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HEISENBERG'S FUNDAMENTAL EQUATION AND QUANTUM FIELD THEORY WITH A FUNDAMENTAL LENGTH

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ABSTRACT. Heisenberg's fundamental equation of the universe [8, 9] has a coupling constant l which has the dimension of length [L]. We consider a model which has a coupling constant l of the same dimension as Heisenberg's fundamental equation of the universe, and solved it. The solution is not an operator-valued tempered distribution but an ultrahyperfunctions. The constant $\ell = l/(\sqrt{2}\pi)$ is the fundamental length in the sense of [1], that is, events occuring within the distance ℓ cannot be distinguished in the framework of Ultrahyperfunction Quantum Field Theory which has been developed recently by the authors.

CONTENTS

1. INTRODUCTION

The basic relativistic equation of quantum mechanics called Dirac equation

$$i\frac{\hbar}{c}\gamma_\mu\frac{\partial}{\partial x_\mu}\psi(x) - m\psi(x) = 0,\ x_0 = ct, x_1 = x, x_2 = y, x_3 = z \tag{1.1}$$

contains a constants c (velocity of light) which is the fundamental constant in relativity theory, and Planck's constant $h = 2\pi\hbar$ which is the fundamental constant in quantum mechanics. The dimension of c is $[\mathrm{LT}^{-1}]$ and that of h is $[\mathrm{ML}^2\mathrm{T}^{-1}]$. W. Heisenberg thought that a fundamental equation of Physics must also contain a constant l with the dimension of length [L]. If such a constant l is introduced, then the dimensions of any other quantity can be expressed in terms of combinations of the basic constants c, h and l, e.g., time $[\mathrm{T}] = [\mathrm{L}]/[\mathrm{LT}^{-1}]$, or mass as $[\mathrm{M}] = [\mathrm{ML}^2\mathrm{T}^{-1}]/([\mathrm{LT}^{-1}][\mathrm{L}])$.

In 1958, Heisenberg and Pauli introduced the equation

$$\frac{\hbar}{c}\gamma_\mu \frac{\partial}{\partial x_\mu}\psi(x) \pm l^2\gamma_\mu\gamma_5\psi(x)\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) = 0, \tag{1.2}$$

which was later called the *equation of the universe* and studied in [4, 9]. The constant l has the dimension [L] and is called the fundamental length of the theory.

Unfortunately, nobody has been able to solve this equation. At present, even in the more advanced framework of ultra hyperfunction quantum field theory, we do not see how this equation could be solved. Accordingly we study a linearized version of this equations which enherits the important property of a fundamental length l and which first has been studied by Okubo [13]. This linearized version is solvable in the sense of classical field theory. We write it in the form

$$\begin{cases} \Box\phi(x) + \left(\frac{cm}{\hbar}\right)^2\phi(x) = 0 \\ \left(i\frac{\hbar}{c}\gamma^\mu\frac{\partial}{\partial x^\mu} - \tilde{m}\right)\psi(x) = -2\gamma^\mu l^2\psi(x)\phi(x)\frac{\partial\phi(x)}{\partial x^\mu} \end{cases} \tag{1.3}$$

and propose to solve these equations by two methods, (i) constructing the Schwinger functions of the fields $\phi(x)$ and $\psi(x)$, and (ii) constructing directly the operator-valued generalized functions which satisfy the system of equations (1.3). In the following we will work with the natural units $c = \hbar = 1$. Then the system of equations (1.3) reads

$$(\Box + m^2)\phi(x) = 0 \tag{1.4}$$

$$(i\gamma^\mu\partial_\mu - \tilde{m})\psi = -2l^2\gamma^\mu\psi(x)\phi(x)\partial_\mu\phi(x). \tag{1.5}$$

and they are the field equations of the following Lagrangian density:

$$L(x) = L_{Ff}(x) + L_{Fb}(x) + L_I(x), \tag{1.6}$$

$$L_{Ff}(x) = \bar{\psi}(x)(i\gamma_\mu\partial^\mu - \tilde{m})\psi(x), \tag{1.7}$$

$$L_{Fb}(x) = \frac{1}{2}\{(\partial^\mu\phi(x))^2 - m^2\phi(x)^2\}, \tag{1.8}$$

$$L_I(x) = 2l^2(\bar{\psi}(x)\gamma_\mu\psi(x))\phi(x)\partial^\mu\phi(x). \tag{1.9}$$

Equation (1.5) has no solutions in the axiomatic framework of Wightman, that is, the field $\psi(x)$ is not an operator-valued tempered distribution. But, as we are going to show, Equation (1.5) has a solution (ϕ, ψ) as operator-valued tempered ultrahyperfunction which satisfy the conditions of the framework of a relativistic quantum field theory with a fundamental length as given in [1].

