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Kyoto University
Borsuk-Ulam Theorems for Set-valued Mappings

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1 Introduction

S.Eilenberg and D. Montgomery [2] gave the fixed point formula of acyclic mappings which is a generalization of Lefschetz's fixed point theorem. L. Górniewicz [6] has studied set-valued mappings and fixed point theorems for acyclic mappings. In this paper, the author shall give a proof of a coincidence theorem for a Vietoris mapping and a compact mapping and prove Borsuk-Ulam type theorems for a class of set-valued mappings.

When a closed subset $\varphi(x)$ in $Y$ is assigned for a point $x$ in $X$, we say that the correspondence is a set-valued mapping and write $\varphi : X \to Y$ by the Greek alphabet. For single-valued mapping, we write $f : X \to Y$ etc. by the Roman alphabet. A set-valued mapping is studied particularly in Chapter 2 in [6]. We assume that any set-valued mapping is upper semi-continuous.

The following theorem is our main theorem (cf. Theorem 2.7). From the theorem we obtain the fixed point theorem for admissible mapping.

**Main Theorem 1.** Let $X$ be an ANR space and $Y$ a paracompact Hausdorff space. Let $\varphi : Y \to X$ be a Vietoris mapping and $q : Y \to X$ be a compact mapping. Then $(p^*)^{-1}q^*$ is a Leray endomorphism. If the Lefschetz number $L((p^*)^{-1}q^*)$ is not zero, there exists a coincidence point $z \in Y$, that is, $p(z) = q(z)$.

Borsuk-Ulam type theorems are proved in the following theorems which are the generalizations of Theorem 43.10 in L.Górniewicz [6]. (cf. Theorem 3.5, Theorem 3.9). The author shall give the related results and the detail proofs in [13].

**Main Theorem 2.** Let $N$ be a paracompact Hausdorff space with a free involution $T$ and $M$ an $m$-dimensional closed topological manifold. If a set-valued mapping $\varphi : N \to M$ is $*$-admissible and satisfies $\varphi^* = 0$ for positive dimension and $c(N,T)^m \neq 0$, then there exists a point $x_0 \in N$ such that $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$. Moreover if $N$ is an $n$-dimensional closed topological manifold, it holds $\dim A(\varphi) \geq n - m$ where $A(\varphi) = \{ x \in N \mid \varphi(x) \cap \varphi(T(x)) \neq \emptyset \}$.

**Main Theorem 3.** Let $N$ be a closed topological manifold with a free involution $T$ which has the homology group of the $n$-dimensional sphere and $M$ be a closed topological manifold. If a set-valued mapping $\varphi : N \to M$ is admissible and $\varphi(N) \neq M$ and $n \geq m$, then there exists a point $x_0 \in N$ such that $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$. Moreover it holds $\dim A(\varphi) \geq n - m$ and $\text{Ind} A(\varphi) \geq n - m$. 

2 Coincidence Theorem

We give some remarks about several cohomology theories. Alexander-Spanier cohomology theory $\check{H}^*(-)$ is isomorphic to the singular cohomology theory $H^*(-)$ (cf. Theorem 6.9.1 in [14]), that is,

$$\check{H}^*(X) \cong H^*(X)$$

if the singular cohomology theory satisfies the continuity: $\lim_{\overline{U}} H^*(U) = H^*(x)$ where $\{U\}$ is a system of neighborhood of $x$.

For a paracompact Hausdorff space $X$, it holds also the isomorphism between Čech cohomology theory $\check{H}^*(-)$ with a constant sheaf and Alexander cohomology theory $\check{H}^*(-)$ (cf. Theorem 6.8.8 in [14])

$$\check{H}^*(X) \cong \check{H}^*(X).$$

For a locally compact subset $A$ of Euclidean neighborhood retract $X$ (cf. Chapter 4 in [1]), it holds also the isomorphism between Čech cohomology theory $\check{H}^*(-)$ and the singular cohomology theory $H^*(-)$

$$\check{H}^*(A) = \lim_{\overline{U}} H^*(U)$$

where $U$ is a neighborhood of $A$ in $X$. For Euclidean neighborhood retract $X$, it holds also the isomorphism $\check{H}^*(X) \cong H^*(X)$. Hereafter we use Alexander-Spanier (co)homology theory with a field as the coefficient and use the notation $H^*(X)$ instead of $\check{H}^*(X)$. When we have to distinguish them, we use the corresponding notation.

