<table>
<thead>
<tr>
<th>Title</th>
<th>A cohomology group of a $\mathbb{Z}_2$-orbifold model of the symplectic fermionic vertex operator superalgebra (Group Theory and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Abe, Toshiyuki</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2007 1564: 103-111</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2007-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/81139">http://hdl.handle.net/2433/81139</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
A cohomology group of a $\mathbb{Z}_2$-orbifold model of the symplectic fermionic vertex operator superalgebra

Toshiyuki Abe
Ehime university

1 Introduction

In this report we calculate a cohomological group of a model of an irrational $C_2$-cofinite simple vertex operator algebra. The cohomological group is considered by Miyamoto in a study on the category of modules for $C_2$-cofinite vertex operator algebras, and this result is just a calculation of a concrete example. In my talk, I introduced a homology of a certain functor. But the functor we considered is left exact, and hence the homology should be considered as a cohomology. In this report we consider the cohomological group of the simple vertex operator algebra $SF^+$ which is one of examples of irrational $C_2$-cofinite vertex operator algebra.

2 Preliminaries

We do not state the definition of vertex operator algebras and its modules. For them, please refer to the literatures [LL], [MN] and [FHL]. Let $(V, Y(\cdot, x), 1, \omega)$ be a simple vertex operator algebra over $\mathbb{C}$, and $(M, Y(\cdot, x))$ a weak $V$-module. We write $Y(a, x) = \sum_{n \in \mathbb{Z}} a_{(n)} x^{-n-1}$ for $a \in V$ following [MN], where $a_{(n)} \in \text{End } M$. We also write $L_n$ for the $n$-th mode $\omega(n)$ of the Virasoro vector $\omega$. The vacuum vector $1$ satisfies that for any $a \in V$ and $i \in \mathbb{Z}_{\geq 0}$, $a_{(i)} 1 = 0$ and $a_{(-1)} 1 = a$.

A vacuum-like vector $u \in M$ is a vector $u \in M$ satisfying $a_{(i)} u = 0$ for any $a \in V$ and $i \in \mathbb{Z}_{\geq 0}$. We set $\text{Vac}(M)$ to be the set of all vacuum-like vectors in $M$. It is known that

$$\text{Vac}(M) = \ker L_{-1} = \{u \in M \mid L_{-1} u = 0\}.$$ 

Actually, $L_{-1} = \omega_{(0)}$ shows that $\text{Vac}(M) \supset \ker L_{-1}$. On the other hand if $u \in \ker L_{-1}$, then $(-i)^{-1} a_{(i)} u = \frac{1}{k!} L_{-1}^k a_{(i+k)} u$. Since $a_{(j)} u = 0$ for sufficiently

---

1The original title is "A homology group of a $\mathbb{Z}_2$-orbifold model of the symplectic fermionic vertex operator superalgebra.

2After my talk, Professors Matsuo and Arakawa gave me this advice. I apologize that I made audience confused a lot according to my knowledgeless.
large positive integer $j$ and $(-^{i-1}k) \neq 0$ for any $i, k \in \mathbb{Z}_{\geq 0}$, we see that $a_{(i)}u = 0$ and that $u \in \text{Vac}(M)$.

We note that $\text{Vac}(M)$ is included in the $L_0$-eigenspace $M_0$ of weight 0 because $L_0 = \omega_{(1)}$. Thus if $L_0$ does not have any eigenvector in $M$, then $\text{Vac}(M) = 0$.

**Proposition 2.1.** ([Li]) Let $u \in \text{Vac}(M)$, and suppose that $u \neq 0$. Then the $V$-submodule $\langle u \rangle$ of $M$ generated from $u$ is isomorphic to $V$. A linear map $V \rightarrow \langle u \rangle$ defined by $a \mapsto a_{(-1)}u$ is a $V$-module isomorphism.

