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The $L^p$ boundedness of wave operators for Schrödinger operators

Kenji Yajima
Department of Mathematics, Gakushuin University
1-5-1 Mejiro, Toshima-ku, Tokyo 171-8588, Japan

1 Introduction

Let $H = -\Delta + V$ be the Schrödinger operator on $\mathbb{R}^m$, $m \geq 1$, with real valued potential $V(x)$ such that $|V(x)| \leq C\langle x \rangle^{-\delta}$ for some $\delta > 2$, where $\langle x \rangle = (1 + x^2)^{1/2}$. Then, it is well known that

1. $H$ is selfadjoint in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^m)$ with domain $D(H) = H^2(\mathbb{R}^m)$ and $C_0^\infty(\mathbb{R}^m)$ is a core;

2. the spectrum $\sigma(H)$ of $H$ consists of an absolutely continuous part $[0, \infty)$, and at most a finite number of non-positive eigenvalues $\{\lambda_j\}$ of finite multiplicities;

3. the singular continuous spectrum and positive eigenvalues are absent from $\sigma(H)$.

We denote the point and the absolutely continuous spectral subspaces of $\mathcal{H}$ for $H$ by $\mathcal{H}_p$ and $\mathcal{H}_{ac}$ respectively, and the orthogonal projections in $\mathcal{H}$ onto the respective subspaces by $P_p$ and $P_{ac}$. We write $H_0 = -\Delta$ for the free Schrödinger operator.

4. The wave operators $W_\pm$ defined by the following limits in $\mathcal{H}$:
   $$W_\pm = \overset{s}{\lim}_{t \to \pm \infty} e^{itH} e^{-itH_0}$$

   exist and are complete in the sense that $\text{Image } W_\pm = \mathcal{H}_{ac}$.

5. $W_\pm$ satisfy the so called intertwining property and the absolutely continuous part of $H$ is unitarily equivalent to $H_0$ via $W_\pm$: For Borel functions $f$ on $\mathbb{R}$, we have
   $$f(H)P_{ac}(H) = W_\pm f(H_0)W_\pm^*.$$  (1.1)
It follows from the intertwining property (1.1) that, if $X$ and $Y$ are Banach spaces such that $L^2(\mathbb{R}^m) \cap X$ and $L^2(\mathbb{R}^m) \cap Y$ are dense in $X$ and $Y$ respectively, then,

$$
\| f(H)P_{ac}(H) \|_{B(X,Y)}
\leq \| W_\pm \|_{B(Y)} \| f(H_0) \|_{B(X,Y)} \| W_\pm^* \|_{B(X)} = C \| f(H_0) \|_{B(X,Y)}.
$$

(1.2)

Here it is important that the constant $C = \| W_\pm \|_{B(Y)} \| W_\pm^* \|_{B(X)}$ is independent of the function $f$. Thus, the mapping property of $f(H)P_{ac}(H)$ from $X$ to $Y$ may be deduced from that of $f(H_0)$, once we know that $W_\pm$ are bounded in $X$ and in $Y$. Note that the solutions $u(t)$ of the Cauchy problem for the Schrödinger equation

$$
\quad i\partial_t u = (-\Delta + V)u, \quad u(0) = \varphi
$$

and $v(t)$ of the wave equation

$$
\quad \partial_t^2 v = (\Delta - V)v, \quad v(0) = \varphi, \quad \partial_t v(0) = \psi
$$

are given in terms of the functions of $H$, respectively by

$$
\quad u(t) = e^{-itH} \varphi, \quad \text{and} \quad v(t) = \cos(t\sqrt{H}) \varphi + \frac{\sin(t\sqrt{H})}{\sqrt{H}} \psi.
$$

It follows that, if $W_\pm$ are bounded in Lebesgue spaces $L^p(\mathbb{R}^m)$ for $1 \leq p \leq \infty$ and if the initial states $\varphi$ and $\psi$ belong to the continuos spectral subspace $\mathcal{H}_{c}(H)$, then the $L^p$-$L^q$ estimates for the propagators of the respective equations may be deduced from the well known $L^p$-$L^q$ estimates for the free propagators $e^{-itH_0}$ or $\cos(t\sqrt{H_0})$ and $\sin(t\sqrt{H_0})/\sqrt{H_0}$ (if $\varphi$ and $\psi$ are eigenfunctions of $H$, the behavior of $u(t)$ and $v(t)$ are trivial). In particular, we have the so called dispersive estimates for the Schrödinger equation

$$
\| e^{-itH} P_{c}(H) \varphi \|_{\infty} \leq C |t|^{-\frac{m}{2}} \| \varphi \|_{1}.
$$