For a covering $\mathcal{U}$ of $X$, the simplicial complex $K(\mathcal{U})$ called the nerve of $\mathcal{U}$ is defined in §1.6 of Chapter 3 in [14] and the simplicial complex $X(\mathcal{U})$ called the Vietoris simplicial complex of $\mathcal{U}$ is defined in §5 of Chapter 6 in [14]. They are chain equivalent each other (cf. Exercises D of Chapter 6 in [14]). Clearly by the definition of Alexander cohomology theory, we have the isomorphism:

$$\lim_{\overline{U}} H^*(C^*(X(\mathcal{U}))) \cong \check{H}(X).$$

We have the cross products $\overline{\mu} : \check{H}^*(X, A) \otimes \check{H}^*(Y, B) \to \check{H}^*((X, A) \times (Y, B))$ and $\mu : H^*(X, A) \otimes H^*(Y, B) \to H^*((X, A) \times (Y, B))$ and the natural transformation $\tau : \check{H}(-) \to H^*(-)$ which satisfy the commutativity $\mu(\tau \otimes \tau) = \tau \overline{\mu}$.

In this paper, we shall work in the category of paracompact Hausdorff spaces and continuous mappings. We give some definitions and notation. Let $w_{K}^{U} \in H_{n}(U, U-K)$ be the cycle such that $(i_{x})_{*}(w_{K}^{U}) = w_{x} \in H_{n}(\mathbb{R}^{n}, \mathbb{R}^{n}-x)$ where $i_{x} : (U, U-K) \to (\mathbb{R}^{n}, \mathbb{R}^{n}-x)$. Define $\gamma_{0} \in H^{n}(\mathbb{R}^{n}, \mathbb{R}^{n}-0)$ the dual cocycle of $w_{0}$.

Definition 1. Define a class $\gamma_{K}^{U} \in H^{n}((U, U-K) \times K)$ by $\gamma_{K}^{U} = d^{*}(\gamma_{0})$ where $d : (U, U-K) \times K \to (\mathbb{R}^{n}, \mathbb{R}^{n}-0)$ defined by $d(x, y) = x - y$. 


**Definition 2.** A mapping \( f : X \to Y \) is called a Vietoris mapping, if it satisfies the following conditions:

1. \( f \) is proper and onto continuous mapping.
2. \( f^{-1}(y) \) is an acyclic space for any \( y \in Y \), that is, \( \tilde{H}^s(f^{-1}(y)) : G) = 0 \).

When \( f \) is closed and onto continuous mapping and satisfies the condition (2), we call it weak Vietoris mapping (abbrev. \( w \)-Vietoris mapping).

Note that a proper mapping is closed mapping. We need Alexander-Spanier cohomology for the proof of the Vietoris theorem (cf. Theorem 6.9.15 in [14]).

**Theorem 2.1** (Vietoris). Let \( f : X \to Y \) be a \( w \)-Vietoris mapping between paracompact Hausdorff spaces \( X \) and \( Y \). Then,

\[
f^* : H^m(Y : G) \to H^m(X : G)
\]

is an isomorphism for all \( m \geq 0 \).

A mapping \( f : X \to Y \) is called a compact mapping, if \( f(X) \) is contained in a compact set of \( Y \), or equivalently its closure \( \overline{f(Y)} \) is compact.

**Definition 3.** Let \( U \) an open set of the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) and \( Y \) be a paracompact Hausdorff space. For a \( w \)-Vietoris mapping \( p : Y \to U \) and a compact mapping \( q : Y \to U \), the coincidence index \( I(p, q) \) of \( p \) and \( q \) is defined by

\[
I(p, q)w_0 = \overline{q}_*(\overline{p})^{-1}_*(w^U_K)
\]

where \( K \) is a compact set satisfying \( q(Y) \subset K \subset U \), and \( \overline{p} : (Y, Y - p^{-1}(K)) \to (U, U - K) \) and \( \overline{q} : (Y, Y - p^{-1}(K)) \to (\mathbb{R}^n, \mathbb{R}^n - 0) \) are defined by \( \overline{p}(y) = p(y) \) and \( \overline{q}(y) = p(y) - q(y) \) respectively.

**Lemma 2.2.** It holds a formula:

\[
d_*(1 \times q_*(p_*^{-1}))\Delta_*(w^U_K) = I(p, q)w_0
\]

where \( \Delta(x) = (x, x) \), \( d(x, y) = x - y \).

In this section, we give a proof of the coincidence theorem which is different from L.Górniwicz [5, 6] and depends on the line of M. Nakaoka [8]. The following theorem is easily verified.

**Theorem 2.3.** Let \( U \) be an open set of the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) and \( Y \) a paracompact Hausdorff space. For \( p : Y \to U \) a \( w \)-Vietoris mapping and \( q : Y \to U \) a compact mapping, if the index \( I(p, q) \) is not zero, there exists a coincidence point \( z \in Y \), that is, \( p(z) = q(z) \).