**Proof.** Let $f : V \rightarrow \langle u \rangle$ be a linear map given by $f(a) = a_{(-1)}u$. It is known that $\langle u \rangle$ is spanned by vectors of the form $a_{(m)}u$ with $a \in V$ and $m \in \mathbb{Z}$. Since $u \in \text{Vac}(M)$, we see that $\langle u \rangle$ is in fact spanned by $a_{(-m)}u$ with $a \in V$ and $m \in \mathbb{Z}_{>0}$. Thus $f$ is surjective. We also see that $\langle u \rangle = \{a_{(-1)}u | a \in V\}$ because $(m - 1)!a_{(-m)}u = (L_{-1}^{m-1}a)_{(-1)}u$ for $m \in \mathbb{Z}_{>0}$.

Now we see that

$$f(a_{(n)}b) = (a_{(n)}b)_{(-1)}u = \sum_{i=0}^{\infty} \binom{n}{i} (-1)^i (a_{(n-i)}b_{(-1+i)}u - (-1)^n b_{(n-1-i)}a_{(i)}u) = a_{(n)}b_{(-1)}u = a_{(n)}f(b)$$

for $a, b \in V$ and $n \in \mathbb{Z}$. Therefore, $f$ is a $V$-module homomorphism. Finally ker $f$ is a proper ideal of $V$ and hence ker $f = 0$ because $V$ is simple. Thus $f$ is a $V$-module isomorphism. \hfill $\square$

## 3 A cohomological group associated to $V$

Suppose that the adjoint module $V$ has an injective resolution:

$$0 \rightarrow V \rightarrow X^0 \xrightarrow{f_0} \cdots \rightarrow X^n \xrightarrow{f_n} X^{n+1} \xrightarrow{f_{n+1}} \cdots \quad (\text{exact}).$$

Then we have a cochain complex

$$0 \rightarrow \text{Vac}(X^0) \xrightarrow{r_0} \text{Vac}(X^1) \xrightarrow{r_1} \cdots \rightarrow \text{Vac}(X^n) \xrightarrow{r_n} \text{Vac}(X^{n+1}) \xrightarrow{r_{n+1}} \cdots,$$

where $r_n = f_n|_{P^n}$. We denote the corresponding cohomological group by $H(V) = \bigoplus_{n=0}^{\infty} H^n(V)$;

$$H^n(V) = \ker r_n / \text{Im} r_{n-1}$$
for \( n \in \mathbb{Z}_{\geq 0} \), where \( r_{-1} = 0 \). The cohomological group is independent of the choice of injective resolutions.

A vertex operator algebra \( V \) is called \( C_{2}\)-cofinite if the subspace \( C_{2}(V) \) spanned by vectors of the form \( a(-2)b \) with \( a, b \in V \) has finite codimension in \( V \). If \( V \) is \( C_{2}\)-cofinite then we can show that any finitely generated weak \( V \)-module has a projective cover. Therefore, the contragredient module \( V' \) has a projective resolution. In particular, \( V \) has an injective resolution.

4 The vertex operator algebra \( SF^{+} \)

Let \( \mathfrak{h} \) be a finite dimensional vector space of dimension \( 2d \) with a nondegenerate skew-symmetric bilinear form \( \langle \cdot, \cdot \rangle \). Then the vector space \( \hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K \) has a Lie super-algebra structure as follows; the even part is \( \mathbb{C}K \) and the odd part is \( \mathfrak{h} \otimes \mathbb{C}[t^{\pm 1}] \), and the super-commutation relations are

\[
\{ \psi \otimes t^{m}, \phi \otimes t^{n} \} = m \langle \psi, \phi \rangle \delta_{m,-n}K, \quad [K, \hat{\mathfrak{h}}] = 0
\]

for \( \phi, \psi \in \mathfrak{h} \) and \( m, n \in \mathbb{Z} \).

