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The L^p boundedness of wave operators for Schrödinger operators

Kenji Yajima
Department of Mathematics, Gakushuin University
1-5-1 Mejiro, Toshima-ku, Tokyo 171-8588, Japan

1 Introduction

Let $H = -\Delta + V$ be the Schrödinger operator on $\mathbf{R}^m$, $m \geq 1$, with real valued potential $V(x)$ such that $|V(x)| \leq C\langle x\rangle^{-\delta}$ for some $\delta > 2$, where $\langle x\rangle = (1+x^2)^{1/2}$. Then, it is well known that

(1) H is selfadjoint in the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^m)$ with domain $D(H) = H^2(\mathbf{R}^m)$ and $C_0^\infty(\mathbf{R}^m)$ is a core;

(2) the spectrum $\sigma(H)$ of H consists of an absolutely continuous part $[0, \infty)$, and at most a finite number of non-positive eigenvalues $\{\lambda_j\}$ of finite multiplicities;

(3) the singular continuous spectrum and positive eigenvalues are absent from $\sigma(H)$.

We denote the point and the absolutely continuous spectral subspaces of $\mathcal{H}$ for H by $\mathcal{H}_{\rm p}$ and $\mathcal{H}_{\rm ac}$ respectively, and the orthogonal projections in $\mathcal{H}$ onto the respective subspaces by $P_{\rm p}$ and $P_{\rm ac}$. We write $H_0 = -\Delta$ for the free Schrödinger operator.

(4) The wave operators $W_\pm$ defined by the following limits in $\mathcal{H}$:

$$W_\pm = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH}e^{-itH_0}$$

exist and are complete in the sense that Image $W_\pm = \mathcal{H}_{\rm ac}$.

(5) $W_\pm$ satisfy the so called intertwining property and the absolutely continuous part of H is unitarily equivalent to H_0 via $W_\pm$: For Borel functions f on $\mathbf{R}$, we have

$$f(H)P_{\rm ac}(H) = W_\pm f(H_0)W_\pm^*. \tag{1.1}$$

It follows from the intertwining property (1.1) that, if X and Y are Banch spaces such that $L^2(\mathbf{R}^m) \cap X$ and $L^2(\mathbf{R}^m) \cap Y$ are dense in X and Y respectively, then,

$$\begin{aligned} &\|f(H)P_{ac}(H)\|_{\mathbf{B}(X,Y)} \\ &\quad \le \|W_\pm\|_{\mathbf{B}(Y)} \|f(H_0)\|_{\mathbf{B}(X,Y)} \|W_\pm^*\|_{\mathbf{B}(X)} = C\|f(H_0)\|_{\mathbf{B}(X,Y)}. \end{aligned} \tag{1.2}$$

Here it is important that the constant $C = \|W_\pm\|_{\mathbf{B}(Y)}\|W_\pm^*\|_{\mathbf{B}(X)}$ is independent of the function f. Thus, the mapping property of $f(H)P_{ac}(H)$ from X to Y may be deduced from that of $f(H_0)$, once we know that $W_\pm$ are bounded in X and in Y. Note that the solutions $u(t)$ of the Cauchy problem for the Schrödinger equation

$$i\partial_t u = (-\Delta + V)u, \quad u(0) = \varphi$$

and $v(t)$ of the wave equation

$$\partial_t^2 v = (\Delta - V)v, \quad v(0) = \varphi, \quad \partial_t v(0) = \psi$$

are given in terms of the functions of H, respectively by

$$u(t) = e^{-itH}\varphi, \ \ \text{and} \ \ v(t) = \cos(t\sqrt{H})\,\varphi + \frac{\sin(t\sqrt{H})}{\sqrt{H}}\psi.$$

It folllows that, if $W_\pm$ are bounded in Lebegue spaces $L^p(\mathbf{R}^m)$ for $1 \le p \le \infty$ and if the initial states φ and ψ belong to the continuos spectral subspace $\mathcal{H}_c(H)$, then the L^p-L^q estimates for the propagators of the respective equations may be deduced from the well known L^p-L^q estimates for the free propagators e^{-itH_0} or $\cos(t\sqrt{H_0})$ and $\sin(t\sqrt{H_0})/\sqrt{H_0}$ (if φ and ψ are eigenfunctions of H, the behavior of $u(t)$ and $v(t)$ are trivial). In particular, we have the so called dispersive estimates for the Schrödinger equation

$$\|e^{-itH}P_c(H)\varphi\|_\infty \le C|t|^{-\frac{m}{2}}\|\varphi\|_1.$$

In this lecture we would like to briefly survery the current status of the study of the mapping property of $W_\pm$ in Lebesgue spaces $L^p(\mathbf{R}^m)$. We say that 0 is a resonance of H, if there is a solution φ of $(-\Delta + V(x))\varphi(x) = 0$ such that $|\varphi(x)| \le C\langle x\rangle^{2-m}$ but $\varphi \notin \mathcal{H}$ and call such a solution $\varphi(x)$ a resonance function of H; H is of generic type, if 0 is neither an eigenvalue nor a resonance of H, otherwise of exceptional type. Note that there is no zero resonance if $m \ge 5$. We shall see that the mapping property of $W_\pm$ in $L^p(\mathbf{R}^m)$ spaces is fairely well understood when H is of generic type although the conditions on potentials for the L^p-boundedness of $W_\pm$ are far from optimal and the end point problem, viz. the problem for the case $p = 1$ and $p = \infty$ is not settled completely in the cases $m = 1$ and $m = 2$. On the other hand, if H is of exceptional type, the situation is much less satisfactory: We have essentially no results when $m = 2$ and only a partial result for $m = 4$; when dimensions $m = 3$ or $m \geq 5$, we know that $W_\pm$ are bounded in $L^p(\mathbf{R}^m)$ for p between $m/m-2$ and $m/2$, however, we have only partial answers for what happens for p outside this interval. We should also emphasize that these results are obtained only for operators $-\Delta + V(x)$ and, the problem is completely open when magnetic fields are present or when the metric of the space is not flat.

