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QUANTUM CAUSALITY AND QUANTUM CONTROL: A MODEL, DUALITY AND AXIOMS

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ABSTRACT. Quantum mechanical systems exhibit an inherently probabilistic nature upon measurement which excludes in principle the singular direct observability continual case. Quantum theory of time continuous measurements and quantum prediction theory, developed by the author on the basis of an independent-increment model for quantum noise and nondemolition causality principle in the 80's, solves this problem allowing continual quantum predictions and reducing many quantum information and quantum feedback control problems to the classical stochastic ones. Using explicit indirect observation models for diffusive and counting measurements we derive quantum filtering (prediction) equations to describe the stochastic evolution of the open quantum system under the continuous partial observation. The resulting filtering and Bellman equation for the diffusive observation is then applied to the explicitly solvable quantum linear-quadratic-Gaussian (LQG) problem which emphasizes many similarities and differences with the corresponding classical nonlinear filtering and control problems and demonstrates microduality between quantum filtering and classical control.

1. INTRODUCTION

The purpose of this paper is to build on the original work of the author and present an accessible account of the theory of quantum continual measurements, quantum causality and predictions and optimal quantum feedback control. Firstly we introduce the necessary concepts and mathematical tools from modern quantum theory including quantum probability, continuous causal (non-demolition) measurements, quantum stochastic calculus, and quantum filtering. However, we first start from a model example of quantum filtering and the feedback control problem. This is important because it is one of the few exactly solvable control problems which emphasizes the similarities between the corresponding classical and quantum filtering and control theories. It allows us to set up notations and clearly demonstrates not only the similarity but also the difference of classical and quantum feedback control theories which can be observed in *microduality principle*, a more elaborated duality between quantum linear Gaussian filtering and classical linear optimal control.

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2. Model example: quantum free particle

The quantum linear filtering and optimal quadratic control problem with quantum Gaussian noise was first studied and resolved by the author in a series of quantum measurement and filtering papers [5],[28],[16], and based on these quantum feedback control papers [6],[8],[3]. The simplest example of a single quantum Gaussian oscillator matched with a transmission line, [28] as a complex one-dimensional channel, was taken as a quantum feedback model in the starting preprint [6] eventually published in [4]. However, a more similar to the classical case quantum linear models require at least two real dimensions instead of a single complex one; and we may now use the multidimensional quantum LQG control solutions derived in the last Section of this paper for application on higher dimensional systems which do not have such complex representation. The optimal control of a continuously observed quantum free particle with quadratic cost is the simplest such example.

Let $\breve{x}_\bullet = (\breve{x}_1, \breve{x}_2)$ be a pair of phase space operators $\breve{x}_1 = \breve{q}$, $\breve{x}_2 = \breve{p}$ for a quantum particle in one dimension, given by selfadjoint operators of position $\breve{q}$ and momentum $\breve{p}$ satisfying the canonical commutation relation (CCR)

$$[\breve{q}, \breve{p}] := \breve{q}\breve{p} - \breve{p}\breve{q} = \mathrm{i}\hbar\breve{\mathrm{I}}. \tag{2.1}$$

Here $\breve{\mathrm{I}}$ is the identity operator in a Hilbert space $\mathfrak{h}$ of the CCR representation (2.1) and $\hbar$ is called Planck constant, which for our purpose could be any positive constant $\hbar \geq 0$. Let us denote the row of initial expectations $\langle \breve{x}_j \rangle$ of $\breve{x}_j$ in a quantum Gaussian state by $x_\bullet = (q, p)$, and also denote the initial dispersions of $\breve{q}$ and $\breve{p}$ by σ_q and σ_p respectively and the initial symmetric covariance $\mathrm{Re}\,\langle \breve{q}\breve{p} \rangle - qp$ by $\sigma_{qp} = \sigma_{pq}$. The Hamiltonian $\breve{p}^2/2\mu$ of the free particle is perturbed by a controlling force, using the linear potential $\phi(t, \breve{q}) = \beta u(t)\, \breve{q}$ with $u(t) \in \mathbb{R}$, as $H(u) = \breve{p}^2/2\mu + \beta u \breve{q}$ where $\mu > 0$ is the mass of the particle. The particle is assumed to be coupled not only to control which can be realized by a quantum coherent (forward) channel, but also to a coherent observation (estimation) quantum channel such that its open Heisenberg dynamics is described by quantum Langevin equations:

$$\mathrm{d}Q(t) + \lambda Q(t)\,\mathrm{d}t = \frac{1}{\mu}P(t)\,\mathrm{d}t + \mathrm{d}W_q^t,\ Q(0) = \breve{q} \tag{2.2}$$

$$\mathrm{d}P(t) + \lambda P(t)\,\mathrm{d}t = \mathrm{d}V_p^t - \beta u(t)\,\mathrm{d}t,\ P(0) = \breve{p} \tag{2.3}$$

Here $\lambda = \frac{1}{2}(\alpha\varepsilon + \beta\gamma)$ and $V_p^t = \alpha V_e^t + \beta V_f^t$, $W_q^t = -\varepsilon W_e^t - \gamma W_f^t$ are given by two independent pairs $(V_\circ, W_\circ)$ (where $\circ = e, f$ stands for error and force) of Wiener noises $V_\circ = \hbar\Im(A_\circ^+)$, $W_\circ = 2\Re(A_\circ^+)$ due to the interaction with the coupled estimation and feedback channels. Note that these noises do not commute,

$$W_q^s V_p^r - V_p^r W_q^s = (r \wedge s)\,\mathrm{i}\hbar\lambda I, \tag{2.4}$$

if $\lambda \neq 0$, which is necessary and sufficient condition for preservation of the CCR (2.1) by the system (2.2), (2.3). It can be easily found by substituting the solution

$$P(t) = \mathrm{e}^{-\lambda t}\breve{p} + \int_0^t \mathrm{e}^{(s-t)\lambda}\left(\mathrm{d}V_p^s - \beta u(s)\,\mathrm{d}s\right)$$

of the second equation (2.3) into the first one (2.2), that

$$[Q(r), Q(s)] = \frac{\mathrm{i}\hbar}{\mu}|r - s|\,\mathrm{e}^{-\lambda|r-s|} \neq 0.$$

Therefore the family $\{Q(t)\}$ is incompatible and cannot be represented as a classical stochastic process and directly observed. However, it can be indirectly observed by continuous measuring of the coupling operator $\alpha\check{q}$ with an error white noise in the estimation channel as it was suggested in [3],[13]. To this end we measure $W_e^t = 2\Re(A_e^+)$ as an evolved input process, after an interaction with the particle, onto an output classical process given by a commutative family $[Y_e^t : t > 0]$ in the linear estimation channel

$$\mathrm{d}Y_e^t = \alpha Q(t)\,\mathrm{d}t + \mathrm{d}W_e^t. \tag{2.5}$$

Here the input process appears as measurement error noise with commutative independent increments $\mathrm{d}W_e^t$, representing the standard Wiener process such that $(\mathrm{d}W_e)^2 = \mathrm{d}t$, but noncommuting with the perturbative force V_p^t since,

$$\mathrm{d}W_e^t \mathrm{d}V_p^t = \frac{\alpha\hbar}{2\mathrm{i}}\mathrm{d}t,\ \mathrm{d}W_e^t \mathrm{d}W_q^t = -\varepsilon \mathrm{d}t. \tag{2.6}$$