In Sections 2 - 5, we discuss the Wightman axioms for the fields with a fundamental length, which uses the theory of tempered ultrahyperfunctions. In Section 6, the Schwinger functions and Wightman functions are constructed by path integral method. The Wightman functions are tempered ultrahyperfunctions. In Section 7, the operator-valued tempered ultrahyperfunctions are constructed, which satisfy the system of equations (1.3). The Wightman functions constructed from the operator solutions coincide with those which are constructed by path integral method. The detailed calculations are found in [11, 2]

2. Wightman axioms

Wightman's set of axioms consist of the following 7 conditions. A special attention is paid to the locality axiom WVI.
W.I (Relativistic invariance of the state space).
W.II (Spectral property).
W.III (Existence and uniqueness of the vacuum).
W.IV (Fields and temperedness).
W.V (Poincaré-covariance of the fields).
W.VI (Locality, or microcausality).
Any two field components $\phi_j^{(\kappa)}(x)$ and $\phi_\ell^{(\kappa')}(y)$ either commute or anti-commute under a spacelike separation of x and y:
If f and g have space-like separated supports, then

$$\phi_j^{(\kappa)}(f)\phi_\ell^{(\kappa')}(g)\Psi \mp \phi_\ell^{(\kappa')}(g)\phi_j^{(\kappa)}(f)\Psi = 0$$

for all $\Psi \in \mathcal{D}$, the common domain for all operator $\phi_j^{(\kappa)}(f)$. We express this by saying

$$\phi_j^{(\kappa)}(x)\phi_\ell^{(\kappa')}(y)\Psi \mp \phi_\ell^{(\kappa')}(y)\phi_j^{(\kappa)}(x)\Psi = 0 \text{ for } (x-y)^2 < 0.$$

W.VII (Cyclicity of the vacuum).

3. Fundamental length

The axiom W.VI says that two events which are space-likely separated are independent. Even if we replace W.VI by a weaker axiom

$$\phi_j^{(\kappa)}(x)\phi_\ell^{(\kappa')}(y)\Psi \mp \phi_\ell^{(\kappa')}(y)\phi_j^{(\kappa)}(x)\Psi = 0 \text{ for } (x-y)^2 < -\ell^2 < 0, \tag{3.1}$$

which says that the two events which are separated by ℓ are independent, we can prove W.VI by using the other axioms. It is not easy to weaken the condition of locality if the field $\phi_j^{(\kappa)}(x)$ has the localization property of Schwartz distributions. We must introduce generalized functions which have more general localization properties than distributions.
We indicate briefly a way in which localization properties of generalized functions can be 'weakened'. Denote $T(-\ell,\ell) = \mathbb{R} + i(-\ell,\ell) \subset \boldsymbol{C}$, and let $\mathcal{T}(T(-\ell,\ell))$ be the set of functions f holomorphic in $T(-\ell,\ell)$. Then for $|a| < \ell$, we get

$$\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{a^n}{n!}\delta^{(n)}(x)f(x)dx = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} f^{(n)}(0)$$

$$= f(-a) = \int_{-\infty}^{\infty} \delta(x+a)f(x)dx.$$

The above equality implies the following two facts.

(A) $\Delta_N(x) = \sum_{n=0}^{N} \frac{a^n}{n!}\delta^{(n)}(x)$ converges to $\delta(x+a) = \delta_{-a}(x)$ in $\mathcal{T}(T(-\ell,\ell))'$ as $N \to \infty$. Clearly, for all $N \in \mathbb{N}$, $\operatorname{supp}\Delta_N = \{0\}$ while for the limit we find $\operatorname{supp}\delta_{-a} = \{-a\}$.

(B) If $|a| > \ell$, $\Delta_N(x)$ does not converge in $\mathcal{T}(T(-\ell,\ell))'$.

(A) and (B) say: Elements in $\mathcal{T}(T(-\ell,\ell))'$ do not allow to distinguish between $\{0\}$ and $\{-a\}$, if $|a| < \ell$, but if $|a| > \ell$ then elements in $\mathcal{T}(T(-\ell,\ell))'$ can be used to distinguish between the locations $\{0\}$ and $\{-a\}$.

4. Ultrahyperfunction

Tempered ultrahyperfunctios were first introduced by Hasumi, M. [7] in 1961 and developed by Morimoto, M. [10] in 1975. Here we just mention the basic definition. For a subset A of $\mathbb{R}^n$, we denote by $T(A) = \mathbb{R}^n + iA \subset \mathbb{C}^n$ the tubular set with base A. For a convex compact set K of $\mathbb{R}^n$, $\mathcal{T}_b(T(K))$ is, by definition, the space of all continuous functions f on $T(K)$ which are holomorphic in the interior of $T(K)$ and satisfy

$$\|f\|^{T(K),j} = \sup\{|z^p f(z)|; z \in T(K), |p| \leq j\} < \infty, \ j = 0, 1, \ldots$$

where $p = (p_1, \ldots, p_n)$ and $z^p = z_1^{p_1} \cdots z_n^{p_n}$. $\mathcal{T}_b(T(K))$ is a Fréchet space with the semi-norms $\|f\|^{T(K),j}$. If $K_1 \subset K_2$ are two compact convex sets, we have the canonical injections:

$$\mathcal{T}_b(T(K_2)) \to \mathcal{T}_b(T(K_1)).$$

Let O be a convex open set in $\mathbb{R}^n$. We define

$$\mathcal{T}(T(O)) = \varprojlim \mathcal{T}_b(T(K_1)),$$

where K_1 runs through the convex compact sets contained in O, and the projective limit is taken following the restriction mappings.