In this lecture we would like to briefly survey the current status of the study of the mapping property of $W_\pm$ in Lebesgue spaces $L^p(\mathbb{R}^m)$. We say that $0$ is a resonance of $H$, if there is a solution $\varphi$ of $(-\Delta + V(x))\varphi(x) = 0$ such that $|\varphi(x)| \leq C(x)^{2-m}$ but $\varphi \not\in \mathcal{H}$ and call such a solution $\varphi(x)$ a resonance function of $H$; $H$ is of generic type, if $0$ is neither an eigenvalue nor a resonance of $H$, otherwise of exceptional type. Note that there is no zero resonance if $m \geq 5$. We shall see that the mapping property of $W_\pm$ in $L^p(\mathbb{R}^m)$ spaces is fairly well understood when $H$ is of generic type although the conditions on potentials for the $L^p$-boundedness of $W_\pm$ are far
from optimal and the end point problem, viz. the problem for the case $p = 1$ and $p = \infty$ is not settled completely in the cases $m = 1$ and $m = 2$. On the other hand, if $H$ is of exceptional type, the situation is much less satisfactory: We have essentially no results when $m = 2$ and only a partial result for $m = 4$; when dimensions $m = 3$ or $m \geq 5$, we know that $W_{\pm}$ are bounded in $L^p(\mathbb{R}^m)$ for $p$ between $m/m - 2$ and $m/2$, however, we have only partial answers for what happens for $p$ outside this interval. We should also emphasize that these results are obtained only for operators $-\Delta + V(x)$ and, the problem is completely open when magnetic fields are present or when the metric of the space is not flat.

The general reference are as follows: For one dimension $m = 1$ see [3]; [17] and [8] for $m = 2$, [16] and [9] for $m = 4$, [15] and [19] for odd $m \geq 3$, and [16] and [5] for even $m \geq 6$.

2 One dimensional case

In one dimension we have the fairly satisfactory result. The following result is due to D'Ancona and Fanelli ([3], see [14, 1] for earlier results).

**Theorem 2.1.** (1) Suppose $\langle x \rangle^2 V(x) \in L^1(\mathbb{R})$. Then, $W_{\pm}$ are bounded in $L^p$ for all $1 < p < \infty$.

(2) Suppose $\langle x \rangle V(x) \in L^1(\mathbb{R})$ and $H$ is of generic type, then $W_{\pm}$ are bounded in $L^p$ for all $1 < p < \infty$.

**Remark 2.2.** We believe that $W_{\pm}$ are not bounded in $L^1$ nor in $L^\infty$ and that $W_{\pm}$ are bounded from Hardy space $H^1$ into $L^1$ and $L^\infty$ into BMO. However, we do not know the definite answer yet.

The proof of Theorem 2.1 employs the expression of $W_{\pm}$ in terms of the scattering eigenfunctions $\varphi_{\pm}(x, \xi)$ of $H$:

$$W_{\pm}u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi_{\pm}(x, \xi) \hat{u}(\xi) d\xi$$

as in earlier works [14, 1]) and uses some detailed properties of $\varphi_{\pm}(x, \xi)$. The functions $\varphi_{\pm}(x, \xi)$ are obtained by solving the Lippmann-Schwinger equation

$$\varphi_{\pm}(x, \xi) = e^{ix\xi} + \frac{1}{2i\xi} \int_{-\infty}^{\infty} e^{\pm i\xi|x-y|} V(y) \varphi_{\pm}(y, \xi) dy$$

and it can be expressed in terms of Jost functions. We refer [3] for the details.
### 3 Higher dimensional case $m \geq 2$

In higher dimensions $m \geq 2$, the situation is not as satisfactory as in the one dimensional case: We believe that the conditions on the potentials in the following theorems are far from optimal.

When $m \geq 2$, the problem has been studied by using the stationary representation formula of wave operators which expresses $W_\pm$ in terms of the boundary values of the resolvent. We write

$$G(\lambda) = (H - \lambda^2)^{-1}, \quad G_0(\lambda) = (H_0 - \lambda^2)^{-1}, \quad \lambda \in \mathbb{C}^+$$

where $\mathbb{C}^+ = \{ z \in \mathbb{C}: \Im z > 0 \}$ is the upper half plane. We write

$$\mathcal{H}_s = L^2_s(\mathbb{R}^m) = L^2(\mathbb{R}^m, \langle x \rangle^{2s} dx)$$

for the weighted $L^2$ spaces. We recall the well known limiting absorption principle (LAP) for $G_0(\lambda)$ and $G(\lambda)$ due to Agmon and Kuroda (see [11]). For Banach spaces $X, Y$, $B_\infty(X, Y)$ is the space of compact operators from $X$ to $Y$; $a_-$ for $a \in \mathbb{R}$ stands for an arbitrary number smaller than $a$.

#### Lemma 3.1.

1. Let $1/2 < \sigma$. Then, $G_0(\lambda)$ is a $B_\infty(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$ valued function of $\lambda \in \overline{\mathbb{C}^+} \setminus \{0\}$ of class $C^{(\sigma-\frac{1}{2})_-}$. For non-negative integers $j < \sigma - \frac{1}{2}$,

   $$\|G_0^{(j)}(\lambda)\|_{B(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})} \leq C_{j\sigma}|\lambda|^{-1}, \quad |\lambda| \geq 1. \quad (3.1)$$

2. Let $\frac{1}{2} < \sigma, \tau < m - \frac{3}{2}$ satisfy $\sigma + \tau > 2$. Then, $G_0(\lambda)$ is a $B_\infty(\mathcal{H}_\sigma, \mathcal{H}_{-\tau})$-valued function of $\lambda \in \overline{\mathbb{C}^+}$ of class $C^{\rho_*}$, $\rho_* = \min(\tau + \sigma - 2, \tau - 1/2, \sigma - 1/2)$.