Let \( V \) be a vector space and \( f : V \to V \) a linear mapping. Let \( f^k \) be the \( k \) time iterated composition of \( f \). Set \( N(f) = \cup_{k \geq 0} \ker f^k \) a subspace of \( V \) and \( \tilde{V} = V/N(f) \). Then \( f \) induces the linear mapping \( \tilde{f} : \tilde{V} \to \tilde{V} \) which is a monomorphism. When \( \dim \tilde{V} < \infty \), we define \( \text{Tr}(f) \) by \( \text{Tr}(f) \). In the case of \( \dim V < \infty \), it coincides with the classical one \( \text{Tr}(f) \).
Definition 4. Let \( \{V_k\}_k \) be a graded vector space and \( f = \{f_k : V_k \to V_k\}_k \) graded linear mapping. Define the generalized Lefschetz number for the case of \( \sum_{k \geq 0} \dim V_k < \infty \):

\[
L(f) = \sum_{k \geq 0} (-1)^k \text{Tr}(f_k)
\]

In this case, \( f = \{f_k\}_k \) is called a Leray endomorphism.

Lemma 2.4. In the following commutative diagram of graded vector spaces:

\[
\begin{array}{ccc}
V_k & \xrightarrow{\phi_k} & W_k \\
\downarrow f_k & \xleftarrow{\psi_k} & \downarrow g_k \\
V_k & \xrightarrow{\phi_h} & W_k
\end{array}
\]

If one of \( f = \{f_k\}_k \) and \( g = \{g_k\}_k \) is a Leray endomorphism, the other is also a Leray endomorphism, and \( L(f) = L(g) \) holds.

The following theorem is a new proof of a coincidence theorem which is based on M. Nakaoka [8].

Theorem 2.5. Let \( U \) be an open set in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) and \( Y \) a paracompact Hausdorff space. Let \( p : Y \to U \) be a \( w \)-Vietoris mapping and \( q : Y \to U \) be a compact mapping. Then \( (p^*)^{-1}q^* : H^*(U) \to H^*(U) \) is a Leray endomorphism and we have the following formula:

\[
L((p^*)^{-1}q^*) = I(p, q)
\]

Especially, if the Lefschetz number \( L((p^*)^{-1}q^*) \) is not zero, there exists a coincidence point \( z \in Y \) such that \( p(z) = q(z) \).

Proof: At first we remark that there exists a finite complex \( K \) in \( U \) such that \( q(Y) \subset K \subset U \). Here we subdivide \( U \) into small boxes whose faces are parallel to axes and construct the complex \( K \) by collecting small boxes which intersect with \( f(Y) \). Consider the following diagram:

\[
\begin{array}{ccc}
H^*(U) & \xrightarrow{\iota^*} & H^*(K) \\
\downarrow q^* & \xleftarrow{q''^*} & \downarrow q^* \\
H^*(Y) & \xrightarrow{j^*} & H^*(p^{-1}(K)) \\
\downarrow (p^*)^{-1} & \xleftarrow{(p'')^{-1}} & \downarrow (p''')^{-1} \\
H^*(U) & \xrightarrow{\iota^*} & H^*(K)
\end{array}
\]

where \( p', q' \) are restriction mappings of \( p, q \) to the subspace \( p^{-1}(K) \) respectively and \( q'' : Y \to K \) is defined by \( q' = q''j \) and \( q = iq'' \). Since \( (p^*)^{-1}q^* :
$H^*(K) \to H^*(K)$ is a Leray endomorphism, $(p^*)^{-1}q^* : H^*(U) \to H^*(U)$ is also a Leray endomorphism by Lemma 2.4. Then, we have

$$L((p^*)^{-1}q^*) = L((p^*)^{-1}q^*).$$

Consider the following diagram:

$$
\begin{array}{ccc}
H^*(K) & \cong & H^*(K) \\
\downarrow (p^*)^{-1}q'' & & \downarrow (p^*)^{-1}q' \\
H^*(U) & \overset{i^*}{\rightarrow} & H^*(K) \\
\downarrow (-) \cap w_K & & \downarrow (-1)^q \gamma_K/(\cdot) \\
H_*(U, U-K) & \cong & H_*(U, U-K)
\end{array}
$$

Clearly the upper square is commutative. The commutativity of lower square is proved by Lemma 3 in [8] for the singular (co)homology theory, that is, $i^*(x) = (-1)^q \gamma^U_K/(x \cap w^U_K)$ for $x \in H^*(U)$. Here since $K$ is a finite complex, $i^* : H^*(U) \to H^*(K)$ of Alexander-Spanier cohomology coincides with the one of the singular cohomology. We use $i^*$ of the singular cohomology to calculate $i^*$ of Alexander-Spanier cohomology. Note that Alexander-Spanier cohomology groups $H^*(U)$, $H^*(U, U-K)$, $H^*((U, U-K) \times K)$, $H^*(K)$ are coincide with ones of the singular cohomology.