Now we consider the super-algebra \( \mathcal{A} := U(\hat{\mathfrak{h}}) / \langle K - 1 \rangle \), where \( U(\hat{\mathfrak{h}}) \) is the universal enveloping algebra of \( \hat{\mathfrak{h}} \) and \( \langle K - 1 \rangle \) is the two-sided ideal of \( U(\hat{\mathfrak{h}}) \) generated by \( K - 1 \). Let \( I_{\geq 0} \) be the left ideal of \( \mathcal{A} \) generated by \( \psi \otimes t^{n} \) for all \( \psi \in \mathfrak{h} \) and \( n \in \mathbb{Z}_{\geq 0} \). We then have a left \( \mathcal{A} \)-module \( \mathcal{A}/I_{\geq 0} \) and denote it by \( SF \).\(^{3}\) It is clear that \( SF \) is isomorphic to the exterior algebra \( \Lambda(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]) \) as vector spaces. We write \( \psi_{(n)} \) for the left multiplication on \( SF \) by \( \psi \otimes t^{n} \) for \( \psi \in \mathfrak{h} \) and \( n \in \mathbb{Z} \). Let 1 be the image of the unit of \( \mathcal{A} \) in \( SF \). Then \( SF \) is spanned by vectors of the form

\[
\psi_{(-n_{1})}^{1} \psi_{(-n_{2})}^{2} \cdots \psi_{(-n_{r})}^{r} 1, \quad (\psi^{i} \in \mathfrak{h}, n_{i} \in \mathbb{Z}_{>0}).
\]

We define the vertex operator map \( Y(\cdot, z) : SF \to (\text{End} SF)[[z, z^{-1}]] \) by

\[
Y(1, z) = \text{id}_{T},
\]

\[
Y(\psi(-1)\mathbf{1}, z) = \sum_{n \in \mathbb{Z}} \psi_{(n)}z^{-n-1},
\]

\[
Y(\psi_{(-n_{1})}^{1} \psi_{(-n_{2})}^{2} \cdots \psi_{(-n_{r})}^{r} \mathbf{1}, z) = \partial^{(n_{1}-1)}Y(\psi_{(-1)}^{1} \mathbf{1}, z) \cdots \partial^{(n_{r}-1)}Y(\psi_{(-1)}^{r} \mathbf{1}, z).
\]

\(^{3}\)The notation \( SF \) comes from "Symplectic Fermion".
for $\psi, \psi_i \in h, n, n_i \in \mathbb{Z}_{>0}$, where $\partial^{(k)} := \frac{1}{k!} \frac{d^k}{dz^k}$ for $k \in \mathbb{Z}_{\geq 0}$.

Let $\{e^i, f^i\}_{i=1,\ldots,d}$ be a basis of $h$ satisfying

$$\langle e^i, e^j \rangle = \langle f^i, f^j \rangle = 0 \quad \text{and} \quad \langle e^i, f^j \rangle = -\delta_{i,j}$$

for $1 \leq i, j \leq d$. Then the Virasoro element $\omega$ is given by

$$\omega = \sum_{i=1}^{d} e_{(-1)}^i f_{(-1)}^i 1.$$ 

Finally we have a vertex operator superalgebra $(SF, Y(\cdot, z), 1, \omega)$ of central charge $-2d$.

The vertex operator superalgebra $SF$ has canonically an automorphism $\theta$ defined by

$$\theta(\psi^1_{(-n_1)} \psi^2_{(-n_2)} \cdots \psi^r_{(-n_r)} 1) = (-1)^r \psi^1_{(-n_1)} \psi^2_{(-n_2)} \cdots \psi^r_{(-n_r)} 1$$

for any $\psi_i \in h, n_i \in \mathbb{Z}_{>0}$. The fixed point set $SF^+$ of $SF$ for $\theta$ is the even part of the vertex operator superalgebra $SF$ and the $-1$-eigenspace $SF^-$ is the odd one. The even part $SF^+$ becomes a simple vertex operator algebra of central charge $-2d$, and $SF^-$ is an irreducible $SF^+$-module.