#### Lemma 3.2.

1. Assume $|V(x)| \leq C\langle x \rangle^{-\delta}$ for some $\delta > 1$. Let $\frac{1}{2} < \gamma < \delta - \frac{1}{2}$. Then, $G(\lambda)$ is a $B_\infty(\mathcal{H}_\gamma, \mathcal{H}_{-\gamma})$ valued function of $\lambda \in \overline{\mathbb{C}^+} \setminus \{0\}$ of class $C^{(\gamma-\frac{1}{2})_-}$. For $0 \leq j < \gamma - \frac{1}{2}$,

   $$\|G^{(j)}(\lambda)\|_{B(\mathcal{H}_\gamma, \mathcal{H}_{-\gamma})} \leq C_{j\gamma}|\lambda|^{-1}, \quad |\lambda| \geq 1. \quad (3.2)$$

2. Assume $|V(x)| \leq C\langle x \rangle^{-\delta}$ for some $\delta > 2$ and that $H$ is of generic type. Let $1 < \gamma < \delta - 1$. Then $G(\lambda)$ is a $B_\infty(\mathcal{H}_\gamma, \mathcal{H}_{-\gamma})$ valued function of $\lambda \in \overline{\mathbb{C}^+}$ of class $C^{(\gamma-1)_-}$.

Using the boundary values of the resolvents on the real line, wave operators may be written in the following form (see [10]):

$$W_\pm u = u - \frac{1}{\pi i} \int_0^\infty G(\mp \lambda)V(G_0(\lambda) - G_0(-\lambda))\lambda ud\lambda \quad (3.3)$$

In what follows, we shall deal with $W_-$ only and we denote it by $W$ for brevity.
3.1 Born terms

If we formally expand the second resolvent equation into the series
\[
G(\lambda)V = (1 + G_0(\lambda)V)^{-1}G_0(\lambda)V = \sum_{n=1}^{\infty}(-1)^{n-1}(G_0(\lambda)V)^n
\]
and substitute the right side for \(G(\lambda)V\) in the stationary formula (3.3), then we have the formal expansion of \(W\):
\[
W = 1 - \Omega_1 + \Omega_2 - \cdots
\]
where for \(n = 1, 2, \ldots\),
\[
\Omega_n u = \frac{1}{\pi i} \int_0^\infty (G_0(\lambda)V)^n (G_0(\lambda) - G_0(-\lambda)) u \lambda d\lambda.
\]
This is called the Born expansion of the wave operator, the sum
\[
I - \Omega_1 + \cdots + (-1)^n \Omega_n
\]
the \(n\)-th Born approximation of \(W\) and the individual \(\Omega_n\) the \(n\)-th Born term. The Born terms \(\Omega_n\) may be computed more or less explicitly and they can be expressed as superpositions of one-dimensional convolution operators: We write \(\Sigma\) for the \(m-1\)-dimensional unit sphere. Define the function \(K_n(t, \ldots, t_n, \omega, \cdots, \omega_n)\) of \(t_1, \ldots, t_n \in \mathbb{R}\) and \(\omega_1, \ldots, \omega_n \in \Sigma\) by
\[
K_n(t, \ldots, t_n, \omega, \cdots, \omega_n)
= C^n \int_{\mathbb{R}_+^{n-1}} \int_{\Sigma^n} e^{i(t_1 s_1 + \cdots + t_n s_n)/2} (s_1 \cdots s_n)^{m-2} \prod_{j=1}^{n} \hat{V}(s_j \omega_j - s_{j-1} \omega_{j-1}) ds_1 \cdots ds_n
\]
where \(s_0 = 0, \mathbb{R}_+ = (0, \infty)\) and \(C\) is an absolute constant. Then \(\Omega_n u(x)\) may be written in the form
\[
\int_{\mathbb{R}_+^{n-1} \times I} \left( \int_{\Sigma^n} K_n(t, \ldots, t_n, \omega, \ldots, \omega_n) f(\overline{x} + \rho) d\omega_1 \cdots d\omega_n \right) dt_1 \cdots dt_n
\]
where \(I = (2x \cdot \omega_n, \infty)\) is the range of integration with respect to \(t_n\), \(\overline{x} = x - 2(\omega_n, x) \omega_n\) is the reflection of \(x\) along the \(\omega_n\) axis and \(\rho = t_1 \omega_1 + \cdots + t_n \omega_n\).

