Title: FINITE GROUPS POSSESSING SMITH EQUIVALENT, NONISOMORPHIC REPRESENTATIONS

The theory of transformation groups and its applications

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1. INTRODUCTION

Throughout this paper, we assume that groups are always finite groups, group actions are smooth and representations mean real representations.

In 1960, Paul A. Smith [63] posted the following problem:

**Problem.** Let $G$ be a finite group which acts on a homotopy sphere with just two fixed points. Then are the tangential spaces over the fixed points isomorphic as representations or not?

We call two representations which are obtained as the tangential spaces over fixed points from a finite group action on a sphere with just two fixed points are Smith equivalent.

Atiyah-Bott [1] proved that Smith equivalent representations are always isomorphic for a cyclic group of prime order. According to Sanchez [60], they are always isomorphic for a cyclic group of odd prime power order. By character theory, we obtain they are also always isomorphic for the symmetric group on three letters and a cyclic group of order 2, 4, 6. On the other hand, Cappell-Shaneson proved that there exist Smith equivalent representations which are not isomorphic for a cyclic group of order $4q$ for $q \geq 2$ ([6, 7, 8]). For different classes of finite groups, many related results about this problem were obtained by Petrie, Dovermann, Suh, etc. [37, 57, 58, 59, 17, 19, 21, 64, 9, 10, 22] before 1990. After that, Laitinen and Pawałowski [36] obtained that there exists a pair of Smith equivalent nonisomorphic representations for a perfect group whose Laitinen number is greater than or equal to 2. Here a real conjugacy class means $(g)^* := (g) \cup (g^{-1})$ and the Laitinen number $a_G$ of $G$ is a number of all real conjugacy classes of $G$ represented by elements not of prime power order. We assume that the identity is of prime power order. Pawałowski and Solomon [54] showed there exists a pair of Smith equivalent,
nonisomorphic representations for more groups. Most recently, Morimoto [42, 43] presented the concerning results for groups including \( \text{Aut}(A_6) \) and \( P\Sigma L(2, 27) \).

We show that there exists a pair of Smith equivalent, nonisomorphic representations for groups of the other classes. This report is including a joint work with Krzysztof Pawałowski.

**Theorem 1.1.** Suppose that \( G \) is a nonsolvable group with \( a_G \geq 2 \). If two Smith equivalent representations are always isomorphic, then \( G \) is isomorphic to \( \text{Aut}(A_6) \).

**Theorem 1.2.** There exists a solvable Oliver group \( G \) with \( a_G \geq 2 \) which possesses a pair of two Smith equivalent, nonisomorphic representations.

2. **Representations and real conjugacy classes**

In this section, we recall a necessary condition for which two representations become Smith equivalent.

Let \( G \) be a finite group and let \( RO(G) \) be the real representation ring of \( G \). For convenience, we define subgroups of \( RO(G) \). We denote by \( PO(G) \) the subgroup of \( RO(G) \) of \( G \) consisting of the differences \( U - V \) of representations \( U \) and \( V \) such that \( \dim U^G = \dim V^G \) and \( \text{Res}_P^G(U) \cong \text{Res}_P^G(V) \) for any subgroup \( P \) of \( G \) of prime power order. We note that in [54], \( PO(G) \) is denoted by \( IO(G,G) \). Similarly, we denote by \( \overline{PO}(G) \) the subgroup of \( RO(G) \) of \( G \) consisting of the differences \( U - V \) of representations \( U \) and \( V \) such that \( \dim U^G = \dim V^G \) and \( \text{Res}_P^G(U) \cong \text{Res}_P^G(V) \) for any subgroup \( P \) of \( G \) of odd prime power order and order 2, 4. By a theorem of Sanchez [60], the difference of two Smith equivalent representations lies in \( \overline{PO}(G) \) and the difference of two 2-proper Smith equivalent representations lies in \( PO(G) \). The concept of 2-proper is considered by Petrie. We will write the definition of 2-proper Smith equivalence in the section 3.