Definition 4.1. *A tempered ultra-hyperfunction is by definition a continuous linear functional on $\mathcal{T}(T(\mathbb{R}^n))$.*

Remark 4.2. It seems that the space $\mathcal{T}(T(\mathbb{R}^n))$ is quite unique, in the sense that it is not among the many spaces considered in the book of I.M. Gel'fand and G.E. Shilov [5]. There we find function spaces $\mathcal{S}^{1,B}$ and $\mathcal{S}^1 = \lim\limits_{B\to\infty} \mathcal{S}^{1,B} = \lim\limits_{K_1 \to \{0\}} \mathcal{T}_b(T(K_1))$,
but no space $\lim\limits_{0 \leftarrow B} \mathcal{S}^{1,B} = \lim\limits_{\mathbb{R}^n \leftarrow K_1} \mathcal{T}_b(T(K_1)) = \mathcal{T}(T(\mathbb{R}^n))$.

By the reason explained in Section 3, we can formulate relativistic quantum field theory with a fundamental length by using tempered ultrahyperfunctions, which is shown in [1] and in the next section.

5. Axioms for ultrahyperfunction quantum fields of mixed type

Here we state Wightman's axioms for the ultrahyperfunction quantum field theory. For the case of neutral scalar fields, these axioms have been presented in [1].

W.I. **Relativistic invariance of the state space;**

W.II. **Spectral property;**

W.III. **Existence and uniqueness of the vacuum;**

W.IV. **Fields**: The components $\phi_j^{(\kappa)}$ of the quantum field $\phi^{(\kappa)}$ are operator-valued generalized functions $\phi_j^{(\kappa)}(x)$ over the space $\mathcal{T}(T(\mathbb{R}^4))$ with common dense domain $\mathcal{D}$; i.e., for all $\Psi \in \mathcal{D}$ and all $\Phi \in \mathcal{H}$,

$$\mathcal{T}(T(\mathbb{R}^4)) \ni f \to (\Phi, \phi_j^{(\kappa)}(f)\Psi) \in \mathbb{C}$$

is a tempered ultrahyperfunction. It is assumed that the vacuum vector Φ_0 is contained in $\mathcal{D}$ and that $\mathcal{D}$ is taken into itself under the action of the operators $\phi_j^{(\kappa)}(f)$ and $U(a,A)$, i.e., $\phi_j^{(\kappa)}(f)\mathcal{D} \subset \mathcal{D}$, $U(a,A)\mathcal{D} \subset \mathcal{D}$. Moreover it is assumed that there exist indices $\bar{\kappa}$, $\bar{\jmath}$ such that $\phi_{\bar{\jmath}}^{(\bar{\kappa})}(\bar{f}) \subset \phi_j^{(\kappa)}(f)^*$ where * indicates the Hilbert space adjoint of the operator in question.

W.V. **Poincaré-covariance of the fields;**

W.VI. **Extended causality or extended local commutativity**: Any two field components $\phi_j^{(\kappa)}(x)$ and $\phi_l^{(\kappa')}(y)$ either commute or anti-commute if the distance between x and y is greater than ℓ:

a) The functionals

$$\mathcal{T}(T(\mathbb{R}^4)) \otimes \mathcal{T}(T(\mathbb{R}^4)) \ni f \otimes g \to (\Phi, \phi_j^{(\kappa)}(f)\phi_l^{(\kappa')}(g)\Psi) \quad \text{and}$$

$$\mathcal{T}(T(\mathbb{R}^4)) \otimes \mathcal{T}(T(\mathbb{R}^4)) \ni f \otimes g \to (\Phi, \phi_l^{(\kappa')}(g)\phi_j^{(\kappa)}(f)\Psi)$$

can be extended continuously to $\mathcal{T}(T(L^\ell))$ in some Lorentz frame, for arbitrary elements Φ, Ψ in the common domain $\mathcal{D}$ of the field operators $\phi_j^{\kappa}(f)$, where

$$T(L^\ell) = \{(z_1, z_2) \in \mathbb{C}^{4\cdot 2}; |\mathrm{Im}\, z_1 - \mathrm{Im}\, z_2|_1 < \ell\},$$

where $|y|_1 = |y^0| + \sqrt{\sum_{i=1} |y_i|^2}$.

b) The carrier of the functional

$$f \otimes g \to (\Phi, [\phi_j^{(\kappa)}(f), \phi_l^{(\kappa')}(g)]_{\mp}\Psi)$$

on $\mathcal{T}(T(\mathbb{R}^4)) \otimes \mathcal{T}(T(\mathbb{R}^4))$ is contained in the set

$$W^\ell = \{(z_1, z_2) \in \mathbb{C}^{4\cdot 2}; z_1 - z_2 \in V^\ell\},$$

where

$$V^\ell = \{z \in \mathbb{C}^4; \exists\, x \in V, |\mathrm{Re}\, z - x| + |\mathrm{Im}\, z|_1 < \ell\}$$

is a complex neighborhood of light cone V, i.e., this functional can be extended continuously to $\mathcal{T}(W^\ell)$.