We define \(m_*(m-1)/(m-2)\) for \(m \geq 3\). If \(m \geq 3\), we have with \(\sigma > 1/m_*\) that
\[
\|K_1\|_{L^1(\mathbb{R} \times \Sigma)} \leq C \|\mathcal{F}(x^{\sigma} V)\|_{L^{m_*(\mathbb{R}^m)}}^{n},
\]
\[
\|K_n\|_{L^1(\mathbb{R}^n \times \Sigma^n)} \leq C^n \|\mathcal{F}(x^{2\sigma} V)\|_{L^{m_*(\mathbb{R}^m)}}^{n}, \quad n \geq 2,
\]
(see [15], page 569) and we obtain the following lemma.
Lemma 3.3. Let $m \geq 3$ and $\sigma > 1/m_*$. Then, there exists a constant $C > 0$ such that for any $1 \leq p \leq \infty$
\[
\|\Omega_1 u\|_p \leq C \|\mathcal{F}(\langle x \rangle^\sigma V)\|_{L^{m_*}(\mathbb{R}^m)} \|u\|_p,
\]
\[
\|\Omega_n u\|_p \leq C^n \|\mathcal{F}(\langle x \rangle^{2\sigma} V)\|_{L^{m_*}(\mathbb{R}^m)}^n \|u\|_p, \quad n = 2, \ldots
\] (3.9) (3.10)

It follows that the series (3.4) converges in the operator norm of $\mathcal{B}(L^p)$ for any $1 \leq p \leq \infty$ if $\|\mathcal{F}(\langle x \rangle^{2\sigma} V)\|_{L^{m_*}(\mathbb{R}^m)}$ is sufficiently small and we obtain the following theorem.

Theorem 3.4. Suppose $m \geq 3$ and $V$ satisfies $\mathcal{F}(\langle x \rangle^{2\sigma} V) \in L^{m_*}(\mathbb{R}^m)$ for some $\sigma > 1/m_*$. Then, there exists a constant $C > 0$ such that $W_{\pm}$ are bounded in $L^p(\mathbb{R}^m)$ for all $1 \leq p \leq \infty$ provided that $\|\mathcal{F}(\langle x \rangle^{2\sigma} V)\|_{L^{m_*}(\mathbb{R}^m)} < C$.

Note that that $H$ is of generic type if $\|\mathcal{F}(\langle x \rangle^{2\sigma} V)\|_{L^{m_*}(\mathbb{R}^m)}$ is sufficiently small. We remark that the condition $\mathcal{F}(\langle x \rangle^{\sigma} V) \in L^{m_*}(\mathbb{R}^m)$ requires some smoothness of $V$ if the dimension $m$ becomes larger. Recall that a certain smoothness condition on $V$ is necessary for $W_{\pm}$ to be bounded in $L^p$ for all $1 \leq p \leq \infty$ by virtue of the counter-example of Golberg-Vissan ([6]) for the dispersive estimates for dimensions $m \geq 4$.

In dimension $m = 2$, the factor $(s_1 \ldots s_n)^{m-2}$ is missing from (3.5) and it is evident that estimates (3.7) nor (3.8) do not hold. Nonetheless, we have the following result.

Lemma 3.5. Let $m = 2$. Then, for any $s > 1$ and $1 < p < \infty$, we have
\[
\|\Omega_1 u\|_p \leq C_{ps}\|\langle x \rangle^s V\|_2 \|u\|_p.
\]

If $\tilde{\chi}(\lambda) \in C^\infty(\mathbb{R})$ vanishes near $\lambda = 0$, then for any $s > 2$ and $1 < p < \infty$, we have
\[
\|\Omega_2 \tilde{\chi}(H_0) u\|_p \leq C_{ps}\|\langle x \rangle^s V\|_2^2 \|u\|_p.
\]

3.2 High energy estimate

We let $\chi \in C_0^\infty(\mathbb{R})$ and $\tilde{\chi} \in C^\infty(\mathbb{R})$ be such that
\[
\chi(\lambda) = 1 \text{ for } |\lambda| < \varepsilon, \quad \chi(\lambda) = 0 \text{ for } |\lambda| > 2\varepsilon \text{ for some } \varepsilon > 0
\]
and $\chi(\lambda^2) + \tilde{\chi}(\lambda)^2 = 1$ for all $\lambda \in \mathbb{R}$.

Then, the high energy part of the wave operator $W \tilde{\chi}(H_0)$ may be studied by a unified method for all $m \geq 2$ and we may show that $W$ is bounded in $\mathcal{B}(L^p(\mathbb{R}^m))$ for all $1 \leq p \leq \infty$ when $m \geq 3$ and for $1 < p < \infty$ for $m = 2$:
Theorem 3.6. Let $V$ satisfy $|V(x)| \leq C|x|^{-\delta}$ for some $\delta > m + 2$. Suppose, in addition, that $\mathcal{F}(x^{\sigma}V) \in L^{m^*}(\mathbb{R}^{m})$ if $m \geq 4$. Then $W_{\pm}\tilde{\chi}(H_{0})$ is bounded in $B(L^{p}(\mathbb{R}^{m}))$ for all $1 \leq p \leq \infty$ when $m \geq 3$ and for $1 < p < \infty$ for $m = 2$.