The general reference are as follows: For one dimension $m = 1$ see [3]; [17] and [8] for $m = 2$, [16] and [9] for $m = 4$, [15] and [19] for odd $m \geq 3$, and [16] and [5] for even $m \geq 6$.

2 One dimensional case

In one dimension we have the fairly satisfactory result. The following result is due to D'Ancona and Fanelli ([3], see [14, 1] for eariler results).

Theorem 2.1. (1) *Suppose* $\langle x \rangle^2 V(x) \in L^1(\mathbf{R}^1)$. *Then,* $W_\pm$ *are bounded in* L^p *for all* $1 < p < \infty$.

(2) *Suppose* $\langle x \rangle V(x) \in L^1(\mathbf{R}^1)$ *and* H *is of generic type, then* $W_\pm$ *are bounded in* L^p *for all* $1 < p < \infty$.

Remark 2.2. We believe that $W_\pm$ are not bounded in L^1 nor in L^∞ and that $W_\pm$ are bounded from Hardy space H^1 into L^1 and L^∞ into BMO. However, we do not know the definite answer yet.

The proof of Theorem 2.1 employs the expression of $W_\pm$ in terms of the scattering eigenfunctions $\varphi_\pm(x,\xi)$ of H:

$$W_\pm u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \varphi_\pm(x,\xi)\hat{u}(\xi)d\xi$$

as in earlier works [14, 1]) and uses some detailed properties of $\varphi_\pm(x,\xi)$. The functions $\varphi_\pm(x,\xi)$ are obtained by solving the Lippmann-Schwinger equation

$$\varphi_\pm(x,\xi) = e^{ix\xi} + \frac{1}{2i\xi} \int_{-\infty}^{\infty} e^{\pm i\xi|x-y|} V(y)\varphi_\pm(y,\xi)dy$$

and it can be expressed in terms of Jost functions. We refer [3] for the details.

3 Higher dimensional case $m \geq 2$

In higher dimensions $m \geq 2$, the stituation is not as satisfactory as in the one dimensional case: We believe that the conditions on the potentials in the following theorems are far from optimal.

When $m \geq 2$, the probem has been studied by using the stationary representation formula of wave operators which expresses $W_\pm$ in terms of the boundary values of the resolvent. We write

$$G(\lambda) = (H - \lambda^2)^{-1}, \quad G_0(\lambda) = (H_0 - \lambda^2)^{-1}. \quad \lambda \in \mathbf{C}^+$$

where $\mathbf{C}^+ = \{z \in \mathbf{C} \colon \Im z > 0\}$ is the upper half plane. We write

$$\mathcal{H}_s = L^2_s(\mathbf{R}^m) = L^2(\mathbf{R}^m, \langle x \rangle^{2s} dx)$$

for the weighted L^2 spaces. We recall the well known limiting absorption principle (LAP) for $G_0(\lambda)$ and $G(\lambda)$ due to Agmon and Kuroda (see [11]). For Banach spaces X, Y, $\mathbf{B}_\infty(X,Y)$ is the space of compact operators from X to Y; a_- for $a \in \mathbf{R}$ stands for an arbitrary number smaller than a.

Lemma 3.1. (1) *Let* $1/2 < \sigma$. *Then,* $G_0(\lambda)$ *is a* $\mathbf{B}_\infty(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$ *valued function of* $\lambda \in \overline{\mathbf{C}^+} \setminus \{0\}$ *of class* $C^{(\sigma - \frac{1}{2})-}$. *For non-negative integers* $j < \sigma - \frac{1}{2}$,

$$\|G_0^{(j)}(\lambda)\|_{\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})} \leq C_{j\sigma} |\lambda|^{-1}, \quad |\lambda| \geq 1. \tag{3.1}$$

(2) *Let* $\frac{1}{2} < \sigma, \tau < m - \frac{3}{2}$ *satisfy* $\sigma + \tau > 2$. *Then,* $G_0(\lambda)$ *is a* $\mathbf{B}_\infty(\mathcal{H}_\sigma, \mathcal{H}_{-\tau})$*-valued function of* $\lambda \in \overline{\mathbf{C}^+}$ *of class* $C^{\rho_* -}$, $\rho_* = \min(\tau + \sigma - 2, \tau - 1/2, \sigma - 1/2)$.