Let $\{\alpha_\lambda\}, \{\beta_\mu\}, \{\gamma_\nu\}$ be basis of $H^*(U)$, $H^*(U, U-K)$, $H^*(K)$ respectively. We represent $\gamma^U_K \in H^*((U, U-K) \times K)$ as follows:

$$\gamma^U_K = \sum_{\mu, \nu} c_{\mu \nu} \beta_\mu \times \gamma_\nu$$

Since $p^*$ is isomorphic, we set

$$(p^*)^{-1}q''((\gamma_\xi)) = \sum_\lambda m_{\lambda \xi} \alpha_\lambda$$

We calculate the Lefschetz number $L((p^*)^{-1}q^*)$:

$$(-1)^q (p^*)^{-1}q^* (\gamma_\xi) = (-1)^q i^* (p^*)^{-1}q'' (\gamma_\xi)$$

$$= \gamma^U_K / ((p^*)^{-1}q''((\gamma_\xi)) \cap w^U_K)$$

$$= \sum_{\mu, \nu} c_{\mu \nu} (\beta_\mu \times \gamma_\nu) / ((p^*)^{-1}q''((\gamma_\xi)) \cap w^U_K)$$

$$= \sum_{\mu, \nu} c_{\mu \nu} \beta_\mu \cap (p^*)^{-1}q''((\gamma_\xi)) \cap w^U_K > \gamma_\nu$$

$$= \sum_{\mu, \nu} c_{\mu \nu} \beta_\mu \cap (\sum_\lambda m_{\lambda \xi} \alpha_\lambda) \cap w^U_K > \gamma_\nu$$

$$= \sum_{\lambda, \mu, \nu} c_{\mu \nu} m_{\lambda \xi} \beta_\mu \cap \alpha_\lambda \cap w^U_K > \gamma_\nu$$
Hence we obtain a result:

\[ L((p^\ast)^{-1}q^\ast) = \sum_{\lambda, \mu, \xi} c_{\mu \xi} m_{\lambda \xi} < \beta_{\mu} \cup \alpha_{\lambda}, w_K^U > \]

Next we calculate the incidence index \( I(p, q) \):

\[
I(p, q) = < \Delta^*(1 \times (p^\ast)^{-1}q''^\ast)(\gamma_K^U), w_K^U >
\]
\[
= \sum_{\mu, \nu} c_{\mu \nu} < \Delta^*(\beta_{\mu} \times (\sum_{\lambda} m_{\lambda \nu} \alpha_{\lambda}), w_K^U >
\]
\[
= \sum_{\lambda, \mu, \nu} c_{\mu \nu} m_{\lambda \nu} < \beta_{\mu} \cup \alpha_{\lambda}, w_K^U >
\]

From these results, we have \( L((p^\ast)^{-1}q^\ast) = I(p, q) \). Since \( L((p^\ast)^{-1}q^\ast) \) is equal to \( L((p^\ast)^{-1}q^\ast) \), we obtain the result \( L((p^\ast)^{-1}q^\ast) = I(p, q) \).

We obtain the second statement by the above result and Theorem 2.3.

Q.E.D.

We can generalize the result above to the case of ANR spaces through the line of L. Górniewicz [5, 6] by using the approximation theorem of Schauder.

**Theorem 2.6.** Let \( U \) be an open set in a norm space \( E \) and \( Y \) a paracompact Hausdorff space. Let \( p : Y \rightarrow U \) a \( w \)-Vietoris mapping and \( q : Y \rightarrow U \) be a compact mapping. Then \( (p^\ast)^{-1}q^\ast \) is a Leray endomorphism. We assume that the graph of \( qp^{-1} \) is closed. If the Lefschetz number \( L((p^\ast)^{-1}q^\ast) \) is not zero, there exists a coincidence point \( z \in Y \), that is, \( p(z) = q(z) \).

**Theorem 2.7.** Let \( X \) be an ANR space and \( Y \) a paracompact Hausdorff space. Let \( p : Y \rightarrow X \) a Vietoris mapping and \( q : Y \rightarrow X \) be a compact mapping. Then \( (p^\ast)^{-1}q^\ast \) is a Leray endomorphism. If the Lefschetz number \( L((p^\ast)^{-1}q^\ast) \) is not zero, there exists a coincidence point \( z \in Y \), that is, \( p(z) = q(z) \).