Thus the measurement error noise W_e^t satisfies the error-perturbation CCR

$$\left[V_p^r, W_e^s\right] = (r \wedge s)\,\mathrm{i}\hbar\alpha I, \tag{2.7}$$

which is a necessary and sufficient condition for quantum causality (or quantum nondemolition condition) in the form

$$[Y_e^r, Q(s)] = 0 = [Y_e^r, P(s)] \ \forall r \leq s \tag{2.8}$$

requiring the statistical predictability of *quantum hidden in the future trajectories* $\{X_\bullet(s) : s \geq t\}$ with respect to the *classical observed in the past trajectories* $\{Y_e^r : r \leq t\}$ for each t. From this we derive the Heisenberg *error-perturbation uncertainty principle* in the precise *Belavkin inequality* form [28],[16]

$$\left(\mathrm{d}V_p^t\right)^2 \geq \left(\frac{\alpha\hbar}{2}\right)^2 \mathrm{d}t,\ \left(\mathrm{d}W_e^t\right)^2 = \mathrm{d}t \tag{2.9}$$

in terms of the perturbation V_p^t in (2.3) and standard error W_e^t in (2.5). Thus we have the case

$$\mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \boldsymbol{\lambda} = \begin{pmatrix} \frac{\alpha}{2} & \frac{\mathrm{i}\varepsilon}{\hbar} \\ \frac{\beta}{2} & \frac{\mathrm{i}\gamma}{\hbar} \end{pmatrix} \tag{2.10}$$

of the general quantum linear open system considered in the last Section, where $\boldsymbol{\lambda}$ is the direct sum $\boldsymbol{\lambda}_e \oplus \boldsymbol{\lambda}_f$ of two rows $\boldsymbol{\lambda}_e, \boldsymbol{\lambda}_f$ corresponding to $\boldsymbol{b}_e = (\alpha, 0)$, $\boldsymbol{e} = (0, \varepsilon)$, $\boldsymbol{b} = (\beta, 0)$, $\boldsymbol{e}_f = (0, \gamma)$. From this we compute the matrices $\boldsymbol{g}, \boldsymbol{h}$ satisfying the microduality principle, which are turned to be diagonal,

$$\boldsymbol{g} = \begin{pmatrix} \zeta_q & 0 \\ 0 & \zeta_p \end{pmatrix},\ \boldsymbol{h} = \begin{pmatrix} \eta_q & 0 \\ 0 & \eta_p \end{pmatrix},$$

with eigenvalues $\zeta_q = \gamma^2$, $\zeta_p = (\hbar/2)^2\left(\alpha^2 + \beta^2\right) = \eta_q$, $\eta_p = \varepsilon^2$.

3. Quantum feedback control example

We can now apply the results obtained in the last Section to demonstrate optimal quantum filtering and optimal feedback control and their microduality on this model example. The optimal estimates of the position and momentum based on a nondemolition observation of free quantum particle via the continuous measurement of Y_t, originally derived in [3],[13] in the absence of a control channel, are then given by the Belavkin Kalman filter in the form of linear stochastic equations

$$\mathrm{d}\hat{q}_0^t + \hat{q}_0^t \lambda \mathrm{d}t = \frac{1}{\mu}\hat{p}_0^t \mathrm{d}t + \left(\alpha\sigma_q(t) - \varepsilon\right)\mathrm{d}\hat{W}_e^t \tag{3.1}$$

$$\mathrm{d}\hat{p}_0^t + \hat{p}_0^t \lambda \mathrm{d}t = \beta u(t)\,\mathrm{d}t + \alpha\sigma_{qp}(t)\,\mathrm{d}\hat{W}_e^t. \tag{3.2}$$

Here the estimation innovation process $\hat{W}_e^t$ describes the gain of information due to measurement of Y_e^t given by

$$\mathrm{d}\hat{W}_e^t = \mathrm{d}Y_e^t - \alpha\hat{q}_0^t \mathrm{d}t, \tag{3.3}$$

and the error covariances satisfy the Riccati equations

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\sigma_q &= \zeta_q + 2\left(\tfrac{1}{\mu}\sigma_{qp} + \sigma_q\delta\right) - (\alpha\sigma_q)^2 \\ \tfrac{\mathrm{d}}{\mathrm{d}t}\sigma_{qp} &= \tfrac{1}{\mu}\sigma_p - (\lambda - \delta)\,\sigma_{qp} - \alpha^2\sigma_q\sigma_{qp} \\ \tfrac{\mathrm{d}}{\mathrm{d}t}\sigma_p &= \zeta_p - 2\lambda\sigma_p - (\alpha\sigma_p)^2, \end{aligned} \tag{3.4}$$

where we denote $\delta = \frac{1}{2}(\alpha\varepsilon - \gamma\beta)$, with initial conditions

$$\sigma_q(0) = \sigma_q, \quad \sigma_{qp}(0) = \sigma_{qp}, \quad \sigma_p(0) = \sigma_p.$$

The Riccati equations for the error covariance in the filtered free particle dynamics have an exact solution [13] with profound implications for the ultimate quantum limit satisfying the Heisenberg uncertainty relations for the accuracy of optimal quantum state estimation via the continuous indirect quantum particle coordinate measurement.

The dual optimal control problem can be found by identifying the corresponding dual matrices which give the quadratic control parameters

$$\begin{aligned} \check{c}(u) &= (u - \check{z})^2 + \eta_q\check{q}^2 + \eta_p\check{p}^2, \\ \check{s} &= \omega_q\check{q}^2 + \omega_{qp}(\check{p}\check{q} + \check{q}\check{p}) + \omega_p\check{p}^2 \end{aligned} \tag{3.5}$$

corresponding to the dual output process given by $\check{z} = \gamma\check{p}$. For the linear Gaussian system this gives the optimal control strategy

$$u(t) = \beta(\omega_{pq}(t)\,\hat{p}_0^t + \omega_p(t)\,\hat{q}_0^t) \tag{3.6}$$

where the coefficients $\omega(t)$ are the solutions to the Riccati equations

$$\begin{aligned} -\tfrac{\mathrm{d}}{\mathrm{d}t}\omega_q(t) &= \eta_q - 2\lambda\sigma_q - (\beta\sigma_q)^2 \\ -\tfrac{\mathrm{d}}{\mathrm{d}t}\omega_{qp}(t) &= \tfrac{1}{\mu}\omega_q - (\lambda + \delta)\,\omega_{qp} - \beta^2\omega_p\omega_{qp} \\ -\tfrac{\mathrm{d}}{\mathrm{d}t}\omega_p(t) &= \eta_p + 2\left(\tfrac{1}{\mu}\omega_{qp} - \omega_p\delta\right) - (\beta\omega_p)^2 \end{aligned} \tag{3.7}$$

with terminal conditions

$$\omega_p(T) = \omega_p, \quad \omega_{qp}(T) = \omega_{qp}, \quad \omega_q(T) = \omega_q.$$

Note that in this example, as well as identifying the dual matrices by transposition and time reversal according to the duality, one must also symplecticly interchange the phase coordinates $(\check{q}, \check{p}) \leftrightarrow (\check{p}, \check{q})$. This is because the matrix of coefficients **a** is non-symmetric and nilpotent, so it is dual to its transpose only when we interchange the coordinates in the dual picture. Thus the optimal coefficients $\{\omega_p, \omega_{qp}, \omega_q\}(t)$ in the quadratic cost-to-go correspond to the minimal error covariances $\{\sigma_q, \sigma_{qp}, \sigma_p\}(T - t)$ in the dual picture.

The minimal total cost for the experiment can be obtained by substitution of these solutions

$$
\begin{aligned}
\mathsf{S} = {} & \omega_q(q^2+\sigma_q) + 2\omega_{qp}(qp+\sigma_{qp}) \\
& + \omega_p(0)(p^2+\sigma_p) + \int_0^T(\hbar^2\omega_p(t) + \omega_{pq}^2(t)\sigma_q(t))\mathrm{d}t \\
& + \int_0^T(\omega_p^2(t)\sigma_p(t) + 2\omega_{qp}(t)\omega_p(t)\sigma_{pq}(t))\mathrm{d}t
\end{aligned} \tag{3.8}
$$

This demonstrates the linear microduality principle in the following specified form of the table

Filtering $\check{q}$	$\lambda-\mu^{-1}$	α	ε	$\mathbf{j}^{\intercal}\boldsymbol{k}$	$\boldsymbol{g}\mathbf{j}$	$\Sigma\mathbf{j}$
Control $\check{p}$	$\lambda-\mu^{-1}$	β	γ	$\boldsymbol{l}^{\intercal}$	$\mathbf{j}\boldsymbol{h}$	$\mathbf{j}\boldsymbol{\Omega}$

showing the complete symmetry under the time reversal and exchange of (q,p), in which the coordinate observation is seen as completely dual to the feedback of momentum.

4. Quantum Dynamics with Trajectories

This section highlights the differences between quantum and classical systems and introduces the problem of quantum observation and its solution in the framework of open dynamics. In orthodox quantum mechanics, which treats only closed quantum dynamics without observations, there is no such problem. However, it is meaningless to consider quantum feedback control without the solution of this problem. After the appropriate setting of quantum mechanics with observation is given, the measurement problem is then restated as a statistical problem of quantum causality; this can be resolved by optimal dynamical estimation on the output of an open quantum system called quantum filtering.