Lemma 3.2. (1) *Assume* $|V(x)| \leq C\langle x \rangle^{-\delta}$ *for some* $\delta > 1$. *Let* $\frac{1}{2} < \gamma < \delta - \frac{1}{2}$. *Then,* $G(\lambda)$ *is a* $\mathbf{B}_\infty(\mathcal{H}_\gamma, \mathcal{H}_{-\gamma})$ *valued function of* $\lambda \in \overline{\mathbf{C}^+} \setminus \{0\}$ *of class* $C^{(\gamma - \frac{1}{2})-}$. *For* $0 \leq j < \gamma - \frac{1}{2}$,

$$\|G^{(j)}(\lambda)\|_{\mathbf{B}(\mathcal{H}_\gamma, \mathcal{H}_{-\gamma})} \leq C_{j\gamma} |\lambda|^{-1}, \quad |\lambda| \geq 1. \tag{3.2}$$

(2) *Assume* $|V(x)| \leq C\langle x \rangle^{-\delta}$ *for some* $\delta > 2$ *and that* H *is of generic type. Let* $1 < \gamma < \delta - 1$. *Then* $G(\lambda)$ *is a* $\mathbf{B}_\infty(\mathcal{H}_\gamma, \mathcal{H}_{-\gamma})$ *valued function of* $\lambda \in \overline{\mathbf{C}^+}$ *of class* $C^{(\gamma - 1)-}$.

Using the boundary values of the resolvents on the real line, wave operators may be written in the following form (see [10]):

$$W_\pm u = u - \frac{1}{\pi i} \int_0^\infty G(\mp\lambda) V (G_0(\lambda) - G_0(-\lambda)) \lambda u d\lambda \tag{3.3}$$

In what follows, we shall deal with W_- only and we denote it by W for brevity.

3.1 Born terms

If we formally expand the second resolvent equation into the series

$$G(\lambda)V = (1 + G_0(\lambda)V)^{-1}G_0(\lambda)V = \sum_{n=1}^{\infty}(-1)^{n-1}(G_0(\lambda)V)^n$$

and substitute the right side for $G(\lambda)V$ in the stationary formula (3.3), then we have the formal expansion of W:

$$W = 1 - \Omega_1 + \Omega_2 - \cdots \tag{3.4}$$

where for $n = 1, 2, \ldots,$

$$\Omega_n u = \frac{1}{\pi i}\int_0^{\infty}(G_0(\lambda)V)^n(G_0(\lambda) - G_0(-\lambda))u\lambda d\lambda.$$

This is called the Born expansion of the wave operator, the sum

$$I - \Omega_1 + \cdots + (-1)^n\Omega_n$$

the n-th Born approximation of W_- and the individual Ω_n the n-th Born term. The Born terms Ω_n may be computed more or less explicitly and they can be expressed as superpositions of one dimensional convolution operators: We write Σ for the $m-1$ dimensional unit sphere. Define the funtion $K_n(t, \ldots, t_n, \omega, \ldots, \omega_n)$ of $t_1, \ldots, t_n \in \mathbf{R}$ and $\omega_1, \ldots, \omega_n \in \Sigma$ by

$$\begin{aligned} &K_n(t, \ldots, t_n, \omega, \ldots, \omega_n) \\ &= C^n \int_{\mathbf{R}_+^n} e^{i(t_1 s_1 + \cdots + t_n s_n)/2}(s_1 \ldots s_n)^{m-2}\prod_{j=1}^{n}\hat{V}(s_j\omega_j - s_{j-1}\omega_{j-1})ds_1 \ldots ds_n \end{aligned} \tag{3.5}$$

where $s_0 = 0$, $\mathbf{R}_+ = (0, \infty)$ and C is an absolute constant. Then $\Omega_n u(x)$ may be written in the form

$$\int_{\mathbf{R}_+^{n-1}\times I}\left(\int_{\Sigma^n} K_n(t, \ldots, t_n, \omega, \ldots, \omega_n)f(\overline{x} + \rho)d\omega_1 \ldots \omega_n\right)dt_1 \cdots dt_n \tag{3.6}$$

where $I = (2x \cdot \omega_n, \infty)$ is the range of integration with respect to t_n, $\overline{x} = x - 2(\omega_n, x)\omega_n$ is the reflection of x along the ω_n axis and $\rho = t_1\omega_1 + \cdots + t_n\omega_n$.

We define $m_* = (m-1)/(m-2)$ for $m \geq 3$. If $m \geq 3$, we have with $\sigma > 1/m_*$ that

$$\|K_1\|_{L^1(\mathbf{R}\times\Sigma)} \leq C\|\mathcal{F}(\langle x\rangle^{\sigma}V)\|^n_{L^{m_*}(\mathbf{R}^m)}, \tag{3.7}$$

$$\|K_n\|_{L^1(\mathbf{R}^n\times\Sigma^n)} \leq C^n\|\mathcal{F}(\langle x\rangle^{2\sigma}V)\|^n_{L^{m_*}(\mathbf{R}^m)}, \quad n \geq 2, \tag{3.8}$$

(see [15], page 569) and we obtain the following lemma.