### 3 Borsuk-Ulam Type Theorem

When \( M \) has an involution \( T \), the equivariant diagonal \( \Delta : M \rightarrow M \times M \) is given by \( \Delta(x) = (x, T(x)) \). If \( T \) is trivial, \( \Delta \) is the ordinary diagonal. The involution \( T \) on \( M^2 \) is given by \( T(x, x') = (x', x) \). Hence \( \Delta \) is an equivariant mapping. Hereafter, we use the same notation for involutions, if there is not confusion. M.Nakaoka defined the equivariant Thom class in Lemma 2.2 of [12] (cf. §1 in [10]):

\[
\tilde{U}_M \in H^m(S^\infty \times_\pi (M^2, M^2 - \Delta M))
\]

where the involution \( \tilde{T} \) on \( S^\infty \times_\pi M^2 \) is given by \( \tilde{T}(x, y, y') = (Tx, y', y) \).
For a paracompact Hausdorff space $N$ with a free involution $T$, there exists an equivariant mapping $h : N \to S^\infty$. We also define the element:

$$\hat{U}_{N,M} \in H^m(N \times \pi(M^2, M^2 - \Delta M)$$

by $\hat{U}_{N,M} = (h \times \pi id_{M^2})^*(\hat{U}_M)$ for $h \times \pi id_{M^2} : N \times \pi(M^2, M^2 - \Delta M) \to S^\infty \times \pi(M^2, M^2 - \Delta M)$. Set

$$\Delta_N = j^*(\hat{U}_{N,M}) \in H^m(N \times \pi M^2)$$

where $j : N \times \pi M^2 \to N \times \pi(M^2, M^2 - \Delta(M))$. In the case of $N = S^\infty$ and the trivial involution $T$ on $M$, M.Nakaoka determined $\theta_\infty$ by Proposition 3.4 in [11].

A mapping $\hat{f}_\pi : N_\pi \to N \times \pi M^2$ is defined by $\hat{f}_\pi(x) = (x, f(x), f(Tx))$. Since we use Alexander-Spanier cohomology theory in this paper, we must treat carefully the results of M.Nakaoka. The following theorem is given in Theorem 3.5 in [11].

**Theorem 3.1** (Nakaoka). Let $N$ be a paracompact Hausdorff space with a free involution $T$, and $M$ be an $m$-dimensional closed topological manifold. Let $\{\alpha_1, \ldots, \alpha_s\}$ be a basis for $H^*(M)$, and set

$$d_\ast([M]) = \sum_{j,k} \eta_{j,k} a_j \times a_k \quad (\eta_{j,k} \in \mathbb{Z}/2)$$

where $a_i = \alpha_i \cap [M]$. Then, for any continuous mapping $f : N \to M$, it holds

$$\hat{f}_\pi^*(\theta_N) = \sum_{i \geq 0} c^{m-2i} Q(f^*v_i) + \sum_{j < k} (\eta_{j,k} + \eta_{j,j} \eta_{k,k}) \phi^*(f^*(\alpha_j) \cup T^*f^*(\alpha_k))$$

(1)

where $c = c(N, T)$ and $v_i = v_i(M)$ Wu class of $M$ and $\phi^* : H^*(N) \to H^*(N_\pi)$ is the transfer homomorphism.

The next theorem is proved in Proposition 1.3 in [10].

**Theorem 3.2.** Let $N$ be a paracompact Hausdorff space with a free involution $T$ and $M$ a closed topological manifold. If a continuous mapping $f : N \to M$ satisfies $\hat{f}_\pi^*(\theta_N) \neq 0$, the set $A(f) = \{y \in N \mid f(y) = f(Ty)\}$ is not empty set.

**Definition 5.** A set-valued mapping $\varphi : X \to Y$ is called admissible, if there exists a paracompact Hausdorff space $\Gamma$ satisfying the following conditions:

1. there exist a Vietoris mapping $p : \Gamma \to X$ and a continuous mapping $q : \Gamma \to Y$.

2. $\varphi(x) \supset q(p^{-1}(x))$ for each $x \in X$. 


\( \varphi : X \to Y \) is called \( w \)-admissible, if it satisfies the condition (2) and \( p \) is a \( w \)-Vietoris mapping.

A pair \( (p, q) \) of mappings \( p, q \) is called a selected pair of \( \varphi \). If \( \varphi : X \to Y \) satisfies the first condition and \( \varphi(x) = q(p^{-1}(x)) \) for each \( x \in X \), it is called \( s \)-admissible mapping.

**Definition 6.** A set-valued mapping \( \varphi : X \to Y \) is called \( * \)-admissible mapping, if it is admissible and satisfies \( p_{\varphi}^{*} : H^{*}(X) \to H^{*}(T) \).

**Theorem 3.3.** Let \( X \) be an ANR space and \( \varphi : X \to X \) compact admissible mapping. If \( L(\varphi^{*}) \) contains non-zero element, there exists a fixed point \( x_{0} \in X \), that is, \( x_{0} \in \varphi(x_{0}) \).

**Proof.** We can choose a selected pair \( (p, q) \) where a Vietoris mapping \( p : \Gamma \to X \) and a compact mapping \( q : \Gamma \to X \). We may assume \( L((p^{*})^{-1}q^{*}) \neq 0 \). By Theorem 2.7, there exists a coincidence point \( z \in \Gamma \) such that \( p(z) = q(z) \). We obtain the result.

Q.E.D.