W.VII. **Cyclicity of the vacuum.**

Remark 5.1. The condition (3.1) is expressed that the support of the vector-valued distribution $\phi_j^{(\kappa)}(x)\phi_\ell^{(\kappa')}(y)\Psi \mp \phi_\ell^{(\kappa')}(y)\phi_j^{(\kappa)}(x)\Psi$ is contained in the set $W_\ell = \{(x,y) \in \mathbb{R}^{4\cdots 2}; (x-y)^2 \geq -\ell^2\}$. However, the tempered ultrahyperfunction has a carrier but generally no support, the smallest carrier, and therefore we replace the condition (3.1) with the condition b) of W.VI, that is, the functional

$$f \otimes g \to (\Phi, [\phi_j^{(\kappa)}(f), \phi_l^{(\kappa')}(g)]_\mp \Psi)$$

is continuously extended to $\mathcal{T}(W^\ell)$ where W^ℓ is a complex neighborhood of W_ℓ.

6. Path integral quantization

We quantize this model by path integral methods (see [3]). Formally, the time-ordered two point function is calculated as

$$\int \bar{\psi}_\alpha(x_1)\psi_\beta(x_2) \exp i \left\{ \int_{\mathbb{R}^4} L_I(x)dx \right\} d\mathcal{D}(\psi, \bar{\psi}) d\mathcal{G}(\phi)$$

$$\times \left\{ \int \exp i \left\{ \int_{\mathbb{R}^4} L_I(x)dx \right\} d\mathcal{D}(\psi, \bar{\psi}) d\mathcal{G}(\phi) \right\}^{-1},$$

$$d\mathcal{G}(\phi) = \exp i \left\{ \int_{\mathbb{R}^4} L_{Fb}(x)dx \right\} \prod_{x \in \mathbb{R}^4} d\phi(x)$$

$$d\mathcal{D}(\psi, \bar{\psi}) = \exp i \left\{ \int_{\mathbb{R}^4} L_{Ff}(x)dx \right\} \prod_{x \in \mathbb{R}^4} \prod_{\alpha=1}^{4} d\psi_\alpha(x) d\bar{\psi}_\alpha(x).$$

All these integrals have a rigorous meaning if the continuous space-time is replaced by a lattice. For positive integers M, N define $L = MN$, $\Delta = \sqrt{\pi}/M$ and the lattice

$$\Gamma = \{t = j\Delta; j \in \mathbb{Z}, -L < j \leq L\} = \Delta\mathbb{Z}/(2\sqrt{\pi}N).$$

The lattice version of the differential operator $-\Delta + m^2$ on $\mathbb{R}^{\Gamma^4} = \mathbb{R}^{4 \cdot 2L}$ is the following difference operator

$$- \Delta + m^2 : \mathbb{R}^{\Gamma^4} \ni \Phi(x) \to$$

$$-\sum_{\mu=0}^{3} \frac{\Phi(x+e_\mu) + \Phi(x-e_\mu) - 2\Phi(x)}{\Delta^2} + m^2\Phi(x) \in \mathbb{R}^{\Gamma^4}.$$

Let $dG(\Phi)$ be a Gaussian measure on $\mathbb{R}^{4 \cdot 2L}$ defined by

$$dG(\Phi) = C \exp \left\{ \frac{1}{2} \sum_{y \in \Gamma^4} \left[\sum_{\mu=0}^{3} \frac{\Phi(y+e_\mu) + \Phi(y-e_\mu) - 2\Phi(y)}{\Delta^2} \right.\right.$$

$$\left.\left. -m^2\Phi(y) \right] \Delta^4 \right\} \prod_{y \in \Gamma^4} d\Phi(y),$$

where C is the normalization constant such that $\int dG(\Phi) = 1$. The exponent of the measure is the (Euclideanized $x^0 \to -iy^0$, $\boldsymbol{x} \to \boldsymbol{y}$) discretization of Lagrangian $i\int L_{Fb}(x)dx$.
Now we can calculate the covariance of $dG(\Phi)$

$$\int \Phi(y_1)\Phi(y_2)dG(\Phi) = 2(-\Delta + m)^{-1}(y_1, y_2) = 2\mathcal{S}_m(y_1 - y_2)$$

$$\mathcal{S}_m(y_1 - y_2) = (2\pi)^{-4} \sum_{p\in\tilde{\Gamma}^4} e^{ip(y_1-y_2)} \left[\sum_{\mu=0}^{3}(2 - 2\cos p_\mu\Delta)/\Delta^2 + m^2\right]^{-1} \eta^4,$$

$$\tilde{\Gamma} = \{s = j\eta; j \in \mathbb{Z}, -L < j \leq L, \eta = \sqrt{\pi}/N\} = \eta\mathbb{Z}/(2\sqrt{\pi}M).$$