## 5 Projective and injective resolutions of $SF^+$

It is known that $SF^+$ has four irreducible modules (see [A]). These are given by $SF^\pm$ and irreducible components of the unique irreducible $\theta$-twisted $SF$-module. The lowest weights of $SF^+$ and $SF^-$ are $0$ and $1$ respectively. Those of other two irreducible $SF^+$-modules are $\frac{-d}{8}$ and $\frac{4-d}{8}$.

The two irreducible modules given as submodules of the irreducible $\theta$-twisted $SF$-module are projective and injective. This fact is not so easy but can be shown by using the structure of Zhu’s algebra of $SF^+$ studied in [A]. On the other hand, $SF^\pm$ are not projective nor injective. Their projective covers can be constructed as follows.

First we consider the $SF$-module $\widehat{SF} = \mathcal{A}/I_{>0}$, where $I_{>0}$ is a left ideal of $\mathcal{A}$ generated by $\psi \otimes t^n$ with $\psi \in h$ and $n \in \mathbb{Z}_{>0}$. We see that $\widehat{SF}$ is generated from the vector $\hat{1} = 1 + I_{>0}$ and that $\widehat{SF} \cong \Lambda(h \otimes \mathbb{C}[t^{-1}])$ as vector spaces. We define the action of $\theta$ on $\widehat{T}$ by

$$\theta(\psi^1_{(-n_1)} \psi^2_{(-n_2)} \cdots \psi^r_{(-n_r)} \hat{1}) = (-1)^r \psi^1_{(-n_1)} \psi^2_{(-n_2)} \cdots \psi^r_{(-n_r)} \hat{1}$$
for any \( \psi_i \in \mathfrak{h}, n_i \in \mathbb{Z}_{\geq 0} \). We denote by \( S\tilde{F}^\pm \) by the \( \pm 1 \)-eigenspace for \( \theta \). We note that they are \( SF^+ \)-modules and \( (S\tilde{F}^\pm)' \cong S\tilde{F}^\pm \) respectively. We use the following conjecture.

**Conjecture.** The \( SF \)-modules \( S\tilde{F}^\pm \) are projective and injective.

Assuming this conjecture is true, we can find that \( S\tilde{F}^\pm \) are projective covers of the \( SF^+ \)-module \( SF^\pm \) respectively as follows. By construction, we have an \( SF \)-module epimorphism \( \phi_0 : S\tilde{F} \to SF \) defined by

\[
\phi_0(\psi^1_{(-n_1)} \psi^2_{(-n_2)} \cdots \psi^r_{(-n_r)} 1) = \psi^1_{(-n_1)} \psi^2_{(-n_2)} \cdots \psi^r_{(-n_r)} 1
\]

for \( \psi_i \in \mathfrak{h}, n_i \in \mathbb{Z}_{\geq 0} \). By definition \( \phi_0 \) gives epimorphisms \( S\tilde{F}^\pm \to SF^\pm \) respectively. We set \( W_0 = \ker \phi_0 \). Then \( W_0 \) is an \( SF \)-submodule of \( S\tilde{F} \) generated from \( e^i_{(0)} 1 \) and \( f^i_{(0)} 1 \) for \( 1 \leq i \leq d \). We also see that \( W_0 = (W_0 \cap S\tilde{F}^+) \oplus (W_0 \cap S\tilde{F}^-) \) and the submodules \( W_0 \cap S\tilde{F}^\pm \) are indecomposable. Hence \( S\tilde{F}^\pm \) are projective covers of \( SF^\pm \) respectively.

We now state that \( SF \) has the following projective resolution.

**Theorem 5.1.** The \( SF^+ \)-module \( SF \) has a projective resolution

\[
\cdots \to P^{n+1} \to P^n \to \cdots \to P^0 \to SF \to 0,
\]

with \( P^n = S\tilde{F}^{\oplus h(n+1)} \); the direct sum of \( h(n) \)-copies of \( S\tilde{F} \).