We outline the proof. We write $\nu = (m - 2)/2$. Iterating the resolvent equation, we have

$$G(\lambda)V = \sum_{1}^{2n}(-1)^{j-1}(G_{0}(\lambda)V)^{j} + G_{0}(\lambda)N_{n}(\lambda)$$

where $N_{n}(\lambda) = (VG_{0}(\lambda))^{n-1}VG(\lambda)V(G_{0}(\lambda)V)^{n}$. If we substitute this for $G(\lambda)V$ in the stationary formula (3.3), we obtain

$$W\tilde{\chi}(H_{0})^{2} = \tilde{\chi}(H_{0})^{2} + \sum_{j=1}^{2n}(-1)^{j}\Omega_{j}\tilde{\chi}(H_{0})^{2} - \tilde{\Omega}_{2n+1}, \quad (3.11)$$

$$\tilde{\Omega}_{2n+1} = \frac{1}{i\pi} \int_{0}^{\infty} G_{0}(\lambda)N_{n}(G_{0}(\lambda) - G_{0}(-\lambda))\tilde{\Psi}(\lambda)d\lambda, \quad (3.12)$$

where $\tilde{\Psi}(\lambda) = \lambda\overline{\chi}(\lambda^{2})^{2}$. The operators $\tilde{\chi}(H_{0})$ and $\Omega_{1}\tilde{\chi}(H_{0})^{2}, \ldots, \Omega_{2n}\tilde{\chi}(H_{0})^{2}$ are bounded in $L^{p}(\mathbb{R}^{m})$ for any $1 \leq p \leq \infty$ if $m \geq 3$ and for $1 < p < \infty$ if $m = 2$ by virtue of Lemma 3.3 and Lemma 3.5, since $\tilde{\chi}(H_{0})$ is clearly bounded in $L^{p}(\mathbb{R}^{m})$ for all $1 \leq p \leq \infty$ and $m \geq 2$. We then show that, for sufficiently large $n$, $\tilde{\Omega}_{2n+1}$ is also bounded in $L^{p}(\mathbb{R}^{m})$ for all $1 \leq p \leq \infty$ and $m \geq 2$ by showing that its integral kernel

$$\tilde{\Omega}_{2n+1}(x, y) = \frac{1}{\pi i} \int_{0}^{\infty} \langle N_{n}(\lambda)(G_{0}(\lambda) - G_{0}(-\lambda))\delta_{y}, G_{0}(-\lambda)\delta_{x}\rangle \lambda\Psi^{2}(\lambda^{2})d\lambda,$$

where $\delta_{a} = \delta(x-a)$ is the unit mass at the point $x = a$, satisfies the estimate that

$$\sup_{x \in \mathbb{R}^{m}} \int_{\mathbb{R}^{m}} |\tilde{\Omega}_{2n+1}(x, y)|dy < \infty \quad \text{and} \quad \sup_{y \in \mathbb{R}^{m}} \int_{\mathbb{R}^{m}} |\tilde{\Omega}_{2n+1}(x, y)|dx < \infty. \quad (3.13)$$

It is a result of Schur's lemma that estimates (3.13) imply that $\tilde{\Omega}_{2n+1}$ is bounded in $L^{p}(\mathbb{R}^{m})$ for all $1 \leq p \leq \infty$. Note that $[G_{0}(\lambda)\delta_{y}](x) = G_{0}(\lambda, x-y)$ is the integral kernel of $G_{0}(\lambda)$ and $G_{0}(\lambda, x)$ is given by

$$G_{0}(\lambda, x) = \frac{e^{i\lambda|x|}}{2(2\pi)^{\nu+\frac{1}{2}}\Gamma(\nu + \frac{1}{2})|x|^{m-2}} \int_{0}^{\infty} e^{-t|\lambda|^{2}} \left( \frac{t}{2} - i\lambda|x| \right)^{\nu-\frac{1}{2}} dt. \quad (3.14)$$
As a slight modification of the argument is necessary for the case $m = 2$, we restrict ourselves to the case $m \geq 3$ and, for definiteness, we assume $m$ is even in what follows in this subsection. We define

$$
\tilde{G}_0(\lambda, z, x) = e^{-i\lambda|x|}G_0(\lambda, x - z)
$$
and

$$
T_{\pm}(\lambda, x, y) = \langle N_n(\lambda)\tilde{G}_0(\pm\lambda, \cdot, y), \tilde{G}_0(-\lambda, \cdot, x) \rangle
$$
so that

$$
\tilde{\Omega}_{2n+1}(x, y) = \frac{1}{\pi i} \int_0^\infty (e^{i\lambda(|x|+|y|)}T_{+}(\lambda, x, y) - e^{i\lambda(|x|-|y|)}T_{-}(\lambda, x, y)) \tilde{\Psi}(\lambda) d\lambda.
$$