Let \( N \) be a paracompact Hausdorff space with a free involution \( T \) and \( M \) a closed topological manifold without involution. For a set-valued mapping \( \varphi : N \to M, \tilde{N} \) is defined by

\[ \tilde{N} = \{(x, y, y') \in N \times M^{2} \mid x \in N, \ y \in \varphi(x), \ y' \in \varphi(T(x)) \} \]

A free involution \( \tilde{T} \) on \( \tilde{N} \) is given by \( \tilde{T}(x, y, y') = (Tx, y', y) \). \( \tilde{p} : \tilde{N} \to N \) is the projection. The following Lemma is a key result.

**Lemma 3.4.** Let \( \varphi : N \to M \) be an admissible mapping with a selected pair \( p : \Gamma \to N \) and \( q : \Gamma \to M \). Then \( H^{*}(\tilde{N}) \) and \( H^{*}(\tilde{N}_{\pi}) \) have direct summands \( H^{*}(N) \) and \( H^{*}(N_{\pi}) \) respectively. Moreover if \( N \) is a metric space and \( A \) is a \( \pi \)-invariant closed or open subspace of \( N \), then \( H^{*}(\tilde{N} - \tilde{p}^{-1}(A)) \) and \( H^{*}(\tilde{N}_{\pi} - \tilde{p}_{\pi}^{-1}(A_{\pi})) \) have direct summands \( H^{*}(N - A) \) and \( H^{*}(N_{\pi} - A_{\pi}) \) respectively.

**Theorem 3.5.** Let \( N \) be a paracompact Hausdorff space with a free involution \( T \) and \( M \) an \( m \)-dimensional closed topological manifold. If a set-valued mapping \( \varphi : N \to M \) is \( * \)-admissible and satisfies \( \varphi^{*} = 0 \) for positive dimension and \( c(N, T)^{m} \neq 0 \), then there exists a point \( x_{0} \in N \) such that \( \varphi(x_{0}) \cap \varphi(T(x_{0})) \neq \emptyset \). Moreover if \( N \) is an \( n \)-dimensional closed topological manifold, it holds \( \dim A(\varphi) \geq n - m \) where \( A(\varphi) = \{ x \in N \mid \varphi(x) \cap \varphi(T(x)) \neq \emptyset \} \).

**Proof.** We can define a free involution \( \tilde{T} \) on \( \tilde{N} \) by \( \tilde{T}(x, y, y') = (T(x), y', y) \) and a mapping \( \tilde{\varphi} : \tilde{N} \to M \) by \( \tilde{\varphi}(x, y, y') = y \). We note:

\[ A(\tilde{\varphi}) = \{(x, y, y) \in \tilde{N} \mid y \in \tilde{\varphi}(x), \ y \in \tilde{\varphi}(\tilde{T}(x))\} \]

Now consider the following diagram:
where $	ilde{p}(x, y, y') = x$, $\tilde{p}'(x, y, y') = (x, y)$ and $p_\varphi(x, y) = x$, $q_\varphi(x, y) = y$.

We see $\tilde{\varphi}^* = 0$ from $\varphi^* = 0$. The mapping $\tilde{p} : \tilde{N} \to N$ is $\pi$-equivariant, that is $\tilde{p}(T(x, y, y')) = T(\tilde{p}(x, y, y'))$. Since $\tilde{p}_\pi^*$ is injective by Lemma 3.4. We have $\tilde{c}^m = c(\tilde{N}, \tilde{T})^m = \tilde{p}_\pi^*(c^m) \neq 0$ because of $\pi$-equivariant mapping $\tilde{p} : \tilde{N} \to N$.

Now we calculate $\tilde{\varphi}^*(\theta_\pi)$. Since we have $\varphi^*(\varphi_\pi) \cup T^*\tilde{\varphi}^*(\alpha_k) = 0$ and $\tilde{c}^{m-2i}Q(\tilde{\varphi}^*(\alpha_i)) = 0$ for $i > 0$ from our condition and $\tilde{c}^mQ(\tilde{\varphi}^*(\varphi_0)) = \tilde{c}^m \neq 0$, we obtained $\tilde{\varphi}^*(\theta_\pi) = \tilde{c}^m \neq 0$ from the formula (1) in Theorem 3.1. We conclude $A(\tilde{\varphi}) \neq 0$ from Theorem 3.2. Hence we obtain the former result.