The number \( h(n) \) is given as follows: Let

\[
A = \begin{pmatrix}
0 & 0 & \cdots & 0 & \binom{2d}{0} \\
-1 & 0 & \cdots & 0 & \binom{2d}{1} \\
0 & -1 & \cdots & 0 & \binom{2d}{2} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -1 & \binom{2d}{2d-1}
\end{pmatrix}
\]

be a \( 2d \times 2d \)-matrix, and set

\[
u^{(n)} = A^{n-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.
\]
Then $h(n)$ is the $2d$-th component of $v^{(n)}$. Hence

$$h(1) = 1, \quad h(2) = 2d, \quad h(3) = d(2d + 1), \quad \ldots.$$  

In the case $d = 1$, we have $d(n) = n$.

Since $\overline{SF}'$, the contragredient module to $\overline{SF}$, is isomorphic to $\overline{SF}$, by this theorem, we have an injective resolution

$$0 \rightarrow SF \rightarrow P^0 \rightarrow P^0 \rightarrow \cdots \rightarrow P^n \rightarrow \cdots.$$  

By studying the structure of $\overline{SF}$ in detail, we get

**Theorem 5.2.** The irreducible $SF^+$-modules $SF^\pm$ have injective resolutions

$$0 \rightarrow SF^\pm \rightarrow P^{0,\pm} \rightarrow \cdots \rightarrow P^{n,\pm} \rightarrow P^{n+1,\pm} \rightarrow \cdots$$  

respectively, where

$$P^{n,\pm} = \begin{cases} (SF^\pm)^{\oplus h(n+1)} & \text{if } n \text{ is even}, \\ (SF^\mp)^{\oplus h(n+1)} & \text{if } n \text{ is odd}. \end{cases}$$  

### 6. Cohomological group $H^\bullet(SF^+)$

By Theorem 5.2, we get the cochain complex

$$0 \rightarrow \text{Vac}(P^{0,+}) \xrightarrow{r_0} \text{Vac}(P^{1,+}) \xrightarrow{r_1} \cdots \xrightarrow{\cdots \rightarrow \text{Vac}(P^{n,+}) \xrightarrow{r_n} \text{Vac}(P^{n+1,+}) \xrightarrow{r_{n+1}} \cdots .}$$  

We note that $\text{Vac}(\overline{SF}^+) = \mathbb{C}e_0 \cdots e_d f_1(0) \cdots f_d(0) \hat{1}$ and $\text{Vac}(\overline{SF}^-) = 0$. Hence

$$\text{Vac}(P^{n,+}) \cong \begin{cases} \mathbb{C}^{h(n+1)} & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$  

for $n \geq 1$. We can observe that

$$\text{Im } r_n = 0 \quad \text{for } n \in \mathbb{Z}_{\geq 0},$$  

$$\ker r_n = \begin{cases} \text{Vac}(P^{n,+}) & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

Therefore, we have
Theorem 6.1.

\[ H^i(SF^+) \cong \mathbb{C}^{h(i+1)} \quad \text{if } i \text{ is even,} \]
\[ H^i(SF^+) = 0 \quad \text{if } i \text{ is odd.} \]

Remark 6.2. We can also define \( H^i(SF^-) \). Then we have \( H^i(SF^-) \cong 0 \) if \( i \) is even and \( H^i(SF^-) \cong \mathbb{C}^{h(i+1)} \) if \( i \) is odd.

7 A projective resolution in the case \( d = 1 \)

We explain the projective resolution of \( SF \) in the case \( d = 1 \). For simplicity, we set \( \epsilon = e^1 \) and \( f = f^1 \). In this case, the submodule \( \ker \phi_0 = W_0 \) is generated by \( e_{(0)} \hat{1} \) and \( f_{(0)} \hat{1} \), and the submodule generated from \( e_{(0)} f_{(0)} \hat{1} \) is isomorphic to \( SF \) because \( e_{(0)} f_{(0)} \hat{1} \) is a vacuum-like vector. Therefore, we have the following sequence of submodules:

\[ 0 \subset SF \subset W_0 \subset \hat{SF}. \]

One sees that \( \hat{SF}/W_0 \cong SF \) and \( W_0/SF \cong SF \oplus SF \).