We may compute derivatives $\tilde{G}_0^{(j)}(\lambda, z, x)$ with respect to $\lambda$ using Leibniz's formula. If we set $\psi(z, x) = |x-z| - |x|$, they are linear combinations over $(\alpha, \beta)$ such that $\alpha + \beta = j$ of

$$
e^{i\lambda\psi(z,x)}\psi(z,x)^\alpha |x-z|^{m-2-\beta} \int_0^\infty e^{-t}t^{\nu-\frac{1}{2}}(\frac{t}{2} - i\lambda|x-z|)^{\nu-\beta} dt.
$$

Since $|\psi(z, x)|^\alpha \leq \langle z \rangle^j$ for $0 \leq \alpha \leq j$ and

$$
|z - x| \leq C_\epsilon \frac{t}{2} - i\lambda|x - x| \leq C_\epsilon (t + \lambda|x - x|)
$$
when $|\lambda| \geq 1$, we have for $|\lambda| \geq \epsilon$

$$
\left| \left( \frac{\partial}{\partial \lambda} \right)^j \tilde{G}_0(\lambda, z, x) \right| \leq C_j \left( \frac{\langle z \rangle^j}{|x-z|^{m-2}} + \frac{\lambda \frac{m-3}{2} \langle z \rangle^j}{|x-z|^{\frac{m-1}{2}}} \right).
$$

for $j = 0, 1, 2, \ldots$

Note that $\tilde{G}_0(\lambda, z, x) \sim C|x-z|^{2-m}$ near $z = x$ and $\tilde{G}_0(\lambda, z, x) \not\in L_{loc}^2(\mathbb{R}^m)$ for a fixed $x$ if $m \geq 4$. However, the LAP (3.1) implies

$$
\| \langle x \rangle^{-\gamma-j} G_0^{(j)}(\lambda) \langle x \rangle^{-\gamma-j} \|_{B(H^s, H^{s+2})} \leq C_{sj\gamma} |\lambda|, \quad |\lambda| \geq \epsilon
$$
for any $\gamma > 1/2$, $s \in \mathbb{R}$ and $j = 0, 1, \ldots$ and $k$ times application of $G_0(\lambda)V$ to $\tilde{G}_0(\lambda, \cdot, x)$, $k > (m - 2)/2$, makes it into a function in $L^2_{-\gamma}(\mathbb{R}^m)$ for any $\gamma > 1/2$. Thus, if we take $n = k > (m - 2)/2$, $T_{\pm}(\lambda, x, y)$ are well defined continuous functions of $(x, y)$ which are $(m+2)/2$ times continuously differentiable with respect to $\lambda$. This, however, produces the increasing factor $\lambda^k$ by virtue
of the increase of the norm of (3.18). We, therefore, take \( n \) larger so that \( n > m \) and use the fact (3.1) that \( \| \langle x \rangle^{-\gamma-j} G_0^{(j)}(\lambda) \langle x \rangle^{-\gamma-j} \|_{B(L^2, L^2)} \leq C|\lambda|^{-1} \) decays as \( \lambda \to \pm \infty \). Then, the decay property of extra factors \( (G_0^{(j)}(\lambda)V)^{n-k} \) cancels this increasing factor and makes \( T_{\pm}(\lambda, x, y) \) integrable with respect to \( \lambda \). Using also the fact that \( \tilde{G}_0(\lambda, \cdot, x) \sim |x|^{-\frac{m-1}{2}} \) as \( |x| \to \infty \), we in this way obtain the following estimate:

**Lemma 3.7.** Let \( 0 \leq s \leq \frac{m+2}{2} \). We have

\[
\left| \left( \frac{\partial}{\partial \lambda} \right)^s T_{\pm}(\lambda, x, y) \right| \leq C_{ns} \lambda^{-3} \langle x \rangle^{-\frac{m-1}{2}} \langle y \rangle^{-\frac{m-1}{2}} \tag{3.19}
\]

To obtain the desired estimate for \( \tilde{\Omega}_{2n+1}(x, y) \), we apply integration by parts \( 0 \leq s \leq (m+2)/2 \) times with respect to the variable \( \lambda \) in (3.16):

\[
\int_0^\infty e^{i\lambda(|x|\pm|y|)} T_{\pm}(\lambda, x, y) \tilde{\Psi}(\lambda) d\lambda = \frac{1}{(|x| \pm |y|)^s} \int_0^\infty e^{i\lambda(|x|\pm|y|)} \left( \frac{\partial}{\partial \lambda} \right)^s \left( T_{\pm}(\lambda, x, y) \tilde{\Psi}(\lambda) \right) d\lambda
\]

and estimate the right hand side by using (3.19). We obtain

\[
|\tilde{\Omega}_{n+1}(x, y)| \leq C \sum_{\pm} \langle |x| \pm |y| \rangle^{-\frac{m+2}{2}} \langle x \rangle^{-\frac{m-1}{2}} \langle y \rangle^{-\frac{m-1}{2}}.
\]

It is then an easy exercise to show that \( \tilde{\Omega}_{n+1}(x, y) \) satisfies the estimate (3.13).