Since $\tilde{N} - A(\tilde{\varphi})$, $\tilde{N} - \tilde{p}^{-1}A(\varphi)$, $N - A(\varphi)$ have natural involutions induced by $\tilde{T}$, $T$, we obtained $\tilde{N}_\pi - A(\tilde{\varphi})$, $\tilde{N}_\pi - \tilde{p}^{-1}A(\varphi)$, $N - A(\varphi)$. For the latter proof, we consider the following diagram:

$$
\begin{array}{ccc}
H^*(\tilde{N}_\pi, \tilde{N}_\pi - A(\tilde{\varphi})) & \xrightarrow{j_1^*} & H^*(\tilde{N}_\pi) \\
\downarrow k_1' & & \downarrow \text{id} \\
H^*(\tilde{N}_\pi, \tilde{N}_\pi - \tilde{p}^{-1}A(\varphi)) & \xrightarrow{j_2^*} & H^*(\tilde{N}_\pi - \tilde{p}^{-1}A(\tilde{\varphi})) \\
\uparrow \tilde{p}_\pi' & & \uparrow \tilde{p}_\pi \\
H^*(N, N - A(\varphi)) & \xrightarrow{j_3^*} & H^*(N, N - A(\varphi)) \\
\end{array}
$$

where $k_1$, $k_2$ are induced by natural inclusions and $\tilde{p}_1$, $\tilde{p}_2$ are induced by $\tilde{p}$. Here we note $\tilde{H}^*(-) \cong H^*(-)$ for manifolds. Since $A(\varphi)$ is a $\pi$-invariant closed subset of $N$, we have an into-isomorphism $(\tilde{p}_2)^*: H^*(N - A(\varphi)) \to H^*(\tilde{N}_\pi - \tilde{p}^{-1}A(\tilde{\varphi}))$ by Lemma 3.4. We note that $\tilde{\varphi}_\pi^*(\theta_\pi) = \tilde{c}^m \neq 0$ is an image of $c^m \in H^*(N)$, that is, $(\tilde{p}_\pi)^*(c^m) = \tilde{c}^m$. Since $\tilde{c}^m$ is an image of $\tilde{\varphi}_\pi^*(U_{M, U_N})$ under $j_1^*$, it holds $i_3^*(\tilde{c}^m) = 0$. From this, we see $(\tilde{p}_2)^*i_3^*c^m = (i_3^*)^*(\tilde{c}^m) = 0$ in the above diagram and hence $(i_3)^*c^m = 0$ because of the injectivity of $(\tilde{p}_2)^*$. If $H^m(N, N - A(\varphi)) = 0$, we easily see $c^m = 0$ which contradicts $c^m \neq 0$. Hence we obtain $H^m(N, N - A(\varphi)) \neq 0$.

Since $N$ and $N - A(\varphi)$ are manifolds, the singular homology group $H_m(N, N - A(\varphi)) \neq 0$ by the universal coefficient theorem. We obtain that the Čech cohomology group $\tilde{H}^m(N, N - A(\varphi)) \not\cong 0$ by Poincaré duality. In this case $\tilde{H}^m(N, N - A(\varphi))$ is equal to Alexander-Spanier cohomology group $H^m(N, N - A(\varphi))$. We see $\dim A(\varphi) \geq n - m$ and hence $\dim A(\varphi) \geq n - m$.

Q.E.D.

**Corollary 3.6.** Let $N$ be a paracompact Hausdorff space with a free involution $T$ which has a homology group of $n$-dimensional sphere and $M$ be an $m$-dimensional closed topological manifold. If a set-valued mapping $\varphi : N \to M$ is $\varphi$-admissible and satisfies $\varphi^* = 0$ and $n \geq m$, then there exists a point $x_0 \in N$ such that $\varphi(x_0) \cap \varphi(T(x_0)) \neq 0$. Moreover if $N$ is an $n$-dimensional closed topological manifold, it holds $\dim A(\varphi) \geq n - m$.

Let $X$ be a space with a free involution $T$ and $S^k$ the $k$-dimensional sphere with the antipodal involution. Define

$$
\gamma(X) = \inf \{ k \mid X \to S^k \text{ equivariant mapping} \}
$$

$$
\text{Ind}(X) = \sup \{ k \mid c^k \neq 0 \}.
$$
where $c \in H^1(X_\pi)$ is the class $c = f_\pi^*(\omega)$ for an equivariant mapping $f : X \to S^\infty$. If $X$ is a compact space with a free involution, it holds the following formula (cf. §3 in [3]):

$$\text{Ind}(X) \leq \gamma(X) \leq \dim X.$$ 

K. Gęba and L. Górniewicz determined $\text{Ind}A(\varphi)$ of an admissible mapping $\varphi : S^{n+k} \to R^n$ in [3]. We generalize their result.

**Corollary 3.7.** Let $N$ be a closed topological manifold with a free involution $T$ which has a homology group of $n$-dimensional sphere and $M$ be an $m$-dimensional closed topological manifold. If a set-valued mapping $\varphi : N \to M$ is $*-$admissible and $\varphi^* = 0$ and $n \geq m$, it holds $\text{Ind}A(\varphi) \geq n - m$.