The following fact is shown in [11] by using nonstandard analysis: $\mathcal{S}_m(y_1 - y_2) \to S_m(y_1 - y_2)$, $M, N \to \infty$, where

$$S_m(y_1 - y_2) = (2\pi)^{-4} \int_{\mathbb{R}^4} e^{ip(y_1-y_2)} \left[p^2 + m^2\right]^{-1} d^4p$$

is the Schwinger function of neutral scalar field of mass m. In order to deal with the fermion field Ψ, we need the measure $dD(\Psi^1, \Psi^2)$ on the Grassmann algebra generated by $\{\Psi^1_\alpha(y), \Psi^2_\alpha(y); \alpha = 1, \ldots, 4,\ y \in \Gamma^4\}$:

$$dD(\Psi^1, \Psi^2) = C' \exp\left\{-\sum_{y\in\Gamma^4} \Psi^{2T}(y)\left[\sum_{\mu=0}^{3}\gamma^E_\mu\nabla_\mu + \tilde{m}\right]\Psi^1(y)\Delta^4\right\}$$

$$\times \prod_{y\in\Gamma^4}\prod_{\alpha=1}^{4} d\Psi^1_\alpha(y)d\Psi^2_\alpha(y),$$

$$\Psi^1 = (\Psi^1_1, \ldots, \Psi^1_4)^T,\ \Psi^2 = (\Psi^2_1, \ldots, \Psi^2_4)^T,$$

$$\gamma^E_0 = \gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix},\ \gamma^E_j = -i\gamma_j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix},\ j = 1, 2, 3,$$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\nabla_\mu\Psi_k = \begin{cases} \nabla^+\Psi_k(y) = (\Psi_k(y + e_\mu) - \Psi_k(y))/\Delta & \text{if } k = 1, 2, \\ \nabla^-\Psi_k(y) = (\Psi_k(y) - \Psi_k(y - e_\mu))/\Delta & \text{if } k = 3, 4. \end{cases}$$

The idea to replace the partial derivatives in the continuum case by the forward-, respectively backward- difference as describe above, has originally been developed in [12] in order to avoid the doubling problem. Using $P_\pm = (1 \pm \gamma^E_0)/2$ the interaction Lagrangian is defined as:

$$-L_I(y) = \Psi^{2T}(y)e^{il^2\Phi(y)^2}\sum_{\mu=0}^{3}\gamma^E_\mu$$

$$\times[P_+\Psi^1(y + e_\mu)\{e^{-il^2\Phi(y+e_\mu)^2} - e^{-il^2\Phi(y)^2}\}/\Delta$$

$$+P_-\Psi^1(y-e_\mu)\{e^{-il^2\Phi(y)^2}-e^{-il^2\Phi(y-e_\mu)^2}\}/\Delta],$$

Now we calculate the lattice version of the Schwinger functions of the interacting fields. The two point Schwinger function is

$$\int \Psi^1_\alpha(y_1)\Psi^2_\beta(y_2)\exp\left(\sum_{y\in\Gamma^4} L_I(y)\Delta^4\right) dD(\Psi^1,\Psi^2)dG(\Phi)$$

$$\times\left\{\int \exp\left(\sum_{y\in\Gamma^4} L_I(y)\Delta^4\right) dD(\Psi^1,\Psi^2)dG(\Phi)\right\}^{-1}$$

$$=\int e^{il^2\Phi(y_1)^2}\Psi'^1(y_1)e^{-il^2\Phi(y_2)^2}\Psi'^2(y_2)dD(\Psi'^1,\Psi'^2)dG(\Phi)$$

$$=\int \Psi'^1(y_1)\Psi'^2(y_2)dD(\Psi'^1,\Psi'^2)\int e^{il^2\Phi(y_1)^2}e^{-il^2\Phi(y_2)^2}dG(\Phi)$$

where we used the change of variables

$$\Psi^1(y)=e^{il^2\Phi(y)^2}\Psi'^1(y),\ \ \Psi^2(y)=e^{-il^2\Phi(y)^2}\Psi'^2(y).$$

The covariance

$$\int \Psi'^1(y_1)\Psi'^2(y_2)dD(\Psi'^1,\Psi'^2)=\mathcal{R}_{\tilde{m};\alpha,\beta}(y_1-y_2)$$

converges to the Schwinger function

$$R_{\tilde{m};\alpha,\beta}(y)=\left\{-\sum_{\mu=0}^{3}\gamma^E_\mu\left(\frac{\partial}{\partial y_\mu}\right)+\tilde{m}\right\}_{\alpha,\beta} S_{\tilde{m}}(y)$$

of the free Dirac field of mass $\tilde{m}$. The integral

$$\int e^{il^2\Phi(y_1)^2}e^{-il^2\Phi(y_2)^2}dG(\Phi)$$

$$=\left[(1-il^2\mathcal{S}_m(0))(1+il^2\mathcal{S}_m(0))-l^4\mathcal{S}_m(y_1-y_2)^2\right]^{-1/2}.$$

contains $\mathcal{S}_m(0)$ which diverges to ∞ as $N,M\to\infty$. But this divergent quantity is removed by using the Wick products instead of the ordinary product in the Lagrangian. It is defined by (see [6]):

$$:e^{h\Phi(y)}:=\sum_{n=0}^{\infty}[:(h\Phi(y))^n:/n!]=e^{-h^2\mathcal{S}_m(0)}e^{h\Phi(y)}.$$

