the first approximation of the quadratic terms:

$$\left\| \int_\infty^t e^{iHs} B(Uz_1^0 + z_2^0)ds \right\|_{\dot{H}^1} = O(t^{-1}\log^2 t), \tag{1.25}$$

where $B(u)$ denotes the quadratic terms in the equation for z (1.23). We have the same bound in $\dot{H}^s$ for $0 < s < 1$ if we subtract the term $|Uz^0|^2$. The above estimate is proved by a non-stationary phase argument in space-time (t, ξ) away from $\xi = 0$. The U gain in the quadratic terms is used to remove the stationary point $\xi = 0$ and also to compensate the singularity in the derivatives of H appearing in partial integrations. Once we obtain the above estimate, it is easy to solve the equation (1.23) for (z, u) by the iteration argument.

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