Lemma 3.3. *Let $m \geq 3$ and $\sigma > 1/m_*$. Then, there exists a constant $C > 0$ such that for any $1 \leq p \leq \infty$*

$$\|\Omega_1 u\|_p \leq C\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m_*}(\mathbf{R}^m)}\|u\|_p, \tag{3.9}$$
$$\|\Omega_n u\|_p \leq C^n\|\mathcal{F}(\langle x\rangle^{2\sigma} V)\|^n_{L^{m_*}(\mathbf{R}^m)}\|u\|_p, \quad n = 2, \dots . \tag{3.10}$$

It follows that the series (3.4) converges in the operator norm of $\mathbf{B}(L^p)$ for any $1 \leq p \leq \infty$ if $\|\mathcal{F}(\langle x\rangle^{2\sigma} V)\|_{L^{m_*}(\mathbf{R}^m)}$ is sufficiently small and we obtain the following theorem.

Theorem 3.4. *Suppose $m \geq 3$ and V satisfies $\mathcal{F}(\langle x\rangle^{2\sigma} V) \in L^{m_*}(\mathbf{R}^m)$ for some $\sigma > 1/m_*$. Then, there exists a constant $C > 0$ such that $W_\pm$ are bounded in $L^p(\mathbf{R}^m)$ for all $1 \leq p \leq \infty$ provided that $\|\mathcal{F}(\langle x\rangle^{2\sigma} V)\|_{L^{m_*}(\mathbf{R}^m)} < C$.*

Note that that H is of generic type if $\|\mathcal{F}(\langle x\rangle^{2\sigma} V)\|_{L^{m_*}(\mathbf{R}^m)}$ is sufficiently small. We remark that the condition $\mathcal{F}(\langle x\rangle^\sigma V) \in L^{m_*}(\mathbf{R}^m)$ requires some smoothness of V if the dimension m becomes larger. Recall that a certain smoothness condition on V is necessary for $W_\pm$ to be bounded in L^p for all $1 \leq p \leq \infty$ by virtue of the counter-example of Golberg-Vissan ([6]) for the dispersive estimates for dimensions $m \geq 4$.

In dimension $m = 2$, the factor $(s_1 \dots s_n)^{m-2}$ is missing from (3.5) and it is evident that estimates (3.7) nor (3.8) do not hold. Nonethless, we have the following result.

Lemma 3.5. *Let $m = 2$. Then, for any $s > 1$ and $1 < p < \infty$, we have*

$$\|\Omega_1 u\|_p \leq C_{ps}\|\langle x\rangle^s V\|_2\|u\|_p.$$

If $\tilde{\chi}(\lambda) \in C^\infty(\mathbf{R})$ vanishes near $\lambda = 0$, then for any $s > 2$ and $1 < p < \infty$, we have

$$\|\Omega_2 \tilde{\chi}(H_0) u\|_p \leq C_{ps}\|\langle x\rangle^s V\|_2^2\|u\|_p.$$

3.2 High energy estimate

We let $\chi \in C_0^\infty(\mathbf{R})$ and $\tilde{\chi} \in C^\infty(\mathbf{R})$ be such that

$$\chi(\lambda) = 1 \text{ for } |\lambda| < \varepsilon,\ \chi(\lambda) = 0 \text{ for } |\lambda| > 2\varepsilon \text{ for some } \varepsilon > 0$$
$$\text{and } \chi(\lambda^2) + \tilde{\chi}(\lambda)^2 = 1 \text{ for all } \lambda \in \mathbf{R}.$$

Then, the high energy part of the wave operator $W\tilde{\chi}(H_0)$ may be studied by a unified method for all $m \geq 2$ and we may show that W is bounded in $\mathbf{B}(L^p(\mathbf{R}^m))$ for all $1 \leq p \leq \infty$ when $m \geq 3$ and for $1 < p < \infty$ for $m = 2$:

Theorem 3.6. *Let V satisfy $|V(x)| \le C\langle x\rangle^{-\delta}$ for some $\delta > m+2$. Suppose, in addition, that $\mathcal{F}(\langle x\rangle^{\sigma} V) \in L^{m_*}(\mathbf{R}^m)$ if $m \ge 4$. Then $W_{\pm}\tilde{\chi}(H_0)$ is bounded in $\mathbf{B}(L^p(\mathbf{R}^m))$ for all $1 \le p \le \infty$ when $m \ge 3$ and for $1 < p < \infty$ for $m = 2$.*

We outline the proof. We write $\nu = (m-2)/2$. Iterating the resolvent equation, we have

$$G(\lambda)V = \sum_{1}^{2n}(-1)^{j-1}(G_0(\lambda)V)^j + G_0(\lambda)N_n(\lambda)$$

where $N_n(\lambda) = (VG_0(\lambda))^{n-1}VG(\lambda)V(G_0(\lambda)V)^n$. If we substitute this for $G(\lambda)V$ in the stationary formula (3.3), we obtain

$$W\tilde{\chi}(H_0)^2 = \tilde{\chi}(H_0)^2 + \sum_{j=1}^{2n}(-1)^j\Omega_j\tilde{\chi}(H_0)^2 - \tilde{\Omega}_{2n+1}, \tag{3.11}$$

$$\tilde{\Omega}_{2n+1} = \frac{1}{i\pi}\int_0^{\infty} G_0(\lambda)N_n(G_0(\lambda) - G_0(-\lambda))\tilde{\Psi}(\lambda)d\lambda, \tag{3.12}$$