Quantum physics that deals with the unavoidable random nature of the microworld requires a new, more general, noncommutative theory of stochastic processes than the classical one based on Kolmogorov's axioms. The appropriate quantum probability theory was developed through the 70s and 80s by Accardi, Belavkin, Gardiner, Holevo, Hudson and Parthasarthy [29, 5, 7, 2, 30, 31, 32] amongst others.

The essential difference between classical and quantum systems is that classical states, including the mixed states, are defined by probability measures not on *properties* but *events*. This is because the properties of classical systems are described by measurable subsets $\Delta \subseteq \Omega$ forming a Boolean σ-algebra $\mathfrak{A}$ on the space of classical pure states, the points $\omega \in \Omega$ of a phase space. In principle they all can be tested simultaneously and identified with the events represented by the indicator functions $1_\Delta(\omega)$ of $\Delta \in \mathfrak{A}$ on the *universal observation* space Ω. They are building blocks for classical random variables described by essentially measurable functions with respect to a probability measure $\mathbb{P}$ on $\mathfrak{A}$. The algebra of all such complex functions $a : \Omega \to \mathbb{C}$ with pointwise operations is denoted by A, while $\mathrm{L}^p(\Omega,\mathbb{P})$ with $p = 1, 2, \infty$ stands for the subspaces of absolutely integrable, square-integrable and essentially bounded functions $f, g, b \in \mathrm{A}$ respectively. Note that the Banach space $\mathrm{M} = \mathrm{L}^\infty(\Omega,\mathbb{P})$ is a commutative C*-algebra (see Appendix 1.1) of the algebra A with involution $* : \mathrm{A} \ni a \mapsto a^* \in \mathrm{A}$ defined by the complex conjugation $a^* = \bar{a}$. Moreover, it is W*-algebra since M has the preadjoint space $\mathrm{M}_\star = \mathrm{L}^1(\Omega,\mathbb{P})$ such that $\mathrm{M}_\star^\star = \mathrm{M}$ with respect to the standard pairing

$$
(a|f) := \int_\Omega \overline{a(\omega)} f(\omega)\,\mathbb{P}(\mathrm{d}\omega) \equiv \langle a^*, f\rangle \tag{4.1}
$$

defining the expectation on $\mathrm{M}_\star$ as $\mathbb{E}[f] = \langle 1, f\rangle$.

In a quantum world, unfortunately, there are *incompatible* properties corresponding to inconsistent but not orthogonal (i.e. not mutually excluding) *questions* such that, if the infimum $P \wedge Q$ is zero, it does not mean that $P \perp Q$. These questions cannot be *surely* answered simultaneously, i.e. tested with simultaneous events on any universal measurable space Ω, and they cannot be represented in any Boolean algebra. Since the incompatibility is measured by noncommutativity of orthoprojectors P and Q representing these questions as Hermitian idempotents on a Hilbert space $\mathcal{H}$ of quantum *vector-states*. The algebra $\mathcal{A}$ generated by all quantum properties must be noncommutative. The set $\mathfrak{P}(\mathcal{A})$ of all orthoprojectors $P \in \mathcal{A}$, called *property logic* of a noncommutative algebra $\mathcal{A}$, clearly extends any *eventum logic* of commuting orthoprojectors injectively representing the Boolean logic $\mathfrak{A}$ by a σ-homomorphism $E : \mathfrak{A} \to \mathfrak{P}(\mathcal{A})$ such that $\sum E(\Delta_j) = I$ for any measurable σ-partition $\Omega = \sum \Delta_j$. Two normal quantum variables are said to be *compatible* if their orthoprojectors commute, and therefore can be represented classically by measurable functions on their joint spectrum space Ω. However, there is no such Ω if they do not commute. Since there are many incompatible quantum variables, e.g. the position and momentum in quantum mechanics, quantum properties cannot be identified with any commuting set $E(\mathfrak{A})$ representing a Boolean logic $\mathfrak{A}$.

4.1. **Quantum causality and predictions.** Almost simultaneously with Kolmogorov's functional formulation of classical probability theory von Neumann [33] gave another, more general operator formulation, aiming to lay down the foundation of quantum probability theory. It deals not necessarily with commutative W*-algebras, called von Neumann algebras when they are represented as algebras of operators on a Hilbert space $\mathcal{H}$ with involution as Hermitian conjugation $*$ and unit as the identity operator I on $\mathcal{H}$. In order to understand the relation between these two formulations it is useful to reformulate Kolmogorov's axioms in terms of von Neumann's (vice versa is impossible in the case of noncommutativity of the operator algebra). Any random variable $a \in \mathrm{M}$ can be represented by the diagonal operator $\hat{a}$ of pointwise multiplication $\hat{a}g = ag$ in the Hilbert space $\mathrm{H} = \mathrm{L}^2(\Omega, \mathbb{P})$ such that the abelian (commutative) operator algebra $\hat{\mathrm{M}} = \{\hat{a} : a \in \mathrm{M}\}$ is maximal in the algebra $\mathcal{B}(\mathrm{H})$ of all bounded operators on H in the sense that $\hat{\mathrm{M}} = \hat{\mathrm{M}}'$. Here $\hat{\mathrm{M}}' = \left\{B \in \mathcal{B}(\mathrm{H}) : \left[\hat{\mathrm{M}}, B\right] = 0\right\}$ with $[\hat{\mathrm{M}}, B] = \left\{AB - BA : A \in \hat{\mathrm{M}}\right\}$ stands for the *bounded commutant* of $\hat{\mathrm{M}}$, which obviously coincides on H with the commutant

$$\hat{1}'_{\mathfrak{A}} = \{B : [\hat{1}_\Delta, B] = 0, \Delta \in \mathfrak{A}\} \tag{4.2}$$

of the Boolean algebra $\hat{1}_{\mathfrak{A}} = \left\{\hat{1}_\Delta : \Delta \in \mathfrak{A}\right\}$ of all diagonal orthoprojectors (the multiplications by 1_Δ) generating $\hat{\mathrm{M}}$. Note that the commutant $\mathcal{B} = \mathcal{M}'$ of any *nonmaximal* abelian subalgebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a noncommutative W*-algebra with strict inclusion of $\mathcal{M}$ as the *center* $\mathcal{B} \cap \mathcal{B}'$ of $\mathcal{B}$. Thus the *simple algebra* $\mathcal{B} = \mathcal{B}(\mathcal{H})$ is the commutant of the abelian algebra of scalar multipliers $\mathcal{M} = \mathbb{C}I$ which is generated by the trivial Boolean algebra $\mathfrak{A} = \{\emptyset, \Omega\}$ represented by improper orthoprojectors $P_\emptyset = O$, $P_\Omega = I$. The noncommutative algebra $\mathcal{A}$ cannot be generated by any Boolean algebra of orthoprojectors as the commuting Hermitian idempotents $P^2 = P = P^*$ in $\mathcal{B}(\mathcal{H})$.

Quantum causality, assuming the existence of not only properties but also observable events, requires that all quantum properties related to *present and future* at each time-instant t must be compatible with all *passed* events. This makes an allowance for simultaneous predictability of incompatible properties upon the observed events, at least in the statistical sense. However, the usual quantum mechanics, dealing only with irreducible representations $\mathcal{A} = \mathcal{B}(\mathcal{H})$ of quantum properties and not with the events, is causal only for the trivial eventum algebra of improper orthoprojectors $\{O, I\}$ on $\mathcal{H}$. This is why any nontrivial causality requires an extension of the orthodox framework of quantum mechanics to *quantum stochastics* unifying it with the framework of classical stochastics in a minimalistic way allowing the distinction between the future quantum properties and past classical events. This program was completed in [2, 12] on the basis of quantum nondemolition (QND) principle [7, 8, 15] as an algebraic formulation quantum causality: *The past events, corresponding to the measurable histories $\Delta \in \mathfrak{A}_{t]}$ up to each $t \in \mathbb{R}_+$, should be represented in the commutant of a noncommutative subalgebra $\mathcal{A}_{[t} \subseteq \mathcal{A}$ describing the present and future on a universal Hilbert space $\mathcal{H}$.* Thus, instead of a single noncommutative algebra $\mathcal{A}$ extending the eventum W*-algebra $\mathcal{M}$ generated by $E(\mathfrak{A})$ one should consider a decreasing family $(\mathcal{A}_{[t})$ of reduced subalgebras $\mathcal{A}_{[t} \subseteq \mathcal{A}_{[s}$ $\forall s < t$ in the relative commutants $\mathcal{B}_t = \mathcal{A} \cap E(\mathfrak{A}_{t]})'$ of the past eventum logics $E(\mathfrak{A}_{t]}) = \{E(\Delta) : \Delta \in \mathfrak{A}_{t]}\}$ reresenting the consistent histories of increasing probability spaces $(\Omega_{t]}, \mathfrak{A}_{t]}, \mathbb{P}_{t]})$ in nonmaximal abelian W*-algebras $\mathcal{M}_{t]}$ generated by $E(\mathfrak{A}_{t]})$.