This removes the divergent quantity and the result is

$$\int :e^{il^2\Phi(y_1)^2}::e^{-il^2\Phi(y_2)^2}:dG(\Phi)=\left[1-4l^4\mathcal{S}_m(y_1-y_2)^2\right]^{-1/2},$$

and hence the two point Schwinger function of ψ on the lattice

$$\left[1-4l^4\mathcal{S}_m(y_1-y_2)^2\right]^{-1/2}\mathcal{R}_{\tilde{m};\alpha,\beta}(y_1-y_2),$$

converges to the continuum one:

$$\left[1 - 4l^4 S_m(y_1 - y_2)^2\right]^{-1/2} R_{\tilde{m};\alpha,\beta}(y_1 - y_2).$$

Let $D_m^{(-)}(x_0 - i\epsilon, \boldsymbol{x}) = S_m(ix_0 + \epsilon, \boldsymbol{x})$. Then the two point Wightman function $W_{\alpha,\beta}(x_0 - i\epsilon, \boldsymbol{x})$ is

$$\left[1 - 4l^4 D_m^{(-)}(x_0 - i\epsilon, \boldsymbol{x})^2\right]^{-1/2} (i\gamma_\mu \partial^\mu + \tilde{m})_{\alpha,\beta} D_{\tilde{m}}^{(-)}(x_0 - i\epsilon, \boldsymbol{x}).$$

It follows from the relations

$$|D_m^{(-)}(x_0 - i\epsilon, \boldsymbol{x})| \leq (2\pi\epsilon)^{-2}, \quad \epsilon^2 D_m^{(-)}(-i\epsilon, \mathrm{o}) \to (2\pi)^{-2} \ (\epsilon \to 0)$$

that if $\epsilon > \sqrt{2}l/(2\pi)$, then $|4l^4 D_m^{(-)}(x_0 - i\epsilon, \boldsymbol{x})^2| < 1$ and

$$\left[1 - 4l^4 D_m^{(-)}(z_0, \boldsymbol{x})^2\right]^{-1/2} (i\gamma_\mu \partial^\mu + \tilde{m})_{\alpha,\beta} D_{\tilde{m}}^{(-)}(x_0 - i\epsilon, \boldsymbol{x})$$

is holomorphic for $\operatorname{Im} z_0 > \sqrt{2}l/(2\pi)$ and defines a ultrahyperfunction $W_{\alpha,\beta}$ by the formula

$$W_{\alpha,\beta}(f) = \int_{\mathbb{R}^4} W_{\alpha,\beta}(x_0 - i\epsilon, \boldsymbol{x}) f(x_0 - i\epsilon, \boldsymbol{x}) dx$$

for all $f \in \mathcal{T}(T(O_s))$, $O_s = \{x \in \mathbb{R}^4; \|x\| < s\}$ for some $s > \sqrt{2}l/(2\pi)$. $\sqrt{2}l/(2\pi)$ is the fundamental length in the sense of W.VI, i.e., two events within the distance $\sqrt{2}l/(2\pi)$ cannot be distinguished. Thus the parameter l which appears in the equation (1.5) turns out to be the fundamental length. The full sequence of n point Wightman ultrahyperfunctions is calculated in [11].

7. Operator solution

Recall the system of equations (1.3) which which we plan to solve

$$\begin{cases} \Box\phi(x) + \left(\dfrac{cm}{\hbar}\right)^2 \phi(x) = 0 \\ \left(i\dfrac{\hbar}{c}\gamma_\mu \dfrac{\partial}{\partial x_\mu} - \tilde{m}\right)\psi(x) = -2\gamma_\mu l^2 \psi(x)\phi(x)\dfrac{\partial\phi(x)}{\partial x_\mu} \end{cases}.$$

The basic ideas to solve the above system is quite natural: Take the free Klein-Gordon field ϕ and suppose that we can show that (i)

$$\rho(x) =: e^{il^2\phi(x)^2} := \sum_{n=0}^{\infty} i^n l^{2n} : \phi(x)^{2n} : /n!$$

is well-defined as an operator-valued ultrahyperfunction, and that (ii) it satisfies

$$\frac{\partial}{\partial x^\mu}\rho(x) = 2il^2 : e^{il^2\phi(x)^2}\phi(x)\frac{\partial}{\partial x^\mu}\phi(x) := 2il^2 : \rho(x)\phi(x)\frac{\partial}{\partial x^\mu}\phi(x) :,$$

and that (iii) the free Dirac field $\psi_0(x)$ is a multiplier for the field ρ.
Then define the field

$$\psi(x) = \psi_0(x)\rho(x),$$

and calculate

$$\left(i\gamma_\mu \frac{\partial}{\partial x^\mu} - \tilde{m}\right)\psi(x) = \left[\left(i\gamma_\mu \frac{\partial}{\partial x^\mu} - \tilde{m}\right)\psi_0(x)\right]\rho(x) + \gamma_\mu \psi_0(x)\frac{\partial}{\partial x^\mu}\rho(x)$$

$$= -2l^2\gamma_\mu\psi_0(x) : \rho(x)\phi(x)\frac{\partial}{\partial x^\mu}\phi(x) := -2l^2\gamma_\mu : \psi(x)\phi(x)\frac{\partial}{\partial x^\mu}\phi(x) : .$$

Thus the operator-valued ultrahyperfunction $\psi(x)$ satisfies the equation (1.5).