where $\tilde{\Psi}(\lambda) = \lambda\tilde{\chi}(\lambda^2)^2$. The operators $\tilde{\chi}(H_0)$ and $\Omega_1\tilde{\chi}(H_0)^2, \dots, \Omega_{2n}\tilde{\chi}(H_0)^2$ are bounded in $L^p(\mathbf{R}^m)$ for any $1 \le p \le \infty$ if $m \ge 3$ and for $1 < p < \infty$ if $m = 2$ by virtue of Lemma 3.3 and Lemma 3.5, since $\tilde{\chi}(H_0)$ is clearly bounded in $L^p(\mathbf{R}^m)$ for all $1 \le p \le \infty$ and $m \ge 2$. We then show that, for sufficiently large n, $\tilde{\Omega}_{2n+1}$ is also bounded in $L^p(\mathbf{R}^m)$ for all $1 \le p \le \infty$ and $m \ge 2$ by showing that its integral kernel

$$\tilde{\Omega}_{2n+1}(x,y) = \frac{1}{\pi i}\int_0^{\infty}\langle N_n(\lambda)(G_0(\lambda) - G_0(-\lambda))\delta_y, G_0(-\lambda)\delta_x\rangle\lambda\Psi^2(\lambda^2)d\lambda,$$

where $\delta_a = \delta(x-a)$ is the unit mass at the point $x = a$, satisfies the estimate that

$$\sup_{x\in\mathbf{R}^m}\int|\tilde{\Omega}_{2n+1}(x,y)|dy < \infty \quad \text{and} \quad \sup_{y\in\mathbf{R}^m}\int|\tilde{\Omega}_{2n+1}(x,y)|dx < \infty. \tag{3.13}$$

It is a result of Schur's lemma that estimates (3.13) imply that $\tilde{\Omega}_{2n+1}$ is bounded in $L^p(\mathbf{R}^m)$ for all $1 \le p \le \infty$. Note that $[G_0(\lambda)\delta_y](x) = G_0(\lambda, x-y)$ is the integral kernel of $G_0(\lambda)$ and $G_0(\lambda, x)$ is given by

$$G_0(\lambda, x) = \frac{e^{i\lambda|x|}}{2(2\pi)^{\nu+\frac{1}{2}}\Gamma(\nu+\frac{1}{2})|x|^{m-2}}\int_0^{\infty} e^{-t}t^{\nu-\frac{1}{2}}\left(\frac{t}{2} - i\lambda|x|\right)^{\nu-\frac{1}{2}}dt. \tag{3.14}$$

As a slight modification of the argument is necessary for the case $m = 2$, we restrict ourselves to the case $m \geq 3$ and, for definiteness, we assume m is even in what follows in this subsection. We define

$$\tilde{G}_0(\lambda, z, x) = e^{-i\lambda|x|} G_0(\lambda, x - z)$$

and

$$T_\pm(\lambda, x, y) = \langle N_n(\lambda)\tilde{G}_0(\pm\lambda, \cdot, y), \tilde{G}_0(-\lambda, \cdot, x)\rangle \tag{3.15}$$

so that

$$\tilde{\Omega}_{2n+1}(x, y) = \frac{1}{\pi i}\int_0^\infty \left(e^{i\lambda(|x|+|y|)} T_+(\lambda, x, y) - e^{i\lambda(|x|-|y|)} T_-(\lambda, x, y)\right) \tilde{\Psi}(\lambda) d\lambda. \tag{3.16}$$

We may compute derivatives $\tilde{G}_0^{(j)}(\lambda, z, x)$ with respect to λ using Leibniz's formula. If we set $\psi(z, x) = |x - z| - |x|$, they are linear combinations over (α, β) such that $\alpha + \beta = j$ of

$$\frac{e^{i\lambda\psi(z,x)}\psi(z, x)^\alpha}{|x - z|^{m-2-\beta}} \int_0^\infty e^{-t} t^{\nu-\frac{1}{2}} \left(\frac{t}{2} - i\lambda|x - z|\right)^{\nu-\frac{1}{2}-\beta} dt.$$

Since $|\psi(z, x)|^\alpha \leq \langle z\rangle^j$ for $0 \leq \alpha \leq j$ and

$$|z - x| \leq C_\varepsilon \left|\frac{t}{2} - i\lambda|z - x|\right| \leq C_\varepsilon(t + \lambda|z - x|)$$

when $|\lambda| \geq 1$, we have for $|\lambda| \geq \varepsilon$

$$\left|\left(\frac{\partial}{\partial\lambda}\right)^j \tilde{G}_0(\lambda, z, x)\right| \leq C_j \left(\frac{\langle z\rangle^j}{|x - z|^{m-2}} + \frac{\lambda^{\frac{m-3}{2}}\langle z\rangle^j}{|x - z|^{\frac{m-1}{2}}}\right). \tag{3.17}$$

for $j = 0, 1, 2, \ldots$.