The nondemolition principle requiring the choice of time arrow makes quantum causality even microscopically irreversible by allowing future observations represented by decreasing eventum algebras $E(\mathfrak{A}_t) \subset \mathcal{A}_{[t}$ to be incompatible with some nonaticipating questions Q even if $Q \in \mathcal{A}_{[t}$. Although any projectively increasing family of classical probability spaces can be obtained by Kolmogorov construction from a single $(\Omega, \mathfrak{A}, \mathbb{P})$ with projections $\kappa_{s]} : \Omega \to \Omega_{s]}$ inverting the injections $\kappa_{s]}^{-1}(\mathfrak{A}_{s]}) \subseteq \kappa_{t]}^{-1}(\mathfrak{A}_{t]}) \subseteq \mathfrak{A}$ for all $s \leq t$ such that $\mathbb{P}_{s]} = \mathbb{P}_{t]} \circ \kappa_{s]}^{-1} = \mathbb{P} \circ \kappa_{s]}^{-1}$. However, this projective limit may not be compatible with any noncommutative algebra $\mathcal{A}_{[t}$. Thus the maximal W*-algebra $\mathcal{A} = \mathcal{A}_{[0}$, satisfying the compatibility condition $\mathcal{A}' = \mathcal{M}_{0]}$ with the initial central algebra $\mathcal{M}_{0]} = \mathcal{A} \cap \mathcal{A}'$, coincides with the decomposable algebra $\mathcal{M}'_{0]}$ which is not compatible with the total eventum algebra $\mathcal{M} = \vee \mathcal{M}_{t]}$ except the case $\mathcal{M} = \mathcal{M}_{0]}$ of absence of innovation $\mathcal{M}_{r]} = \mathcal{M}_{t]}$ for all r and t. The latter with $\mathcal{M} = \mathbb{C}I$ is a standard assumption in the orthodox quantum mechanics dealing in the absence of observations with the constant $\mathcal{A}_{[t}$ equal to $\mathcal{B}(\mathcal{H})$. In the nonorthodox quantum mechanics with causal observations the eventum algebra $\mathcal{M}$ is nontrivial such that $\mathcal{A}_{[t} \subseteq \mathcal{M}'_{t]}$ can not simply $\mathcal{B}(\mathcal{H}_{[0})$ for all t, however we may assume that $\mathcal{M}_{0]} = \mathbb{C}I$ corresponding to trivial initial history, $\mathfrak{A}_{0]} = \{\varnothing, \Omega\}$ with $\mathbb{P}_{0]} = 1$ on a single-point $\Omega_{0]} = \{0\}$, which allows $\mathcal{A}_{[0} = \mathcal{B}(\mathcal{H}_{[0})$.

Note that since all operators $B \in E(\mathfrak{A}_{t]})'$ commute with $\mathcal{M}_{t]}$, they are jointly decomposable, given in the diagonal representation of $\mathcal{M}_{t]}$ by $(\mathfrak{A}_{t]}, \mathbb{P}_{t]})$-essentially bounded functions $B : \omega \mapsto B(\omega)$ on $\Omega_{t]}$ with operator values $B(\omega) \in \mathcal{B}(\omega)$ on the Hilbert components $\mathcal{H}(\omega)$ of the orthogonal decomposition $\int_{\Omega_{t]}}^{\oplus} \mathcal{H}(\omega) \mathbb{P}_{t]}(\mathrm{d}\omega) \sim \mathcal{H}$ corresponding to the joint spectral representations

$$E(\Delta) \simeq \int_{\Omega_{t]}}^{\oplus} 1_{\Delta}(\omega) I(\omega) \mathbb{P}_{t]}(\mathrm{d}\omega) \equiv I_{t]}(\Delta) \tag{4.3}$$

of commuting orthoprojectors $E(\Delta)$, $\Delta \in \mathfrak{A}_{t]}$.

Quantum state (See Appendix 1.2) consistent with the trajectory probability space $(\Omega, \mathbb{P}, \mathfrak{A})$ is given as the linear positive functional $\langle \varpi, Q \rangle = (\varpi|Q)$ by a Hermitian-positive $\varpi = \varpi^*$ mass-one $\langle \varpi, I \rangle = 1$ operator $\varpi \vdash \mathcal{A}$ ($\in \mathcal{A}$ in usual or a generalized sense as affiliated to $\mathcal{A}$) defining the probability measure $\mathbb{P}$ as the projective limit of

$$\mathbb{P}_{t]}(\Delta) = \langle \varpi, E(\Delta) \rangle = \mathbb{P}(\Delta), \quad \Delta \in \mathfrak{A}_{t]} \tag{4.4}$$

(where $\langle \varpi, B \rangle = \mathrm{tr}[B\varpi]$ for $\mathcal{A} = \mathcal{B}(\mathcal{H})$). It is called the (probability) density operator for $\mathcal{A}$ since it defines the probability $\Pr[Q] = \langle \varpi, Q \rangle \in [0,1]$ of any quantum property described by an orthoprojector $Q \in \mathcal{A}$. Since $Q \in \mathfrak{P}(\mathcal{A}_{[t})$ is compatible with each eventum projector $E(\Delta)$ for $\Delta \in \mathfrak{A}_{t]}$, the property Q is statistically predictable with respect to all past events due to the existence of *a posteriori* conditional probability

$$\Pr[Q|\Delta] = \frac{1}{\mathbb{P}(\Delta)} \langle \varpi, QE(\Delta) \rangle \quad \forall \Delta : \mathbb{P}(\Delta) \neq 0 \tag{4.5}$$

such that $\Pr[Q] = \mathbb{P}(\Delta) \Pr[Q|\Delta] + \mathbb{P}(\Delta^{\perp}) \Pr[Q|\Delta^{\perp}]$. Note that $\langle \varpi, QE \rangle$ is not positive and even not real without the compatibility of Q and E. This leads to the existence of the *posterior quantum states* $\hat{\varpi}_{[t}$ on $\mathcal{A}_{[t}$ given by the conditional expectations ϵ_t as normal projections on W*-algebras $E(\mathfrak{A}_{t]})'$ onto their centers $\mathcal{M}_{t]}$:

$$\langle \hat{\varpi}_{[t}, A \rangle := \epsilon_t(A) \equiv \hat{\varpi}^{\star}_{[t}(A) \quad \forall A \in \mathcal{A}_{[t}. \tag{4.6}$$

The existence of these posterior states as classical stochastic adapted processes $\omega \mapsto \varpi^{\omega}_{[t}$ with density operator values affiliated to the spectral components $\mathcal{A}_{[t}(\omega)$ of $\mathcal{A}_{[t}$ is established in the following theorem.

Theorem 1. *Let ϖ be a normal state on $\mathcal{A}$. Then the induced states $\varpi_{[t}$ on the relative commutants $\mathcal{A}_{[t} \subseteq \mathcal{A} \cap \mathcal{M}'_{t]}$ of the eventum algebra $\mathcal{M}_{t]}$ are given as classical expectations*

$$\langle \varpi_{[t}, A \rangle = \int_{\Omega_{t]}} \langle \hat{\varpi}_{[t}, A \rangle (\omega) \mathbb{P}_{t]}(\mathrm{d}\omega) \equiv \mathbb{E}_{\Omega_{t]}} [\langle \hat{\varpi}_{[t}, A \rangle]$$

in terms of the $\mathrm{M}_{t]} \sim \mathcal{M}_{t]}$*-valued pairings (4.6) by*

$$\langle \hat{\varpi}_{[t}, A \rangle (\omega) = \hat{\varpi}^{\star}_{[t}(A)(\omega) := \left\langle \varpi^{\omega}_{[t}, A(\omega) \right\rangle \tag{4.7}$$

on $\mathcal{A}_{[t}(\omega)$ paired with the posterior density operators $\varpi^{\omega}_{[t} \vdash \mathcal{A}_{[t}(\omega)$ which are a.s. defined as positive weakly integrable functions of $\omega \in \Omega_{t]}$.