**Proof.** At first, we remark commutativity of the following diagram for $n$-dimensional closed topological manifold $X$ and a closed subset $Y$ of $X$:

$$
\begin{array}{ccc}
H_k(X) & \overset{j^*}{\rightarrow} & H_k(X, X - Y) \\
\downarrow -\backslash U_0 & & \downarrow -\backslash U_1 \\
H^{n-k}(X) & \overset{i^*}{\rightarrow} & H^{n-k}(Y)
\end{array}
$$

where $U_0, U_1$ are restrictions of $U \in H^n(X^2, X^2 - d(X))$ for $k : (X^2, \emptyset) \to (X^2, X^2 - d(X))$, $l : (X, X - Y) \times Y \to (X^2, X^2 - d(X))$ respectively. Here the vertical arrows are Poincaré isomorphisms.

We apply the above diagram for the case $X = N_\pi$, $Y = A(\varphi)$. In the proof of the Theorem 3.5, we find a class $\alpha \in H^m(N_\pi, N_\pi - A(\varphi)_\pi)$ such that $j^*(\alpha) = c^m$. Let $b \in H_m(N_\pi)$ be the dual element of $c^m \in H^m(N_\pi)$ and $a \in H_m(N_\pi, N_\pi - A(\varphi)_\pi)$ be the dual class of $\alpha$. Then we obtain $j^*(b) = a \neq 0$. Since the Poincaré dual of $b$ is $c^{n-m}$, we obtain $i^*(c)^{n-m} = i^*(c^{n-m}) \neq 0$ by the above diagram. Hence we obtain the result. 

Q.E.D.

**Theorem 3.8.** Let $N$ be a paracompact Hausdorff space with a free involution $T$ and $M$ be an $m$-dimensional closed topological manifold which has a homology group of $m$-dimensional sphere. If a set-valued mapping $\varphi : N \to M$ is admissible and satisfies $c(N, T)^m \neq 0$ and $\varphi(N) \neq M$, then there exists a point $x_0 \in N$ such that $\varphi(x_0) \cap \varphi(T(x_0)) \neq 0$. Moreover if $N$ is an $n$-dimensional closed topological manifold, it holds $\dim A(\varphi) \geq n - m$.

**Proof.** We use the notation and method in the proof of Theorem 3.5. A homology group of $M' = M - \{a\}$ is trivial for positive dimensions by a homology group of $M$. From the fact and $\varphi(N) \neq M$, we have $\tilde{\varphi}^* = 0$ for positive dimensions. We see that $\tilde{c}^m = c(N, T)^m \neq 0$ by our assumption. By the similar method of Theorem 3.5, we see $\tilde{\varphi}^*(\theta_\tilde{N}) = \tilde{c}^m \neq 0$ by $\tilde{\varphi}^* = 0$ for positive dimension and $c(N, T)^m \neq 0$. Hence there exists a point $z_0 \in \tilde{N}$ such that $\tilde{\varphi}(z_0) = \tilde{\varphi}(\tilde{T}(z_0))$. We obtain $\varphi(x_0) \cap \varphi(T(x_0)) \neq 0$ for $x_0 \in N$. 


We can prove the last statement as in the proof of Theorem 3.5. We omit the proof. Q.E.D.

**Theorem 3.9.** Let $N$ be a closed topological manifold with a free involution $T$ which has the homology group of the $n$-dimensional sphere and $M$ be a closed topological manifold. If a set-valued mapping $\varphi : N \to M$ is admissible and $\varphi(N) \neq M$ and $n \geq m$, then there exists a point $x_0 \in N$ such that $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$. Moreover it holds $\dim A(\varphi) \geq n - m$ and $\text{Ind} A(\varphi) \geq n - m$.

**Proof.** We use the notation and method in the proof of Theorem 3.5. We remark $v_i(M) = 0$ for $i > \frac{n}{2}$ by the definition of Wu class. Therefore we see $\check{\varphi}(v_i(M)) = 0$ for $i > 0$ because of $H^*(N) = H^*(S^n)$. We see also $\phi^*(\check{\varphi}^*(\alpha_i) \cup \check{T}^*\check{\varphi}^*(\alpha_j)) = 0$ by $H^*(N) = H^*(S^n)$ and $\deg \alpha_i + \deg \alpha_j = m$ and $\check{\varphi}^*(\alpha_0) = 0$ for the class $\alpha_0$ such that $\deg \alpha_0 = m$. Note $\check{c}^m = c(N, T)^m \neq 0$ by our assumption. From this remark we see

$$\check{\varphi}^*(\theta_N) = \check{c}^m = c(N, T)^m \neq 0.$$ 

Therefore there exists a point $z_0 \in \check{N}$ such that $\check{\varphi}(z_0) = \check{\varphi}(\check{T}(z_0))$. We obtain $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$ for $x_0 \in N$. We can prove the last statement as in the proof of Theorem 3.5 and Corollary 3.7. We omit the proof. Q.E.D.

**References**