Now we consider the \( SF \)-module epimorphism \( \phi_1 : \hat{SF} \oplus \hat{SF} \to W_0 \) defined by

\[ \phi_1(u \hat{1}, v \hat{1}) = u \epsilon_{(0)} \hat{1} + v f_{(0)} \hat{1}, \]

where \( u, v \in \Lambda(\mathfrak{h} \otimes \mathbb{C}[t^{-1}]) \). Then we see that the kernel of \( \phi_1 \), denoted by \( W_1 \), is the \( SF \)-submodule of \( \hat{SF} \oplus \hat{SF} \) generated by the vectors \( (\epsilon_{(0)} \hat{1}, 0) \), \( (f_{(0)} \hat{1}, \epsilon_{(0)} \hat{1}) \) and \( (0, f_{(0)} \hat{1}) \).

If we draw an extension of \( X \) by \( Y \) as

\[
\begin{array}{c}
X \\
\downarrow \\
Y
\end{array}
\]

then we have the following pictures;

\[
\begin{array}{c}
SF \\
\downarrow \\
\hat{SF} = \begin{array}{c}
SF \\
\downarrow \\
SF
\end{array}
\end{array}
\]
We also see that
\[
\begin{array}{c}
\xymatrix{
& SF 
& SF \\
SF 
& SF 
& SF 
& SF \\
SF 
& SF 
& SF 
& SF
}
\end{array}
\]
and
\[
\begin{array}{c}
\xymatrix{
& SF 
& SF \\
SF 
& SF 
& SF 
& SF \\
SF 
& SF 
& SF 
& SF
}
\end{array}
\]

By the same way, for \( n \in \mathbb{Z}_{>0} \), we consider a \( SF \)-module homomorphism \( \phi_{n-1} : SF^\oplus n \to SF^\oplus (n-1) \) defined by
\[
\phi_{n-1}(u^1, \ldots, u^n) = (u^1 e_{(0)} + u^2 f_{(0)} + \ldots + u^n f_{(0)})
\]
with \( u^1, \ldots, u^n \in \Lambda(\mathfrak{h} \otimes \mathbb{C}[t^{-1}]) \). Then we can show that
\[
\text{Im} \phi_n = \ker \phi_{n-1}
\]
for \( n \in \mathbb{Z}_{>0} \) and we have the exact sequence
\[
\cdots \xrightarrow{\phi_{n+1}} SF^\oplus (n+1) \xrightarrow{\phi_n} SF^\oplus n \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} SF^\oplus 0 \to SF \to 0.
\]

We recall the action of \( \theta \) on \( SF \). We extend the action of \( \theta \) to that on \( SF^\oplus n \) with diagonal action. Then it is easy to see that \( \theta \circ \phi_n \circ \theta = -\phi_n \) for any \( n \in \mathbb{Z}_{>0} \). Therefore, the projective resolution above gives rise to two projective resolutions
\[
\cdots \xrightarrow{\phi_{n+1}} (SF^\oplus (n+1))^\oplus \xrightarrow{\phi_n} (SF^\oplus n)^\oplus \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} SF^\oplus \to SF^\oplus \to 0,
\]
\[
\cdots \to (SF^\oplus)^\oplus 2 \xrightarrow{\phi_1} SF^\oplus \xrightarrow{\phi_0} SF^\oplus \to 0,
\]
where $\varepsilon_n^{\pm}$ is defined by

$$
\varepsilon_n^{\pm} = \begin{cases} 
\mp & \text{if } n \text{ is even} \\
\pm & \text{if } n \text{ is odd}.
\end{cases}
$$

References

[A] SF. Abe, A $\mathbb{Z}_2$-orbifold model of the symplectic fermionic vertex operator superalgebra, Math. Z., online.