Note that $\tilde{G}_0(\lambda, z, x) \sim C|x-z|^{2-m}$ near $z = x$ and $\tilde{G}_0(\lambda, z, x) \notin L^2_{\mathrm{loc}}(\mathbf{R}^m_z)$ for a fixed x if $m \geq 4$. However, the LAP (3.1) implies

$$\|\langle x\rangle^{-\gamma-j} G_0^{(j)}(\lambda)\langle x\rangle^{-\gamma-j}\|_{\mathbf{B}(H^s, H^{s+2})} \leq C_{sj\gamma}|\lambda|, \quad |\lambda| \geq \varepsilon \tag{3.18}$$

for any $\gamma > 1/2$, $s \in \mathbf{R}$ and $j = 0, 1, \ldots$ and k times application of $G_0(\lambda)V$ to $\tilde{G}_0(\lambda, \cdot, x)$, $k > (m-2)/2$, makes it into a function in $L^2_{-\gamma}(\mathbf{R}^m_z)$ for any $\gamma > 1/2$. Thus, if we take $n = k > (m-2)/2$, $T_\pm(\lambda, x, y)$ are well defined continuous functions of (x, y) which are $(m+2)/2$ times continuously differentiable with respect to λ. This, however, produces the increasing factor λ^k by virtue of the increase of the norm of (3.18). We, therefore, take n larger so that $n > m$ and use the fact (3.1) that $\|\langle x\rangle^{-\gamma-j}G_0^{(j)}(\lambda)\langle x\rangle^{-\gamma-j}\|_{\mathbf{B}(L^2,L^2)} \le C|\lambda|^{-1}$ decays as $\lambda \to \pm\infty$. Then, the decay property of extra factors $(G_0^{(j)}(\lambda)V)^{n-k}$ cancels this increasing factor and makes $T_\pm(\lambda,x,y)$ integrable with respect to λ. Using also the fact that $\tilde{G}_0(\lambda,\cdot,x) \sim |x|^{-\frac{m-1}{2}}$ as $|x| \to \infty$, we in this way obtain the following estimate:

Lemma 3.7. *Let* $0 \le s \le \frac{m+2}{2}$. *We have*

$$\left|\left(\frac{\partial}{\partial\lambda}\right)^s T_\pm(\lambda,x,y)\right| \le C_{ns}\lambda^{-3}\langle x\rangle^{-\frac{m-1}{2}}\langle y\rangle^{-\frac{m-1}{2}} \tag{3.19}$$

To obtain the desired estimate for $\tilde{\Omega}_{2n+1}(x,y)$, we apply integration by parts $0 \le s \le (m+2)/2$ times with respect to the variable λ in (3.16):

$$\begin{aligned}&\int_0^\infty e^{i\lambda(|x|\pm|y|)}T_\pm(\lambda,x,y)\tilde{\Psi}(\lambda)d\lambda\\ &= \frac{1}{(|x|\pm|y|)^s}\int_0^\infty e^{i\lambda(|x|\pm|y|)}\left(\frac{\partial}{\partial\lambda}\right)^s\left(T_\pm(\lambda,x,y)\tilde{\Psi}(\lambda)\right)d\lambda\end{aligned}$$

and estimate the right hand side by using (3.19). We obtain

$$|\tilde{\Omega}_{n+1}(x,y)| \le C\sum_\pm\langle|x|\pm|y|\rangle^{-\frac{m+2}{2}}\langle x\rangle^{-\frac{m-1}{2}}\langle y\rangle^{-\frac{m-1}{2}}.$$

It is then an easy exercise to show that $\tilde{\Omega}_{n+1}(x,y)$ satisfies the estimate (3.13).

3.3 Low energy estimate, generic case

By virtue of the intertwining property we have $W_\pm\chi(H_0)^2 = \chi(H)W_\pm\chi(H_0)$ and, from (3.3), we may write the low energy part $W_\pm\chi(H_0)^2$ as the sum of $\chi(H)\chi(H_0)$ and

$$\Omega = \frac{i}{\pi}\int_0^\infty \chi(H)G_0(\lambda)V(1+G_0(\lambda)V)^{-1}(G_0(\lambda)-G_0(-\lambda))\chi(H_0)\,\lambda d\lambda. \tag{3.20}$$

Here $\chi(H_0)$ and $\chi(H)$ both are integral operators of which the integral kernels satisfy for any $N > 0$

$$|\chi(H_0)(x,y)| \le C_N\langle x-y\rangle^{-N}, \quad |\chi(H)(x,y)| \le C_N\langle x-y\rangle^{-N} \tag{3.21}$$

and are, a fortiori, bounded in $L^p(\mathbf{R}^m)$ (see [16]). If H is of generic type and $m \geq 3$ is odd, then $(1 + G_0(\lambda)V)^{-1}$ has no singularities at $\lambda = 0$ and we may prove that Ω is bounded in $L^p(\mathbf{R}^m)$ for all $1 \leq p \leq \infty$ by proving that its integral kernel $\Omega(x, y)$ satisfies the estimate (3.13) by a method similar to the one used for the high energy part. The argument is simpler in the point that we do not have to expand $(1 + G_0(\lambda)V)^{-1}$ since the range of the integration with respect to λ in (3.20) is compact and since the integral kernels of $G_0(\lambda)\chi(H_0)$ and $G_0(\lambda)\chi(H)$ have no singularalities at the diagonal set by virtue of (3.21). It is, however, more complicated than in the high energy case in that the integral kernels of

$$\frac{i}{\pi}\int_0^\infty \chi(H)G_0(\lambda)V(1 + G_0(\lambda)V)^{-1}G_0(\pm\lambda)\chi(H_0)\,\lambda d\lambda,$$

do not separately satisfy the estimate (3.13) but only their difference does.