### 3.3 Low energy estimate, generic case

By virtue of the intertwining property we have \( W_{\pm}(H_0)^2 = \chi(H) W_{\pm} \chi(H_0) \) and, from (3.3), we may write the low energy part \( W_{\pm}(H_0)^2 \) as the sum of \( \chi(H) \chi(H_0) \) and

\[
\Omega = \frac{i}{\pi} \int_0^\infty \chi(H) G_0(\lambda) V (1 + G_0(\lambda)V)^{-1} (G_0(\lambda) - G_0(-\lambda)) \chi(H_0) \lambda d\lambda. \tag{3.20}
\]

Here \( \chi(H_0) \) and \( \chi(H) \) both are integral operators of which the integral kernels satisfy for any \( N > 0 \)

\[
|\chi(H_0)(x, y)| \leq C_N \langle x - y \rangle^{-N}, \quad |\chi(H)(x, y)| \leq C_N \langle x - y \rangle^{-N} \tag{3.21}
\]
and are, a fortiori, bounded in $L^p(\mathbb{R}^m)$ (see [16]). If $H$ is of generic type and $m \geq 3$ is odd, then $(1 + G_0(\lambda)V)^{-1}$ has no singularities at $\lambda = 0$ and we may prove that $\Omega$ is bounded in $L^p(\mathbb{R}^m)$ for all $1 \leq p \leq \infty$ by proving that its integral kernel $\Omega(x,y)$ satisfies the estimate (3.13) by a method similar to the one used for the high energy part. The argument is simpler in the point that we do not have to expand $(1 + G_0(\lambda)V)^{-1}$ since the range of the integration with respect to $\lambda$ in (3.20) is compact and since the integral kernels of $G_0(\lambda)\chi(H_0)$ and $G_0(\lambda)\chi(H)$ have no singularities at the diagonal set by virtue of (3.21). It is, however, more complicated than in the high energy case in that the integral kernels of

$$\frac{i}{\pi} \int_0^\infty \chi(H)G_0(\lambda)V(1 + G_0(\lambda)V)^{-1}G_0(\pm\lambda)\chi(H_0) \lambda d\lambda,$$

do not separately satisfy the estimate (3.13) but only their difference does.

If $H$ is of generic type and $m$ is even, then $(1 + G_0(\lambda)V)^{-1}$ or its derivatives contain logarithmic singularities at $\lambda = 0$ which are stronger when the dimensions are lower. Thus, the analysis becomes more involved than the odd case particularly when $m = 2$ and $m = 4$. However, basically the idea as in the odd dimensional case works and we obtain the following theorem. We write $B(x, 1) = \{y \in \mathbb{R}^m : |y - x| < 1\}$.

**Theorem 3.8.** Suppose that $H$ is of generic type:

1. Let $m = 2$. Suppose that $V$ satisfies $|V(x)| \leq C\langle x \rangle^{-6-\epsilon}$ for some $\epsilon > 0$. Then, $W_{\pm}$ are bounded in $L^p$ for all $1 < p < \infty$.

2. Let $m = 3$. Suppose that $V$ satisfies $|V(x)| \leq C\langle x \rangle^{-5-\epsilon}$ for some $\epsilon > 0$. Then, $W_{\pm}$ are bounded in $L^p$ for all $1 \leq p \leq \infty$.

3. Let $m = 4$. Suppose that $V$ satisfies for some $q > 2$

$$\|V\|_{L^q(B(x, 1))} + \|\nabla V\|_{L^q(B(x, 1))} \leq C\langle x \rangle^{-7-\epsilon}$$

for some $\epsilon > 0$. Then, $W_{\pm}$ are bounded in $L^p$ for all $1 \leq p \leq \infty$.

4. Let $m \geq 5$. Suppose that $V$ satisfies $|V(x)| \leq C\langle x \rangle^{-m-2-\epsilon}$ for some $\epsilon > 0$ in addition to $F(\langle x \rangle^{2\sigma}V) \in L^m(\mathbb{R}^m)$ for some $\sigma > 1/m_*$. Then, $W_{\pm}$ are bounded in $L^p$ for all $1 \leq p \leq \infty$.

**Remark 3.9.** When $m = 2$, at the end point, the same remark as in the one dimension applies: We believe $W_{\pm}$ are not bounded in $L^1$ nor in $L^\infty$ at the end point and they are bounded from Hardy space $H^1$ into $L^1$ and $L^\infty$ to BMO. However, we have no proofs.
3.4 Low energy estimate, exceptional case