Proof. Since $\varpi_{[t}$ is normal state on $\mathcal{A}_{[t}$, equivalent to the space of essentially bounded functions on $(\Omega_{t]}, \mathfrak{A}_{t]}, \mathbb{P}_{t]})$ with operator values in $\mathcal{A}_{[t}(\omega) \subseteq \mathcal{B}(\omega)$, it is uniquely defined as the expectation of (4.7) by an essentially integrable function $\omega \mapsto \varpi^{\omega}_{[t}$ with values affiliated to $\mathcal{A}_{[t}(\omega)$. Each $\varpi^{\omega}_{[t}$ is a density operator of the posterior state as the conditional expectations defined on $\mathcal{A}_{[t}$ with respect to the central Abelian subalgebra $\mathcal{M}_{t]} \sim \mathrm{L}^{\infty}(\Omega_{t]}, \mathbb{P}_{t]})$ by the Radon-Nikodym derivatives

$$\epsilon_t[A|\omega] := \lim_{\mathfrak{A}_{t]} \ni \Delta \searrow \{\omega\}} \frac{\langle \varpi_t, AE(\Delta) \rangle}{\mathbb{P}(\Delta)}, \quad A \in \mathcal{A}_{[t} \tag{4.8}$$

where the limit is understood for almost all $\omega \in \Omega_{t]}$ in the same way as in the classical case. □

Note that in the most important product case $\mathcal{H} \sim \mathrm{H}_{t]} \otimes \mathcal{H}_{[t}$ considered in next sections, all $\mathcal{H}(\omega)$ with $\omega \in \Omega_{t]}$, corresponding to a complete in the sense $\mathrm{H}_{t]} = \mathrm{L}^2(\Omega_{t]}, \mathbb{P}_{t]})$ adapted observation, are isomorphic to a single Hilbert space $\mathcal{H}_{[t}$ of a decreasing family $(\mathcal{H}_{[t})$ such that $B(\omega) \in \mathcal{B}(\mathcal{H}_{[t})$ if $\omega \in \Omega_{t]}$. Defining the subalgebra $\mathcal{A}_{[t}$ of $E(\mathfrak{A}_{t]})' = \hat{\mathrm{M}}_{t]}\bar{\otimes}\mathcal{B}(\mathcal{H}_{[t})$ as $\mathcal{A}_{[t} \sim \mathcal{B}(\mathcal{H}_{[t})$ for any t, the posterior states are described then by the positive trace-normalized operators $\varpi^{\omega}_{[t} \in \mathcal{B}(\mathcal{H}_{[t})$ for almost each $\omega \in \Omega_{t]}$ and can be considered as the conditional states on the present-plus-futuare operator algebras $\mathcal{B}(\mathcal{H}_{[t})$, controlled by the history trajectory ω such that the weak expectation $\int_{\Omega_{t]}} \varpi^{\omega}_{[t}\mathbb{P}_{t]}(\mathrm{d}\omega)$ is the prior marginal state $\varpi_{[t}$ for $\mathcal{A}_{[t}$. Thus the above quantum causality setting gives immediately the posterior states $\varpi^{\omega}_{[t}$ for quantum present-plus- future conditioned by the classical past without any reference to the projection or other phenomenological reduction postulate of quantum measurement. This is the main advantage of the extended event enhanced quantum mechanics, or *eventum mechanics*. It allows treatment of the observable events on an equal basis with other quantum properties of the system. It can be shown, see (5.5), that any reduction postulate of the operational quantum mechanics can all be derived from QND causality, and this principle is also applicable to the continuous measurements in both time and spectrum where projection postulate fails.

We now describe an appropriate dynamical model for the time-continuous interactions between the open quantum system and the field.

4.2. **Quantum open dynamics and input-output.** Quantum Markovian dynamics with observable trajectories, which entered into physics in the 90's in terms of stochastic transfer-operators or stochastic Master equations, define the phenomenological "instruments" of observation without giving any microscopic dynamical model in terms of the fundamental Hamiltonian interactions. In fact such approach is equivalent to the earlier operational approach based on the instrumental transfer-measures (See Appendix 1.3), and its starting point corresponds to already filtered Markov dynamics in the classical case. Here we describe the general scheme for underlying Hamiltonian interaction models with continuous observation for open quantum dynamical objects in terms of quantum stochastic evolutions in parallel to the classical stochastic models with partial observation, following the original Belavkin approach suggested in [2, 3, 12].

Let us fix a quantum probability space $(\mathcal{H}, \mathcal{A}, \varpi)$ and an increasing family $\mathcal{A}_{s]} \subseteq \mathcal{A}_{t]}$, $\forall s < t$ of W*-subalgebras $\mathcal{A}_{t]} \subseteq \mathcal{A}$ containing the *compatible histories* $E(\mathfrak{A}_{t]}) \subset \mathcal{A}'_{[t}$ and *nontrivial present* $\mathcal{A}_{[t]} = \mathcal{A}_{t]} \cap \mathcal{A}_{[t}$ for each t, assuming that each $\mathcal{A}_{t]}$ commutes with *future* $\mathcal{A}_t \subset \mathcal{A}_{[t}$, $\mathcal{A}_{t]} \subseteq \mathcal{A}'_t$, being generated by only *nonanticipating questions* $Q \in \mathcal{A}'_t \cap \mathcal{A}$. The future is described by W*-subalgebras $\mathcal{A}_t \subseteq \mathcal{A}$ forming a decreasing family $\mathcal{A}_t \subseteq \mathcal{A}_{t+s}\ \forall t, s > 0$ with trivial intersection such that we may assume that $\vee\mathcal{A}_{t]} = \mathcal{A}$. Moreover, we shall assume that the family $(\mathcal{A}_{t]})$ as well as $(\mathcal{M}_{t]})$ form the *W*-product systems* in the sense of W*-isomorphisms

$$\mathcal{A}_{t]}\bar{\otimes}\mathcal{A}^s_t \sim \mathcal{A}_{t+s]},\quad \mathcal{M}_{t]}\bar{\otimes}\mathcal{M}^s_t \sim \mathcal{M}_{t+s]}, \tag{4.9}$$

where $\mathcal{A}^s_t = \mathcal{A}_t \cap \mathcal{A}_{t+s]}$, $\mathcal{M}^s_t = \mathcal{M}_t \cap \mathcal{M}_{t+s]}$ and $(\mathcal{M}_t)$ is decreasing family of W*-algebras $\mathcal{M}_t \subset \mathcal{A}_{[t}$ generated by future events $E(\mathfrak{A}_t)$. This implies that the family $(\mathcal{A}^s_t)$ satisfies the product condition such that $\mathcal{A}_t \sim \mathcal{A}^s_t\bar{\otimes}\mathcal{A}_{t+s}$ for any t and $s > 0$, and similar for $(\mathcal{M}_t^s)$ corresponding to the split property

$$(4.10)\qquad \Omega = \Omega_{t]} \times \Omega_t^s \times \Omega_{t+s}, \ \mathfrak{A} = \mathfrak{A}_{t]} \otimes \mathfrak{A}_t^s \otimes \mathfrak{A}_{t+s}$$

of the measurable trajectory space.

Quantum open object under the observation is represented at each time t by a past-future boundary W*-subalgebra $\mathfrak{a}(t) \subseteq \mathcal{A}_{[t]}$ such that it is a quantum stochastic process (in the genral sense [2]), adapted with respect to the family $(\mathcal{A}_{t]})$, nonanticipating the futures $(\mathcal{A}_t)$, and satisfying causality condition with respect to the histories $(\mathcal{M}_{t]})$. We may assume that each $\mathfrak{a}(t)$ represents a fixed $\mathfrak{a}$, or a variable boundary W*-algebra $\mathfrak{a}_t$ by a W*-homomorphism π^t of $\mathfrak{a}_t$ onto $\mathfrak{a}(t)$, with $\mathfrak{a}_t$ taken in the initial algebra $\mathcal{A}_{[0]}$, say. Due to the causality condition the product

$$\Pi^t(\Delta, \check{q}) = E(\Delta)\pi^t(\check{q}) \ \forall \Delta \in \mathfrak{A}_{t]}$$

defines for each t an adapted transfer-measure $\Pi^t(\Delta) : \mathfrak{a}_t \to \mathcal{A}_{t]}$ (see Appendix 1.4) with W*-homomorphic values, normalized to the history eventum projectors $E(\Delta)$. Obviously W*-algebras $\mathfrak{a}(t)$ and $\mathcal{A}_t^s$ are both in $\mathcal{A}_{[t}^s = \mathcal{A}_{t+s]} \cap \mathcal{A}_{[t}$, as well as $\mathfrak{a}(t+s)$ and W*-algebras $\mathcal{M}_t^s$.