If H is of generic type and m is even, then $(1 + G_0(\lambda)V)^{-1}$ or its derivatives contain logarithmic singulaities at $\lambda = 0$ which are stronger when the dimensions are lower. Thus, the anaysis becomes more involved than the odd caseparticularly when $m = 2$ and $m = 4$. However, basically the idea as in the odd dimensional case works and we obtain the following theorem. We write $B(x, 1) = \{y \in \mathbf{R}^m \colon |y - x| < 1\}$.

Theorem 3.8. *Suppose that H is of generic type:*

(1) *Let $m = 2$. Suppose that V satisfies $|V(x)| \leq C\langle x\rangle^{-6-\varepsilon}$ for some $\varepsilon > 0$. Then, $W_\pm$ are bounded in L^p for all $1 < p < \infty$.*

(2) *Let $m = 3$. Suppose that V satisfies $|V(x)| \leq C\langle x\rangle^{-5-\varepsilon}$ for some $\varepsilon > 0$. Then, $W_\pm$ are bounded in L^p for all $1 \leq p \leq \infty$.*

(3) *Let $m = 4$. Suppopse that V satisfies for some $q > 2$*

$$\|V\|_{L^q(B(x,1))} + \|\nabla V\|_{L^q(B(x,1)} \leq C\langle x\rangle^{-7-\varepsilon}$$

for some $\varepsilon > 0$. Then, $W_\pm$ are bounded in L^p for all $1 \leq p \leq \infty$.

(4) *Let $m \geq 5$. Suppose that V satisfies $|V(x)| \leq C\langle x\rangle^{-m-2-\varepsilon}$ for some $\varepsilon > 0$ in addition to $\mathcal{F}(\langle x\rangle^{2\sigma}V) \in L^{m_*}(\mathbf{R}^m)$ for some $\sigma > 1/m_*$. Then, $W_\pm$ are bounded in L^p for all $1 \leq p \leq \infty$.*

Remark 3.9. When $m = 2$, at the end point, the same remark as in the one dimension applies: We believe $W_\pm$ are not bounded in L^1 nor in L^∞ at the end point and they are bounded from Hardy space H^1 into L^1 and L^∞ to BMO. However, we have no proofs.

3.4 Low energy estimate, exceptional case

We assume H is of exceptional type in this subsection. Then, $(1+G_0(\lambda)V)^{-1}$ of (3.20) is not invertible at $\lambda = 0$ and it has singularities at $\lambda = 0$. As we have no result when $m = 2$ and only a partial result when $m = 4$ which we mention at the end of this subsection, we assume $m = 3$ or $m \geq 5$ before the statement of Theorem 3.12. We study the singularities of $(1 + G_0(\lambda)V)^{-1}$ as $\lambda \to 0$ by expanding $1 + G_0(\lambda)V$ with respect to λ around $\lambda = 0$ and examining the structure of $1 + G_0(0)V$. The result is: If $m \geq 3$ is odd, we have

$$(1 + G_0(\lambda)V)^{-1} = \lambda^{-2}P_0V + \lambda^{-1}A_{-1} + 1 + A_0(\lambda)$$

where A_{-1} is a finite rank operator involving 0 eigenfunctions and the resonance function and $A_0(\lambda)$ has no singularities; if $m \geq 6$ is even, then

$$(1 + G_0(\lambda)V)^{-1} = \frac{P_0V}{\lambda^2} + \sum_{j=0}^{2}\sum_{k=1}^{2}\lambda^j(\log\lambda)^k D_{jk} + I + A_0(\lambda), \qquad (3.22)$$

where D_{jk} are finite rank operators involving 0 eigenfunctions and $A_0(\lambda)$ has no singularities. We substitute this expression for $(1 + G_0(\lambda)V)^{-1}$ in (3.20). Then, the operator produced by $I + A_0(\lambda)$ may be treated as in the previous section for the case when H is of gereric type. The operators produced by singular terms may be treated by using the machinaries of harmonic analysis, the wighted inequalities for the Hilbert transform and the Hardy-Littlewood maximal functions, which is a little too complicated to explain here. In this way we otain the following theorem. We refer the readers to [19] and [5] for the proof respectively for odd and even dimensional case.

Theorem 3.10. *Suppose that H is of exceptional type.*

(1) *Let $m \geq 3$ be odd. Suppose that V satisfies $|V(x)| \leq C\langle x\rangle^{-m-3-\varepsilon}$ for some $\varepsilon > 0$ and $\mathcal{F}(\langle x\rangle^{2\sigma}V) \in L^{m_*}(\mathbf{R}^m)$ in addition for some $\sigma > 1/m_*$. Then, $W_\pm$ are bounded in $L^p(\mathbf{R}^m)$ between $m/(m-2)$ and $m/2$.*

(2) *Let $m \geq 6$ be even. Suppose that V satisfies $|V(x)| \leq C\langle x\rangle^{-m-3-\varepsilon}$ if $m \geq 8$, $|V(x)| \leq C\langle x\rangle^{-m-4-\varepsilon}$ if $m = 6$ for some $\varepsilon > 0$ and $\mathcal{F}(\langle x\rangle^{2\sigma}V) \in L^{m_*}(\mathbf{R}^m)$ for some $\sigma > 1/m_*$ in addition. Then, $W_\pm$ are bounded in $L^p(\mathbf{R}^m)$ for $m/(m-2) < p < m/2$.*

Remark 3.11. When H is of exceptional type, the $W_\pm$ are not bounded in $L^p(\mathbf{R}^m)$ if $p > m/2$ and $m \geq 5$, or if $p > 3$ and $m = 3$. This can be deduced from the results on the decay in time property of the propagator

$e^{-itH}P_{ac}$ in the weighted L^2 spaces [12, 7], or in L^p spaces [4, 18]. We believe the same is true for p's on the other side of the interval given in (b), viz. $1 \le p \le m/(m-2)$ if $m \ge 5$ and $1 \le p \le 3/2$ if $m = 3$, but we have again no proofs.