The rank of \( PO(G) \) is equal to maximum of 0 and the Laitinen number \( a_G \) minus 1. Moreover the rank of \( \overline{PO}(G) \) is equal to the rank of \( PO(G) \) plus the number of all real conjugacy classes represented by 2-elements of order \( \geq 8 \). Now, let \( H \) be a normal subgroup of \( G \). We denote by \( PO(G,H) \) the subgroup of \( RO(G) \) consisting of the differences \( U - V \) of representations \( U \) and \( V \) such that \( U^H \cong V^H \) as representations over \( G/H \), and \( \text{Res}_P^G(U) \cong \text{Res}_P^G(V) \) for any subgroup \( P \) of prime power order. Again, we note that in [54], \( PO(G,H) \) is denoted by \( IO(G,H) \). It holds that \( PO(G) = PO(G,G) \). Let \( b_G \) be the number of all real conjugacy classes in \( G/H \) which are sent from real conjugacy classes of \( G \) represented by elements not of prime power order by the surjection \( G \rightarrow G/H \). Then the rank of \( PO(G,H) \) is equal to \( a_G - b_{G/H} \) (See [54]).

For each prime \( p \), let \( O^p(G) \) be the minimal subgroup among normal subgroups \( N \) of \( G \) with index a power of \( p \). Let \( L(G) \) be the set of subgroups \( L \) of \( G \) containing \( O^p(G) \) for some prime \( p \). A representation \( U \) is said to be \( L(G) \)-free if \( \dim U^L = 0 \)
for any $L \in \mathcal{L}(G)$. We denote by $LO(G)$ the subgroup of $PO(G)$ consisting of the differences $U - V$ of representations $U$ and $V$ which are both $\mathcal{L}(G)$-free. Then it holds that

$$LO(G) \leq PO(G) \leq \overline{PO}(G) \leq RO(G)$$

and Pawałowski and Solomon showed

$$PO(G, G^{nil}) \leq LO(G),$$

where $G^{nil}$ is the minimal subgroup among normal subgroups $N$ of $G$ such that $G/N$ is nilpotent. Note that $G^{nil} = \cap_{p} O^{p}(G)$.

We denote by $QO(G)$ the subgroup of $PO(G)$ consisting of the differences $U - V$ of representations $U$ and $V$ such that $Res_{H}^{G} U \cong Res_{H}^{G} V$ for any proper subgroup $H$ of $G$.

**Lemma 2.1.** $PO(G) \otimes \mathbb{Q}$ is spanned by elements of $\text{Ind}_{C}^{G} QO(C)$ for all cyclic subgroups $C$ of $G$ not of prime power order.

**Corollary 2.2.** Let $C_{1}$ and $C_{2}$ be cyclic subgroups of $G$ not of prime power order. If $C_{1}$ and $C_{2}$ are not conjugate then

$$\text{Ind}_{C_{1}}^{G} QO(C_{1}) \cap \text{Ind}_{C_{2}}^{G} QO(C_{2}) = \{0\}.$$

3. **Finite group actions on spheres with exactly two fixed points**

We denote by $Sm(G)$ the subset of $RO(G)$ consisting the differences of two Smith equivalent representations. A group action of a sphere $\Sigma$ is 2-proper, if $\Sigma^{(g)}$ is connected for any 2-element $g$ of $G$ of order $\geq 8$. In accordance with Petrie's definition, two representations $U$ and $V$ are 2-proper Smith equivalent if there exists a 2-proper action of $G$ on a sphere with exactly two fixed points at which tangential spaces are isomorphic to $U$ and $V$ respectively. We denote by $LSm(G)$ the subset of $Sm(G)$ consisting the differences of two 2-proper Smith equivalent representations. Since $LSm(G) \subset PO(G)$, $a_{G} \leq 1$ implies $LSm(G) = 0$.