Following [2] we shall say that quantum open object $\mathfrak{a}(t)$ with eventum history $E(\mathfrak{A}_{t]})$ is *dynamical* with respect to $(\mathcal{A}_t)$ if

$$\mathcal{M}_t^s \vee \mathfrak{a}(t+s) \subseteq \mathfrak{a}(t) \vee \mathcal{A}_t^s \ \forall t, s > 0.$$

This is equivalent [2] to the existence of *quantum flow with observations* described as follows on the co-images $\mathfrak{a}_t$ of the boundary algebras $\mathfrak{a}(t)$. Assuming that the W*-algebras $\mathcal{A}_{[t}^s$ are generated by $\mathfrak{a}(t)$ and $\mathcal{A}_t^s$, we can always consider the dynamical quantum open object with $\mathfrak{a}(t) = \mathcal{A}_{[t]}$.

Theorem 2. *Let $\pi_t : \mathfrak{a}(t) \to \mathfrak{a}_t$ be normal injections inverted by the dynamical representations π^t, and let $A_{t]}, \mathcal{M}_{t]}$ form the product systems (4.9). Then there exists a transitional spectral measure*

$$(4.11)\qquad \Upsilon_t^r(\Delta, \check{q}) = E_t(\Delta)\,\alpha_t^r(\check{q}), \quad \check{q} \in \mathfrak{a}_{t+r}$$

on $\mathfrak{A}_t^r$ with values in $\mathfrak{a}_t \bar{\otimes} \mathcal{A}_t^r$ given by adapted σ-homomorphisms $E_t : \mathfrak{A}_t \to \mathfrak{a}_t \bar{\otimes} \mathcal{A}_t$ and a Heisenberg flow (α_t^r) of causal tensor-adapted W-homomorphisms α_t^r : $\mathfrak{a}_{t+r} \to E_t(\mathfrak{A}_t^r)'$ such that*

$$(4.12)\qquad \alpha_{t-r}^r \circ \alpha_t^s = \alpha_{t-r}^{r+s} \ \forall r, s > 0,$$

$$(4.13)\qquad \alpha_{t-r}^r(E_t(\Delta)) = E_{t-r}(\Delta) \ \forall t > r$$

under the trivial extensions onto $\mathfrak{a}_t \bar{\otimes} \mathcal{A}_t$.

Proof. The representations π^t as well as π_{t-r} can be trivially extended to the adapted W*-homomorphisms with respect to the identity maps id respectively on $\mathcal{A}_t$ and $\mathcal{A}_{t-r}$ by virtue of commutativity $\mathcal{A}_{t]} \subseteq \mathcal{A}_t'$ as $\pi^t(\check{q}_t \otimes A_t) = \pi^t(\check{q}_t)A_t$ and $\pi_{t-r}(\check{q}A_{t-r}) = \pi_{t-r}(\check{q}) \otimes A_{t-r}$ respectively for all $\check{q}_t \in \mathfrak{a}_t$ and $\check{q}_{t-r} \in \mathfrak{a}_{t-r}$. This defines the compositions $\alpha_{t-r}^r = \pi_{t-r} \circ \pi^t$ of thus extended W*-representations as tensor-adapted W*-homomorphisms $\alpha_{t-r}^r : \mathcal{A}_{[t} \to \mathcal{A}_{[t-r}$ for W*-algebras $\mathcal{A}_{[t} = \mathfrak{a}\bar{\otimes}\mathcal{A}_t$ trivially extending the map $\mathfrak{a}_t \to \mathcal{A}_{[t-r}^r$ and satisfying the hemigroup condition (4.12) such that $\alpha_t^0 = \mathrm{id}(\mathcal{A}_{[t})$ for each t. Obviously these extensions satisfy causality condition

$$\pi^t(\mathcal{A}_{[t}) \subseteq E_t(\mathfrak{A}_{t]})', \ \alpha_t^r(\mathcal{A}_{[t+r}) \subseteq E_t(\mathfrak{A}_t^r)',$$

where the eventum projectors $E_t(\Delta) \in \mathcal{A}_{[t}$ are defined for any $\Delta \in \mathfrak{A}_t$ as $\pi_t(E(\Delta))$ by the extended injections $\pi_t : \mathfrak{a}(t) \vee \mathcal{A}_t \to \mathfrak{a}_t \bar{\otimes} \mathcal{A}_t$ inverted by π^t extended on $E_t = \pi_t(E)$. The second condition (4.13) simply follows from $\pi^t \circ \alpha_t^r = \pi^{t+r}$ due to $\pi^{t+r}(E_t) = E$ for any $r \in [0, t]$ and $E \in E(\mathfrak{A}_t)$. Thus the QS flow with nondemolition observations can be described in terms of the homomorphic transitional measures (4.11) with (4.12) and (4.13) satisfying the hemigroup composition law

$$\Upsilon_{t-r}^r\left(\Delta_{t-r}^r, \Upsilon_t^s\left(\Delta_t^s, \check{q}\right)\right) = \Upsilon_{t-r}^{r+s}\left(\Delta_{t-r}^{r+s}, \check{q}\right) \tag{4.14}$$

where $\Delta_{t-r}^{r+s} = \Delta_{t-r}^r \times \Delta_t^s \in \mathfrak{A}_{t-r}^{r+s}$ and $\check{q} \in \mathfrak{a}_{t+s}$. □

Corollary 1. *The dynamical QS object is Markovian in the usual sense* [2] *if the initial state $\varpi = \varpi_{[0}$ on $\mathcal{A} = \mathcal{A}_{[0}$ is product state $\varpi \sim \varpi_{t]} \otimes \varepsilon_t$ for any t such that*

$$\left\langle \varpi, A_{t]} A_t^s \right\rangle = \left\langle \varpi, A_{t]} \right\rangle \left\langle \varepsilon_t, A_t^s \right\rangle .$$

It is operationally described in such a state by the hemigroup of reduced transitional measures

$$\mathcal{T}_t^s(\Delta, \check{q}) = \varepsilon_t^\star\left[\Upsilon_t^s(\Delta, \check{q})\right], \tag{4.15}$$

where $\varepsilon_t^\star : \mathcal{A}_{[0} \to \mathcal{A}_{t]}$ is conditional expectation defined as

$$\left\langle \varpi_{[0}, A_{t]} \right\rangle = \left\langle \varpi_{t]}, \varepsilon_t^\star\left[A_{t]}\right] \right\rangle \quad \forall \varpi_{t]}, A_{t]} \in \mathcal{A}_{t]}. \tag{4.16}$$

They satisfy the operational Chapman-Kolmogorov equation

$$\mathcal{T}_{t-r}^r\left(\Delta_{t-r}^r, \mathcal{T}_t^s\left(\Delta_t^s, \check{q}\right)\right) = \mathcal{T}_{t-r}^{r+s}\left(\Delta_{t-r}^{r+s}, \check{q}\right) \tag{4.17}$$

as a normal completely positive map $\mathfrak{a}_{t+s} \to \mathfrak{a}_{t-r}$ for each product Δ_{t-r}^{r+s} of $\Delta_{t-r}^r \in \mathfrak{A}_{t-r}^r$ and $\Delta_t^s \in \mathfrak{A}_t^s$, and are normalized as $\mathcal{T}_t^s(\Omega_t^s) = \tau_t^s$ to normal unital CP maps $\tau_t^s = \varepsilon_t^\star \circ \alpha_t^s$ of $\mathfrak{a}_{t+s}$ onto $\mathfrak{a}_t$, forming a dynamical hemigroup (τ_t^s) over the family $(\mathfrak{a}_t)$.