In the case when $m = 2$ or $m = 4$, and if 0 is a resonance of H, then the results of [12] and [7] mentioned above imply that the $W_\pm$ are not bounded in $L^p(\mathbf{R}^m)$ for $p > 2$ and, though proof is missing, we believe that this is the case for all p's except $p = 2$. However, when $m = 4$ and if 0 is a pure eigenvalue of H and not a resonance, the $W_\pm$ are bounded in $L^p(\mathbf{R}^4)$ for $4/3 < p < 4$:

Theorem 3.12. *Let $|V(x)| + |\nabla V(x)| \le C\langle x\rangle^{-\delta}$ for some $\delta > 7$. Suppose that 0 is an eigenvalue of H, but not a resonance. Then the $W_\pm$ extend to bounded operators in the Sobolev spaces $W^{k,p}(\mathbf{R}^4)$ for any $0 \le k \le 2$ and $4/3 < p < 4$:*

$$\|W_\pm u\|_{W^{k,p}} \le C_p \|u\|_{W^{k,p}}, \qquad u \in W^{k,p}(\mathbf{R}^4) \cap L^2(\mathbf{R}^4). \tag{3.23}$$

We do not explain the proof of this theorem and refer the readres to the recent preprint [8]. Again, the results of [12, 7] imply that the $W_\pm$ are unbounded in $L^p(\mathbf{R}^4)$ if $p > 4$ under the assumption of Theorem 3.12. We believe that this is the case also for $1 \le p < 4/3$, though we do not have proofs.

References

[1] G. Artbazar and K. Yajima, *The L^p-continuity of wave operators for one dimensional Schrödinger operators*, J. Math. Sci. Univ. Tokyo **7** (2000), 221-240.

[2] J. Bergh and J. Löfström, *Interpolation spaces, an introduction*, Springer Verlag, Berlin-Heidelberg-New York (1976).

[3] P. D'Ancona and L. Fanelli, *L^p-boundedness of the wave operator for the one dimensional Schrödinger operator*, preprint (2005).

[4] M. B. Erdoğan and W. Schlag, *Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three I.* Dynamics of PDE, **1** (2004), 359.

[5] D. Finco and K. Yajima, *The L^p boundedness of wave operators for Schrödinger operators with threshold singularities II. Even dimensional case*, to appear in J. Math. Sci. Univ. Tokyo.

[6] M. Goldberg and M. Visan, *A counter example to dispersive estimates for Schrödinger operators in higher dimensions*, preprint (2005).

[7] A. Jensen, *Spectral properties of Schrödinger operators and time-decay of the wave functions. Results in $L^2(\mathbf{R}^4)$*, J. Math. Anal. Appl. **101** (1984), 397–422.

[8] A. Jensen and K. Yajima, *A remark on L^p-boundedness of wave operators for two dimensional Schrödinger operators*, Commun. Math. Phys. **225** (2002), 633–637.

[9] A. Jensen and K. Yajima, *A remark on L^p boundedness of wave operators for four dimensional Schrödinger operators with threshold singularities.*

[10] T. Kato, *Wave operators and similarity for non-selfadjoint operators*, Ann. Math. **162** (1966), 258–279.

[11] S. T. Kuroda, *Introduction to Scattering Theory*, Lecture Notes, Aarhus University

[12] M. Murata, *Asymptotic expansions in time for solutions of Schrödinger-type equations*, J. Funct. Anal., **49** (1982), 10–56.

[13] E. M. Stein, *Harmonic analysis, real-variable methods, orthogonality, and oscillatory integrals*, Princeton Univ. Press, Princeton, NJ. (1993).

[14] R. Weder, *L^p-$L^{p'}$ estimates for the Schrödinger equations on the line and inverse scattering for the nonliner Schrödinger equation with a potential*, J. Funct. Anal. **170** (2000), 37–68.

[15] K. Yajima, *The $W^{k,p}$-continuity of wave operators for Schrödinger operators*, J. Math. Soc. Japan **47** (1995), 551-581.

[16] K. Yajima, *The $W^{k,p}$-continuity of wave operators for Schrödinger operators III*, J. Math. Sci. Univ. Tokyo **2** (1995), 311–346.

[17] K. Yajima, *The L^p-boundedness of wave operators for two dimensional Schrödinger operators*, Commun. Math. Phys. **208** (1999), 125–152.

[18] K. Yajima, *Dispersive estimates for Schrödinger equations with threshold resonance and eigenvalue*, Commun. Math. Phys. **259** (2005), 475-509.

[19] K. Yajima, *The L^p boundedness of wave operators for Schrödinger operators with threshold singularities I, Odd dimensional case*, J. Math. Sci. Univ. Tokyo **7** (2006), 43–93.