We assume $H$ is of exceptional type in this subsection. Then, $(1 + G_0(\lambda)V)^{-1}$ of (3.20) is not invertible at $\lambda = 0$ and it has singularities at $\lambda = 0$. As we have no result when $m = 2$ and only a partial result when $m = 4$ which we mention at the end of this subsection, we assume $m = 3$ or $m \geq 5$ before the statement of Theorem 3.12. We study the singularities of $(1 + G_0(\lambda)V)^{-1}$ as $\lambda \to 0$ by expanding $1 + G_0(\lambda)V$ with respect to $\lambda$ around $\lambda = 0$ and examining the structure of $1 + G_0(0)V$. The result is: If $m \geq 3$ is odd, we have

$$(1 + G_0(\lambda)V)^{-1} = \lambda^{-2}P_0V + \lambda^{-1}A_{-1} + 1 + A_0(\lambda)$$

where $A_{-1}$ is a finite rank operator involving 0 eigenfunctions and the resonance function and $A_0(\lambda)$ has no singularities; if $m \geq 6$ is even, then

$$(1 + G_0(\lambda)V)^{-1} = \frac{P_0V}{\lambda^2} + \sum_{j=0}^{2} \sum_{k=1}^{2} \lambda^{j}(\log \lambda)^{k}D_{jk} + I + A_0(\lambda), \quad (3.22)$$

where $D_{jk}$ are finite rank operators involving 0 eigenfunctions and $A_0(\lambda)$ has no singularities. We substitute this expression for $(1 + G_0(\lambda)V)^{-1}$ in (3.20). Then, the operator produced by $I + A_0(\lambda)$ may be treated as in the previous section for the case when $H$ is of generic type. The operators produced by singular terms may be treated by using the machinaries of harmonic analysis, the weighted inequalities for the Hilbert transform and the Hardy-Littlewood maximal functions, which is a little too complicated to explain here. In this way we obtain the following theorem. We refer the readers to [19] and [5] for the proof respectively for odd and even dimensional case.

**Theorem 3.10.** Suppose that $H$ is of exceptional type.

(1) Let $m \geq 3$ be odd. Suppose that $V$ satisfies $|V(x)| \leq C\langle x \rangle^{-m-3-\epsilon}$ for some $\epsilon > 0$ and $\mathcal{F}(\langle x \rangle^{2\sigma}V) \in L^{m,*}(\mathbb{R}^m)$ in addition for some $\sigma > 1/m_*$. Then, $W_{\pm}$ are bounded in $L^p(\mathbb{R}^m)$ between $m/(m-2)$ and $m/2$.

(2) Let $m \geq 6$ be even. Suppose that $V$ satisfies $|V(x)| \leq C\langle x \rangle^{-m-3-\epsilon}$ if $m \geq 8$, $|V(x)| \leq C\langle x \rangle^{-m-4-\epsilon} \text{ if } m = 6$ for some $\epsilon > 0$ and $\mathcal{F}(\langle x \rangle^{2\sigma}V) \in L^{m,*}(\mathbb{R}^m)$ for some $\sigma > 1/m_*$ in addition. Then, $W_{\pm}$ are bounded in $L^p(\mathbb{R}^m)$ for $m/(m-2) < p < m/2$.

**Remark 3.11.** When $H$ is of exceptional type, the $W_{\pm}$ are not bounded in $L^p(\mathbb{R}^m)$ if $p > m/2$ and $m \geq 5$, or if $p > 3$ and $m = 3$. This can be deduced from the results on the decay in time property of the propagator...
\[ e^{-itH} P_{ac} \] in the weighted \( L^2 \) spaces \([12, 7]\), or in \( L^p \) spaces \([4, 18]\). We believe the same is true for \( p \)'s on the other side of the interval given in (b), viz. \( 1 \leq p \leq m/(m-2) \) if \( m \geq 5 \) and \( 1 \leq p \leq 3/2 \) if \( m = 3 \), but we have again no proofs.

In the case when \( m = 2 \) or \( m = 4 \), and if 0 is a resonance of \( H \), then the results of \([12]\) and \([7]\) mentioned above imply that the \( W_{\pm} \) are not bounded in \( L^p(\mathbb{R}^m) \) for \( p > 2 \) and, though proof is missing, we believe that this is the case for all \( p \)'s except \( p = 2 \). However, when \( m = 4 \) and if 0 is a pure eigenvalue of \( H \) and not a resonance, the \( W_{\pm} \) are bounded in \( L^p(\mathbb{R}^4) \) for \( 4/3 < p < 4 \):

**Theorem 3.12.** Let \( |V(x)| + |\nabla V(x)| \leq C (x)^{-\delta} \) for some \( \delta > 7 \). Suppose that 0 is an eigenvalue of \( H \), but not a resonance. Then the \( W_{\pm} \) extend to bounded operators in the Sobolev spaces \( W^{k,p}(\mathbb{R}^4) \) for any \( 0 \leq k \leq 2 \) and \( 4/3 < p < 4 \):

\[
\|W_{\pm} u\|_{W^{k,p}} \leq C_p \|u\|_{W^{k,p}}, \quad u \in W^{k,p}(\mathbb{R}^4) \cap L^2(\mathbb{R}^4). \tag{3.23}
\]

We do not explain the proof of this theorem and refer the readers to the recent preprint \([8]\). Again, the results of \([12, 7]\) imply that the \( W_{\pm} \) are unbounded in \( L^p(\mathbb{R}^4) \) if \( p > 4 \) under the assumption of Theorem 3.12. We believe that this is the case also for \( 1 \leq p < 4/3 \), though we do not have proofs.

**References**