Pawalowski and Solomon showed that if $G$ is a gap Oliver group then $LO(G) \subseteq LSm(G)$, and moreover, if $G$ is a gap nonsolvable group with $a_{G} \geq 2$ and $G \neq \text{Aut}(A_{6})$, $P\Sigma L(2, 27)$ then $PO(G, G^{nil}) \neq 0$ and thus $LSm(G) \neq 0$. Recent works by Morimoto gave us that $Sm(\text{Aut}(A_{6})) = 0$ and $LSm(P\Sigma L(2, 27)) \neq 0$.

Now we recall the weak gap condition ([41]). A representation $V$ satisfies the weak gap condition if it satisfies the following properties.

1. If $P \in \mathcal{P}(G)$ and $H > P$, then $2 \dim V^{H} \leq \dim V^{P}$.
2. If $P \in \mathcal{P}(G)$, $H > P$ and $2 \dim V^{H} = \dim V^{P}$, then $[H : P] = 2$, $\dim V^{H} > \dim V^{K} + 1$ for any $K > H$.
3. If $P \in \mathcal{P}(G)$, $[H : P] = 2$ and $2 \dim V^{H} = \dim V^{P}$, then $V^{H}$ is orientable so that $g : V^{H} \rightarrow V^{H}$ is orientation preserving for any $g \in N_{G}(H)$. 
(4) If $P \in \mathcal{P}(G), H \succ P, K \succ P$ and $2 \dim V^H = 2 \dim V^K = \dim V^P$, then the smallest subgroup $\langle H, K \rangle$ including $H$ and $K$ does not belong to $\mathcal{L}(G)$.

Here, $\mathcal{P}(G)$ is the set of all subgroups of $G$ of prime power order.

We denote by $WLO(G)$ the subgroup of $LO(G)$ consisting of the differences $U - V$ of representations $U$ and $V$ such that both $U \oplus W$ and $V \oplus W$ are $\mathcal{L}(G)$-free and satisfy the weak gap condition. Note that $WLO(G) = LO(G)$ if $G$ is a gap group.

**Lemma 3.1.** It holds $WLO(G) \subseteq LS m(G)$ for an Oliver group $G$.

From now on, we investigate conditions for which Oliver groups $G$ satisfy that $WLO(G) \neq 0$.

### 4. A SUFFICIENT CONDITION

We introduce a sufficient condition for Oliver groups $G$ to hold $WLO(G) \neq 0$ by using elements of the groups.

A pair $(x, y)$ of elements $x, y \in G$ is called *basic* if the following two condition hold.

1. $x$ and $y$ are not of prime power order, and $x$ and $y$ are not real conjugate in $G$ (and thus $a_G \geq 2$).
2. $x$ and $y$ are in some gap subgroup of $G$, or the orders $|x|$ and $|y|$ are even and the involutions of $\langle x \rangle$ and $\langle y \rangle$ are conjugate in $G$.

Moreover, we say that $(x, y)$ is an *$H$-pair* for a subgroup $H$ of $G$, if $xH = yH$.

**Theorem 4.1.** If an Oliver group $G$ has a basic $G^{nil}$-pair, then $WLO(G) \neq 0$ and thus $LS m(G) \neq 0$.

It is easy to see that $G$ has a basic $G^{nil}$-pair in some assumptions. The next theorem is obtained by combining Theorem 5.1.

**Theorem 4.2.** If an Oliver group $G$ has an element of the center whose order is divisible by at least 3 distinct primes then $G$ has a basic $G^{nil}$-pair.

In the case when $G$ has nontrivial center, if $G$ has no basic $G^{nil}$-pair then the structure of $G$ is almost determined. In this paper we omit it.

### 5. OUTLINE OF A PROOF OF THEOREM 1.1

We introduce outline of a proof of Theorem 1.1. The following result is one of keys.

**Theorem 5.1.** Let $G$ be an Oliver group with $a_G \geq 2$. If $G/G^{nil}$ is isomorphic to none of the following groups then $WLO(G) \neq 0$.