While (i) is shown in E. Brüning and S. Nagamachi [1] we use the fundamental formula of A.S. Wightman and L. Gårding [14]

$$(: D^{\alpha^{(1)}}\phi D^{\alpha^{(2)}}\phi \cdots D^{\alpha^{(l)}}\phi : (f)\Phi)^{(n)}(\xi_1, \ldots, \xi_n)$$

$$= \frac{\pi^{l/2}}{(2\pi)^{2(l-1)}} \sum_{j=0}^{l} \left[\frac{(n-l+2j)!}{n!}\right]^{1/2} \int \cdots \int \left(\prod_{k=1}^{j} d\Omega_m(\eta_k)\right)$$

$$\times \sum_{1 \le k_1 < k_2 < \ldots < k_{l-j} \le n} (j!)^{-1} \sum_P P\left((-i\eta_1)^{\alpha^{(1)}} \cdots (-i\eta_j)^{\alpha^{(j)}} (i\xi_{k_1})^{\alpha^{(j+1)}} \cdots (i\xi_{k_{l-j}})^{\alpha^{(l)}}\right.$$

$$\left.\times \tilde{f}\left(\sum_{r=1}^{j}\eta_r - \sum_{r=1}^{l-j}\xi_{k_r}\right)\right)\Phi^{(n-l+2j)}(\eta_1, \ldots, \eta_j, \xi_1,$$

$$\ldots, \hat{\xi}_{k_1}, \ldots, \hat{\xi}_{k_{l-j}}, \ldots, \xi_n).$$

to show (ii). From this formula we get

$$(: \phi^l : (-\frac{\partial}{\partial x^\mu}f)\Phi)^{(n)} = l(: (\frac{\partial}{\partial x^\mu}\phi)\phi^{l-1} : (f)\Phi)^{(n)},$$

$$\frac{\partial}{\partial x^\mu} : \phi(x)^l : \Phi = l : (\frac{\partial}{\partial x^\mu}\phi(x))\phi^{l-1}(x) : \Phi.$$

$\Phi = \rho(f_1)\cdots\rho(f_n)\Phi_0$, $f_j \in \mathcal{T}(T(\mathbb{R}^4))$, Φ_0: vacuum for ρ (and ϕ). It follows

$$\sum_{l=0}^{\infty}(ig)^l : \phi^{2l} : (-\frac{\partial}{\partial x^\mu}f)\Phi/l!$$

$$= \sum_{l=1}^{\infty}(ig)^l 2 : (\frac{\partial}{\partial x^\mu}\phi)\phi\phi^{2(l-1)} : (f)\Phi/(l-1)!$$

$$= 2(ig) : (\frac{\partial}{\partial x^\mu}\phi)\phi\sum_{l=0}^{\infty}(ig)^l\phi^{2l} : (f)\Phi/l! = 2ig : (\frac{\partial}{\partial x^\mu}\phi)\phi\rho : (f)\Phi.$$

$$\frac{\partial}{\partial x^\mu}\rho(x)\Phi = 2ig : (\frac{\partial}{\partial x^\mu}\phi(x))\phi(x)\rho(x) : \Phi.$$

(iii): From the system of axioms one proves that the vector-valued function $\rho(z_1)\cdots\rho(z_n)\Phi_0$ is holomorphic in

$$\{(z_1, \ldots, z_n) \in \mathbb{C}^{4n}; \mathrm{Im}\, z_1 \in V_+ + (\ell_1, \mathrm{o}), \mathrm{Im}\,(z_j - z_{j-1}) \in V_+ + (\ell_j, \mathrm{o})\}$$

for some $\ell_j > \ell > 0$ $(j = 1, \ldots, n)$ (see [2]). Let Ψ_0 is the vacuum for ψ_0. Then $\psi_{0,\alpha_1}(z_1) \cdots \psi_{0,\alpha_n}(z_n)\Psi_0$ is holomorphic in

$$\{(z_1, \ldots, z_n) \in \mathbb{C}^{4n}; \operatorname{Im} z_1 \in V_+, \operatorname{Im}(z_j - z_{j-1}) \in V_+\}$$