Remark 1. *The event representations $E_t = \pi_t(E)$ are usually given as $E_t(\Delta) = \alpha_t(I(\Delta))$ by an* input *σ-homomorphism $I : \mathfrak{A}_t^s \to \mathcal{A}_t^s$, $I(\Delta) = \iota(1_\Delta)$, corresponding to a two side adapted W^*-representation $\iota : \mathrm{M}_t^s \to \mathcal{A}_t^s$ and an* output representation $\alpha_t = \lim_s \alpha_t^s$ *given by a hemigroup (α_t^s) of* interaction isomorphisms $\alpha_t^s : \mathfrak{a}_{t+s} \bar{\otimes} \mathcal{A}_t^s \to \mathfrak{a}_t \bar{\otimes} \mathcal{A}_t^s$. *Thus extended α_t^s define the* output representation α_t *as the projective limit $\alpha_t^\infty | \mathcal{A}_t^s = \alpha_t^s$ which is well defined on $\mathcal{A}_t = \vee_s \mathcal{A}_t^s$ due to the* localization property $\alpha_t^{r+s} | \mathcal{A}_t^s = \alpha_t^s | \mathcal{A}_t^s$ *for any $r > 0$ on the* input eventum algebra $\iota(\mathrm{M}_t) \subset \mathcal{A}_t$ *for $\mathrm{M}_t = \mathrm{L}^\infty(\Omega_t, \mathbb{P}_t)$. Note that this localization property simply follows from the hemigroup condition $\alpha_t^s \circ \alpha_{t+s}^r = \alpha_t^{r+s}$ and the normalization $\alpha_t^s(I_0) = I_0$ of α_t^s, extended adaptively also on $A_0^t \in \mathcal{A}_0^t$ and $A_{t+s} \in \mathcal{A}_{t+s}$ such that*

$$\alpha_t^s\left(A_0^t \otimes B \otimes A_{t+s}\right) = A_0^t \otimes \alpha_t^s(B) \otimes A_{t+s}.$$

The quantum *free evolution* is usually described by a semigroup $(\theta_r)_{r \geq 0}$ of endomorphisms $\theta_r : \mathcal{A}_{[0} \to \mathcal{A}_{[0}$ shifting isomorphically any $\mathcal{A}_t^s$ onto $\mathcal{A}_{t+r}^s$, with the trivial action on $\mathfrak{a}$. QS Heisenberg flow (α_t^s) over a constant algebra $\mathfrak{a}_t = \mathfrak{a}$ with observation (E_t) is called *covariant* with respect to the semigroup (θ_r) acting also on $\mathfrak{A}_0$ by translation of each σ-subalgebra $\theta_t(\mathfrak{A}_0^s) \sim \mathfrak{A}_t^s$ onto $\mathfrak{A}_{t+r}^s$, if

$$\alpha_t^s \circ \theta_{t+s} = \theta_t \circ \vartheta_s, \quad E_t \circ \theta_t = \theta_t \circ E_0, \tag{4.18}$$

where $\vartheta_s = \alpha_0^s \circ \theta_s$ and θ_s is extended on the W*-algebra $\mathcal{A}_{[0} = \mathfrak{a} \bar{\otimes} \mathcal{A}_0$ by $\theta_s(\check{q} \otimes A_0) = \check{q} \otimes \theta_s(A_0)$. This defines a Heisenberg θ-cocycle α_0^s corresponding to the dynamical semigroup (ϑ_s) of injective W*-endomorphisms $\vartheta_s : \mathcal{A}_{[0} \to \mathcal{A}_{[0}$ satisfying causality condition $\vartheta_s(\mathcal{A}) \subseteq E(\mathfrak{A}_{s]})'$. Note that the shift semigroup can be extended to a shift group $\{\theta_t : t \in \mathbb{R}\}$ on $\mathcal{A} = \widetilde{\mathcal{A}}_0 \bar{\otimes} \mathfrak{a} \bar{\otimes} \mathcal{A}_0$ as $\theta_t = \theta_{-t}^{-1}$ by defining $\theta_{-s} = \widetilde{\theta}_s$ on the opposite copy $\widetilde{\mathcal{A}}_0$ of the algebra $\mathcal{A}_0$ similar to θ_s on $\mathcal{A}_0$, inverting θ_s on $\mathcal{A}_s$ and reflecting $\mathcal{A}_0^s$ onto $\widetilde{\mathcal{A}}_0^s$ such that $\mathcal{A}_0^r \bar{\otimes} \mathcal{A}_r^s$ is reflected onto $\widetilde{\mathcal{A}}_0^r \bar{\otimes} \widetilde{\mathcal{A}}_r^s$ by θ_{r+s}. This reversible free dynamics defines an interaction group dynamics $\{\vartheta_t : t \in \mathbb{R}\}$ of automorphisms $\vartheta_t = \alpha_0^t \circ \theta_t$ on the whole $\mathcal{A}$ by the trivial action of α_0^t on $\widetilde{\mathcal{A}}_0$ for $t > 0$ and by $\vartheta_t = \vartheta_{-t}^{-1}$ for for $t < 0$. However, the reversible quantum dynamics on such noncommutative $\mathcal{A}$ cannot satisfy the causality in both directions of time with respect to a nontrivial eventum algebra $E(\mathfrak{A})$, except the case of absence innovation, as it is in the conservative quantum mechanics without observation. To keep the causality in the positive direction of time one must replace the nonabelian $\widetilde{\mathcal{A}}_0$ by the smaller, abelian subalgebra $\check{\mathrm{M}}_0$, a copy of the eventum algebra $\hat{\mathrm{M}}_0 = I(\mathrm{M}_0)$, which makes θ_s and ϑ_s irreversible on $\mathcal{A} = \check{\mathrm{M}}_0 \bar{\otimes} \mathfrak{a} \bar{\otimes} \mathcal{A}_0$ in the case of a larger, noncommutative future algebra $\mathcal{A}_0$ than the past observable algebra $\hat{\mathrm{M}}_0$.

5. Appendix

5.1. A. Some definitions and facts on W*-algebras.

(1) A complex Banach algebra A with involution $a \mapsto a^*$ such that $\|a^*a\| = \|a\|^2$ is called C*-algebra, and W*-algebra if it is dual to a linear subspace $\mathrm{L} \subseteq \mathrm{A}^*$ (called preadjoint of $\mathrm{A} = \mathrm{L}^*$ if it is closed, denoted as $\mathrm{L} = \mathrm{A}_\star$). They all can be realized as operator algebras on a complex Hilbert space $\mathcal{H}$, and an operator W*-algebra is called von Neumann algebra if its unit is the identity operator I in $\mathcal{H}$. The simplest example of W*-algebra is the von Neumann algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators acting in a complex Hilbert space $\mathcal{H}$. A von Neumann algebra $\mathcal{A}$ is called semisimple if $\mathcal{H}$ has an orthogonal decomposition into invariant subspaces $\mathcal{H}_i$ in which $\mathcal{A}$ is $\mathcal{B}(\mathcal{H}_i)$. Let $\{Q_i\}$ (or $\{\mathcal{A}_i\}$) be a family of self-adjoint operators (operator algebras $\mathcal{A}_i$) acting in $\mathcal{H}$, e.g. orthoprojectors $Q_i^2 = Q_i = Q_i^*$. The W*-algebra generated by this family is defined as the smallest weakly closed self-adjoint sub-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ containing these operators, or the spectral projectors of these operators if Q_i are unbounded in $\mathcal{H}$ (or the algebras $\mathcal{A}_i$, $\mathcal{A} = \vee \mathcal{A}_i$). In the case $I \in \mathcal{A}$ it consists of all bounded operators that commute with the bounded commutant $\mathcal{B} = \{B \in \mathcal{B}(\mathcal{H}) : BQ_i = Q_iB \quad \forall i\} \equiv \{Q_i\}'$ (or with $\mathcal{B} = \cap \mathcal{A}_i'$), i.e., it is the second commutant $\mathcal{A} = \{Q_i\}'' = \mathcal{B}'$ of the family $\{Q_i\}$. The latter can be taken as the definition of the von Neumann algebra generated by the family $\{Q_i\}$. Note that the commutant $\mathcal{B}$ is a von Neumann algebra, and $\mathcal{B} = \mathcal{A}'$ is semisimple iff it is commutant of an abelian algebra $\mathcal{A}$ [48].