1. a $p$-group for a prime $p$
2. $C_2 \times P$ for an odd prime $p$ and a $p$-group $P$
(3) \( P \times C_3 \) for a 2-group \( P \) such that any element of \( P \) is self-conjugate

Conversely, we obtain

**Proposition 5.2.** Let \( N \) be a nilpotent group with \( LO(N) = 0 \). Then \( N \) is isomorphic to (1), (2) or (3) in Theorem 5.1.

Let \( G \) be a nonsolvable group with \( a_G \geq 2 \). We point out again that Morimoto obtained \( Sm(\text{Aut}(A_6)) = 0 \) and \( Sm(PSL(2,27)) \neq 0 \). So, suppose that \( G \neq \text{Aut}(A_6), PSL(2,27) \).

Pawalowski and Solomon obtained that \( a_G > b_{G/G^{nil}} \) and \( LO(G) \supset PO(G,G^{nil}) \neq 0 \). Clearly the existence of a basic \( G^{nil} \)-pair yields \( a_G > b_{G/G^{nil}} \).

If \( G \) is isomorphic to (1), (2) or (3) in Theorem 5.1, then we can show that there exists a basic \( G^{nil} \)-pair by the similar argument of the section 2 in [54] and then \( LSm(G) \neq 0 \) follows.

**Remark 5.3.** For a nonsolvable group with \( a_G \geq 2 \), \( LO(G) = 0 \) implies that \( G \) is isomorphic to either \( \text{Aut}(A_6) \) or \( PSL(2,27) \).

6. **Computation by GAP**

We computed solvable Oliver groups \( G \) with \( LO(G) = 0 \) and \( a_G \geq 2 \) of order up to 2000 by using a software GAP [23] and found twelve groups of which ten groups are gap groups and the others are not.

**Proposition 6.1.** If \( G \) is an Oliver group then the order of \( G \) is divisible by at least 3 distinct primes.

We obtain 4 counterexamples to Laitinen’s Conjecture:

**Laitinen’s Conjecture.** It might hold \( LSm(G) \neq 0 \) for an Oliver group \( G \) with \( a_G \geq 2 \).

A counterexample is found first by Morimoto for \( G = \text{Aut}(A_6) \). The key point is next.

**Lemma 6.2 ([42]).** If \( U \) and \( V \) are Smith equivalent representations then \( U^N \) and \( V^N \) are isomorphic for each subgroup \( N \) of \( G \) with index 1 or 2.

This means that

\[ Sm(G) \leq \bigcap_N PO(G, N) \quad \text{and} \quad LSm(G) \leq \bigcap_N PO(G, N) \]

where \( N \) runs over subgroups of \( G \) with index 2.

**Proposition 6.3.** If \( G/G^{nil} \) is an elementary abelian 2-group then \( LSm(G) \subseteq LO(G) \). In addition if \( G \) is a gap Oliver group, it holds the equality \( LSm(G) = LO(G) \).
SG(72, 44), SG(288, 1025), SG(432, 734), SG(576, 8654) are our counterexamples. Here \(SG(ord, type)\) is denoted the group \(\text{SmallGroup}(ord, type)\) in the software GAP of order \(ord\). Note that \(SG(72, 44)\) and \(SG(288, 1025)\) are gap groups and the others are not.

<table>
<thead>
<tr>
<th>(G)</th>
<th>(a_G)</th>
<th>(LS m(G) = 0)</th>
<th>8 condition</th>
<th>(Sm(G) = 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(SG(72, 44))</td>
<td>2</td>
<td>True</td>
<td>Hold</td>
<td>True</td>
</tr>
<tr>
<td>(SG(288, 1025))</td>
<td>2</td>
<td>True</td>
<td>Hold</td>
<td>True</td>
</tr>
<tr>
<td>(SG(432, 734))</td>
<td>2</td>
<td>True</td>
<td>Not hold</td>
<td>True</td>
</tr>
<tr>
<td>(SG(576, 8654))</td>
<td>3</td>
<td>True</td>
<td>Hold</td>
<td>True</td>
</tr>
</tbody>
</table>