Therefore, $\psi_{0,\alpha_1}(z)\Psi$ for $\Psi = \psi_{0,\alpha_2}(g_2) \cdots \psi_{0,\alpha_n}(g_n)\Psi_0$, $g_j \in \mathcal{S}(\mathbb{R}^4)$ and $\rho(z)\Phi$ for $\Phi = \rho(f_2) \cdots \rho(f_n)\Phi_0$, $f_j \in \mathcal{T}(T(\mathbb{R}^4))$ are holomorphic in

$$\{z \in \mathbb{C}^4; \operatorname{Im} z \in V_+ + (\ell_1, \mathbf{o})\}.$$

The product $(\psi_{0,\alpha}\rho)(f)$ is defined by

$$(\psi_{0,\alpha}\rho)(f)(\Psi \otimes \Phi) =$$

$$= \int_{\Gamma_N} f(z)\psi_{0,\alpha}(z)\Psi \otimes \rho(z)\Phi dz, \ \Gamma_N = \{z \in \mathbb{C}^4; z = x + i(N, \mathbf{o})\}$$

for suitable $N > 0$. Moreover, we have $\quad (\frac{\partial}{\partial x^\mu}(\psi_{0,\alpha}\rho))(f)\Psi \otimes \Phi$

$$= (\psi_{0,\alpha}\rho)(-\frac{\partial}{\partial x^\mu} f)\Psi \otimes \Phi = \int_{\Gamma_N} (-\frac{\partial}{\partial x^\mu} f(z))\psi_{0,\alpha}(z)\Psi \otimes \rho(z)\Phi dz$$

$$= \int_{\Gamma_N} f(z)\{(\frac{\partial}{\partial x^\mu}\psi_{0,\alpha}(z)\Psi) \otimes \rho(z)\Phi + \psi_{0,\alpha}(z)\Psi \otimes \frac{\partial}{\partial x^\mu}\rho(z)\Phi\}dz$$

$$= ((\frac{\partial}{\partial x^\mu}\psi_{0,\alpha})\rho)(f)\Psi \otimes \Phi + (\psi_{0,\alpha}\frac{\partial}{\partial x^\mu}\rho)(f)\Psi \otimes \Phi.$$

Thus it follows

$$\frac{\partial}{\partial x^\mu}(\psi_{0,\alpha}(x)\rho(x))(\Psi \otimes \Phi) =$$

$$= (\frac{\partial}{\partial x^\mu}\psi_{0,\alpha}(x))\rho(x)\Psi \otimes \Phi + \psi_{0,\alpha}(x)\frac{\partial}{\partial x^\mu}\rho(x)\Psi \otimes \Phi,$$

and we deduce that Equation (1.5) holds. The Wightman functions for the field $\psi(x) = \psi_0(x)\rho(x)$ is calculated as

$$(\Psi_0 \otimes \Phi_0, \psi_{\alpha_1}(z_1) \cdots \psi_{\alpha_n}(z_n)\Psi_0 \otimes \Phi_0)$$

$$= (\Psi_0, \psi_{0,\alpha_1}(z_1) \cdots \psi_{0,\alpha_n}(z_n)\Psi_0)(\Phi_0, \rho(z_1) \cdots \rho(z_n)\Phi_0).$$

These Wightman functions coincide with those which are calculated by the path integral method in Section 6. A detailed analysis is found in [2].

References

[1] E. Brüning and S. Nagamachi. Relativistic quantum field theory with a fundamental length. *J. Math. Phys.*, 45:2199–2231, 2004.

[2] E. Brüning and S. Nagamachi. Solutions of a linearized model of Heisenberg's fundamental equation II. 2007.

[3] A. Das. *Field Theory: A path integral approach.* World Scientific, Singapre New Yersey London Hong Kong, 1993.

[4] H.-P. Dürr, W. Heisenberg, H. Mitter, S. Schlieder, and K. Yamazaki. Zur Theorie der Elementarteilchen. *Z. Naturf.*, 14a:441–485, 1959.

5 I.M. Gelfand and G.E. Silov. *Spaces of Fundamental and Generalized Functions*, volume 2 of *Generalized Functions*. Academic Press, New York London, 1968.

6 J. Glimm and A. Jaffe. *Quantum Phyiscs: A functional integral point of view*. Springer-Verlag, New York Heidelberg Berlin, 1981.

7 M. Hasumi. Note on the n-dimensional tempered ultra-distributions. *Tohoku Math. J.*, 13:94–104, 1961.

8 W. Heisenberg. Über die quantentheoretische Umdeutung kinematischer und mechnischer Beziehungen. *Zeitsch. f. Phys.*, 33:879, 1925.

9 W. Heisenberg. *Introduction to the Unified Field Theory of Elementary Particles*. John Wiley & Sons, 1966.

10 M. Morimoto. Theory of Tempered Ultrahyyperfunctions I. *Proc. Japan Acad.*, 51:87–81, 1975.

11 S. Nagamachi and E. Brüning. Solutions of a linearized model of Heisenberg's fundamental equation I. 2007.

12 S. Nagamachi and N. Mugibayashi. Nonstandard analysis of Euclidean Fermi fields. 1986.

13 S. Okubo. Green's Function in Some Covariant Soluble Problems. *Nuovo Cimeno*, 19:574–585, 1961.

14 A. S. Wightman and L. Gårding. Fields as operator-valued distributions in relativistic quantum theory. *Arkiv för Fysik*, 28:129–184, 1964.

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