(2) A (normal) state on a von Neumann algebra $\mathcal{B}$ is defined as a linear ultraweakly continuous functional $\mathcal{B} \ni Q \mapsto \langle \varrho, Q \rangle \in \mathbb{C}$, satisfying the positivity and normalization conditions

$$\langle \varrho, Q \rangle \geq 0, \qquad \forall Q \geqq 0, \qquad \langle \varrho, I \rangle = 1 \tag{5.1}$$

[$Q \geqq 0$ signifies the nonnegative definiteness $\langle \psi | Q \psi \rangle \geqq 0 \; \forall \psi \in \mathcal{H}$ called Hermitian positivity of Q]. The linear span of all *normal* states is isometric with the preadjoint space $\mathcal{B}_\star$. The latter is usually described as the space of density operators ϱ uniquely defined as (generalized, or affiliated) elements of the algebra $\mathcal{B}$ with respect to a *standard Hermitian pairing* $\langle \varrho^*, Q\rangle \equiv (\varrho|Q) =: \varrho^*(Q)$ of $\mathcal{B}_\star$ and $\mathcal{B}$ given by the mass $\mu(\varrho) = \langle \varrho, I\rangle$ on the positive ϱ such that $\varrho^* = \varrho \geq 0$ is state density iff $\langle \varrho, I\rangle = 1$. A state ϱ is called vector state if $(\varrho|Q) = \langle \psi|Q\psi\rangle$ (denoted $\varrho = \varrho_\psi$) for some $\psi \in \mathcal{H}$, and pure state if it is an extreme point of the convex set $\mathfrak{s}(\mathcal{B})$ of all normal states on $\mathcal{B}$. Every normal state is in the closed convex hull of vector states ϱ_ψ with $\|\psi\| = 1$ but there might be no pure state in $\mathfrak{s}(\mathcal{B})$. If algebra $\mathcal{B}$ is semifinite (there exists a faithful normal semi-finite trace $Q \mapsto \mathrm{tr}Q$, then the states on $\mathcal{B}$ can be described by unit trace operators $\varrho \in \mathcal{B}$ (or $\varrho \vdash \mathcal{B}$ if the are only affiliated to $\mathcal{B}$), by means of the tracial pairing $\langle \varrho, Q\rangle = \mathrm{tr}[\varrho Q]$. In the simple case $\mathcal{B} = \mathcal{B}(\mathcal{H})$ the density operator ϱ is any nuclear positive operator normalized with respect to the usual trace [48].

(3) Let $\mathcal{A}$, $\mathcal{B}$ be von Neumann algebras in respective Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$, and let $\Phi : \mathcal{B} \to \mathcal{A}$ be a linear map that transforms the operators $B \in \mathcal{B}$ into operators $A \in \mathcal{A}$ (called sometimes superoperator). The map Φ is called a *transfer map* if it is ultraweakly continuous, completely positive (CP) in the sense

$$\sum_{i,k=1}^{\infty} \langle \psi_i|\Phi(B_i^* B_k)\psi_k\rangle \geqq 0, \qquad \forall B_j, \psi_j \tag{5.2}$$

$(i = 1, \ldots, d_e < \infty)$, and unity-preserving: $\Phi(I_1) = I_0$ (or $\Phi(I_1) \leq I_0$). The CP condition is obviously satisfied if Φ is normal homomorphism (or W*-representation) $\pi : \mathcal{B} \to \mathcal{A}$, which is defined by the additional multiplicativity property $\pi(B^*B) = \pi(B)^*\pi(B)$. A composition $\langle \varrho_0, \Phi(B)\rangle$ with any state $\varrho_0 \in \mathfrak{s}(\mathcal{A})$ is a state $\varrho_1 \in \mathfrak{s}(\mathcal{B})$ described by the preadjoint action of the superoperator Φ on ϱ_0,

$$\langle \varrho_0, \Phi(B)\rangle = \langle \Phi_\star(\varrho_0), B\rangle, \forall B \in \mathcal{B}, \varrho_0 \in \mathcal{A}_\star,$$

where $\Phi_\star = \Phi^*|\mathcal{A}_\star$ in terms of $(\Phi^*(\varrho)|B) = (\varrho|\Phi(B))$ is such that $\Phi_\star^* = \Phi$. A transfer map Φ is called spatial if

$$\Phi(B) = FBF^* \quad \text{or} \quad \Phi_\star(\varrho_0) = F_\star \varrho_0 F_\star^*, \tag{5.3}$$

where F is a linear coisometric operator $\mathcal{H}_1 \to \mathcal{H}_0$, $FF^* = I_0$ (or $FI_1F^* \leq I_0$) called the *propagator* and $F_\star = F^\sharp$ is defined as left adjoint $(F^\sharp \varrho|Q) = (\varrho|FQ)$ with respect to the standard pairings (which is usual adjoint, $F^\sharp = F^*$, in the case of tracial pairing $(\varrho|Q) = \mathrm{tr}[\varrho^* Q]$). Every transfer map is in the closed convex hull of spatial transfer maps, but there might be no extreme point in this hull.

(4) Let $\mathbb{V}$ be a measurable space, and $\mathfrak{B}$ its Borel σ-algebra. A mapping Π : $\mathrm{d}v \in \mathfrak{B} \mapsto \Pi(\mathrm{d}v)$ with values $\Pi(\mathrm{d}v)$ in ultraweakly continuous, completely positive superoperators $\mathcal{B} \to \mathcal{A}$ is called a *transfer measure* if for any $\varrho \in \mathcal{A}_\star$, $B \in \mathcal{B}$ the $\mathbb{C}$-valued function

$$\langle \Pi(\mathrm{d}v)_\star \varrho, B\rangle = \langle \varrho, \Pi(\mathrm{d}v) B\rangle$$

of the set $\mathrm{d}v \subseteq \mathbb{V}$ is a countably additive measure normalized to unity for $B = I$. In other words, $\Pi(\mathrm{d}v)$ is a CP map valued measure that is σ-additive in the weak (strong) operator sense and for $\mathrm{d}v = \mathbb{V}$ is equal to some transfer-map Φ. In particular, $\Pi(\mathrm{d}v, B) = M(\mathrm{d}v)\Phi(B)$ with $M(\mathrm{d}v) = \Pi(\mathrm{d}v, I)$ is transfer map iff $[M(\mathrm{d}v), \Phi(B)] = 0$ for all $\mathrm{d}v \in \mathfrak{B}$

and $B \in \mathcal{B}$ as it is the case of the *nondemolition measurements* given by representations $M = E$ of $\mathfrak{B}$ in $\mathcal{A}$ and $\Phi = \pi$ in $E(\mathfrak{B})' \cap \mathcal{A}$. The quantum state transformations $\varrho \mapsto \Pi_\star(\Delta, \varrho)$ corresponding to the results $v \in \Delta$ of an *ideal measurement* are described by transfer-operator measures of the form

$$\Pi(\Delta, B) = \int_\Delta F(v) B F(v)^* \mathrm{d}v, \tag{5.4}$$

where $F(v)$ denote linear operators $\mathcal{H}_1 \to \mathcal{H}_0$, the integral with respect to a positive Borel measure λ on $\mathbb{V}$ is interpreted in strong operator topology, and $\int F(v) F(v)^* \mathrm{d}v = I_0$. Every transfer-operator $\Phi : \mathcal{B} \to \mathcal{A}$ can be represented by the integral (5.4) on $\Delta = \mathbb{V}$ of an ideal measurement $\Pi(\mathrm{d}v)$ as the compression

$$\Pi(\Delta, B) = FE(\Delta)\pi(B)F^* \tag{5.5}$$

of the nondemolition measurement $E(\Delta)\pi(B) = \hat{1}_\Delta \otimes B$ on the extended Hilbert space $\mathcal{H} = \int_{\mathbb{V}}^{\oplus} F(v)^* \mathcal{H}_0 \mathrm{d}v$ with the isometric embedding $F^*\psi_0 = \int_{\mathbb{V}}^{\oplus} F(v)^* \psi_0 \lambda(\mathrm{d}v)$ of $\psi_0 \in \mathcal{H}_0$ into $\mathcal{H}$ adjoint to the coisometry $F\psi = \int_{\mathbb{V}} F(\nu)\psi(\nu)\lambda(\mathrm{d}v)$ of $\psi \in \mathcal{H}$ into $\mathcal{H}_0$.

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