**Table 1.** Counterexamples to Laitinen's Conjecture

For a subset \(S\) of \(RO(G)\), we define rank \(S\) by

\[
\text{rank } S = \max \{ \text{rank } A \mid A \text{ is a subgroup and } A \subseteq S \}.
\]

By definition it holds \(WLO(G) \leq \text{rank } LS m(G) \leq \text{rank } PO(G, O^p(G))\) for each prime \(p\).

Morimoto shows \(LS m(G) \neq 0\) for \(G = SG(864, 2666), SG(864, 4666)\) as well as \(P\Sigma L(2, 27)\) and then it is unknown whether \(LS m(G) = 0\) or not for the following six gap groups \(G\).

<table>
<thead>
<tr>
<th>(G)</th>
<th>(a_G)</th>
<th>(\text{rank } LS m(G))</th>
<th>(G/G^\text{nil})</th>
<th>(LS m(G) = Sm(G))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(SG(864, 4663))</td>
<td>3</td>
<td>0, 1, 2.</td>
<td>(C_8)</td>
<td>False</td>
</tr>
<tr>
<td>(SG(864, 4672))</td>
<td>5</td>
<td>0, 1</td>
<td>(Q_8 \times C_3)</td>
<td>True</td>
</tr>
<tr>
<td>(SG(1176, 220))</td>
<td>2</td>
<td>0, 1</td>
<td>(C_3)</td>
<td>True</td>
</tr>
<tr>
<td>(SG(1176, 221))</td>
<td>2</td>
<td>0, 1</td>
<td>(C_3)</td>
<td>True</td>
</tr>
<tr>
<td>(SG(1152, 155470))</td>
<td>3</td>
<td>0, 1</td>
<td>(C_6)</td>
<td>True</td>
</tr>
<tr>
<td>(SG(1152, 157859))</td>
<td>3</td>
<td>0, 1</td>
<td>(C_6)</td>
<td>True</td>
</tr>
</tbody>
</table>

7. **Problem**

In the section we post a problem with respect to an approach to show \(LO(G) \subseteq LS m(G)\).
Problem 7.1. Let \( G \) be an Oliver group which is not a gap group and let \( K \) be a subgroup of \( G \) with \( K > O^2(G) \). Is either \( C_K(x) \) or \( C_K(y) \) a 2-group for involutions \( x \) and \( y \) of \( K \) outside of \( O^2(K) \) which are not conjugate in \( G \)?

The author confirmed that this problem is affirmative for all groups of order less than 2000.

Theorem 7.2. Let \( G \) be an Oliver group which is not a gap group. Suppose that the problem is affirmative for each \( K \). Then it holds \( 2LO(G) \subseteq WLO(G) \subseteq LO(G) \). In particular, it holds that \( \text{rank } LO(G) \leq \text{rank } LS m(G) \).

Note that \( LO(G) = WLO(G) \) if \( G \) is a gap group.

Putting together with Proposition 6.3, we obtain

Corollary 7.3. Let \( G \) be an Oliver group which is not a gap group. Suppose that the problem is affirmative for each \( K \). If \( G/G^{\text{nil}} \) is an elementary abelian 2-group then it holds \( WLO(G) = LO(G) = LS m(G) \). In particular, \( LS m(G) \) is a group.

Finally we point out that the problem is affirmative if and only if there exists \( U - V \in LO(G) \) such that both two representations \( U \oplus W \) and \( V \oplus W \) satisfy (1) of the weak gap condition for any representation \( W \). The author hope the problem will be solved affirmative.

References


[54] Pawątowski, K., Solomon, R., Smith equivalence and finite Oliver groups with Laitinen number 0 or 1, Algebraic and Geometric Topology 2 (2002), 843–895.


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