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<th>Construction of smooth actions on spheres for Smith equivalent representations (The theory of transformation groups and its applications)</th>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2007), 1569: 52-58</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2007-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/81251">http://hdl.handle.net/2433/81251</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Construction of smooth actions on spheres
for Smith equivalent representations

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1. Problems and results

Throughout this paper, let $G$ be a finite group. A real $G$-representation of finite
dimension is meant by a real $G$-module, a smooth manifold is meant by a manifold,
and a smooth $G$-manifold is meant by a $G$-manifold. For a $G$-manifold $X$, let $\mathcal{TR}(X)$
denote the set of all isomorphism classes (as real $G$-modules) of tangential representations
$T_x(X)$, where $x$ runs over the $G$-fixed point set $X^G$. We are interested in $\mathcal{TR}(X)$
for manifolds $X$ such that $X^G$ consists of exactly two points. In particular, the case where
$X$ are homotopy spheres has been studied as Smith Problem.

Smith Problem. Let $\Sigma$ be a homotopy sphere with $G$-action such that the $G$-fixed
point set consists of exactly two points $a$, $b$. Are the tangential representations $T_a(\Sigma)$
and $T_b(\Sigma)$ isomorphic to each other (namely $|\mathcal{TR}(\Sigma)| = 1$) ?

We have affirmative answers (e.g. Atiyah-Bott, Milnor, Sanchez) as well as negative
answers (e.g. Petrie, Cappell-Shaneson, Petrie-Randall, Petrie-Dovermann, Dovermann-
Washington, Dovermann-Suh, Laitinen-Pawalowski, Pawalowski-Solomon), to Smith Problem
under various hypotheses. There are surveys relevant to studies on Smith Problem
in [24], [18] and [6].

To study the problem, we define the following relations $\sim_{\mathcal{D}}, \sim_{\emptyset}$ and $\sim_{\mathcal{D}\emptyset}$. In the
definition below, $V$ and $W$ are real $G$-modules.

1. $V \sim_{\mathcal{D}} W$ if there exists a disk $D$ with $G$-action such that $D^G = \{a, b\}$ and
   $\{[V], [W]\} = \mathcal{TR}(D)$.
2. $V \sim_{\emptyset} W$ if there exists a homotopy sphere $\Sigma$ with $G$-action such that $\Sigma^G = \{a, b\}$
   and $\{[V], [W]\} = \mathcal{TR}(\Sigma)$.
3. $V \sim_{\mathcal{D}\emptyset} W$ if $V \sim_{\mathcal{D}} W$ and $V \sim_{\emptyset} W$.

2000 Mathematics Subject Classification. Primary: 57S17, 57S25, 55M35. Secondary: 20C05.
Here $\sim_\mathcal{D}$ and $\sim_\mathfrak{DS}$ may not be equivalence relations, although they stably yield equivalence relations. We have been interested in the relation $\sim_\mathcal{E}$ (namely the Smith equivalence), but in the present paper we will mainly pay our attention to the relation $\sim_\mathfrak{DS}$.

Let $\text{RO}(G)$ denote the real representation ring. We define the subsets $\mathcal{D}(G)$, $\mathcal{S}(G)$ and $\mathfrak{DS}(G)$ of $\text{RO}(G)$ by

$$
\mathcal{D}(G) = \{V - W \in \text{RO}(G) \mid V \sim_\mathcal{D} W\}
$$

$$
\mathcal{S}(G) = \{V - W \in \text{RO}(G) \mid V \sim_\mathcal{S} W\}
$$

$$
\mathfrak{DS}(G) = \mathcal{D}(G) \cap \mathcal{S}(G)
$$

The set $\mathcal{S}(G)$ was usually denoted by $\text{Sm}(G)$. By R. Oliver [16], there exists a disk with $G$-action with $|D^G| = 2$ if and only if $G$ is an Oliver group (namely, $G$ is not a mod $\mathcal{P}$ hyperelementary group). Thus it is worthwhile to study $\mathcal{D}(G)$ and $\mathfrak{DS}(G)$ only for Oliver groups $G$.

If $M$ is a subset of $\text{RO}(G)$ then for families $\mathcal{A}$, $\mathcal{B}$ consisting of subgroups of $G$ we define

$$
M_{\mathcal{A}} \overset{\text{def}}{=} \{x \in M \mid \text{res}_H^G x = 0 \ \forall \ H \in \mathcal{A}\}
$$

$$
M_{\mathcal{B}} \overset{\text{def}}{=} \{x = V - W \in M \mid V^K = 0 = W^K \ \forall \ K \in \mathcal{B}\}
$$

$$
M_{\mathcal{A}}^{\mathcal{B}} \overset{\text{def}}{=} \{x = V - W \in M_{\mathcal{A}} \mid V^K = 0 = W^K \ \forall \ K \in \mathcal{B}\}.
$$

Using the notation with the families

$$
\mathcal{P} = \mathcal{P}(G) \overset{\text{def}}{=} \{P \leq G \mid |P| = p^a \ (p \ \text{a prime})\}
$$

$$
\mathcal{N}_2 = \mathcal{N}_2(G) \overset{\text{def}}{=} \{N \leq G \mid |G/N| = 1, 2\}
$$

$$
\mathcal{N} = \mathcal{N}(G) \overset{\text{def}}{=} \{N \leq G \mid |G/N| = 1 \text{ or a prime}\}
$$

$$
\mathcal{L} = \mathcal{L}(G) \overset{\text{def}}{=} \{L \leq G \mid L \supseteq G^{\{p\}} \text{ for some prime } p\},
$$

we study the subsets $\mathcal{D}(G)$, $\mathcal{S}(G)$ and $\mathfrak{DS}(G)$ of $\text{RO}(G)$. Here the group $G^{\{p\}}$ is the smallest normal subgroup of $G$ with prime power index, namely

$$
G^{\{p\}} = \bigcap_{H \leq G : |G/H| = p^a \text{ for some } a} H.
$$

An element in $\mathcal{L}$ defined above is called a large subgroup of $G$.

Many authors (e.g. Petrie-Randall, Petrie-Dovermann, Dovermann-Washington, Dovermann-Suh, Laitinen-Pawlowski, Pawlowski-Solomon) found various pairs $(V, W)$ of nonisomorphic $\mathfrak{DS}$-related real $G$-modules $V$, $W$. But their $(V, W)$ with $V \sim_\mathfrak{DS} W$ satisfy $V^N = 0 = W^N$ for all $N \triangleleft G$ with prime index. In other words, they showed

$$
\mathfrak{DS}(G)^N \neq 0
$$
for various $G$. Now we recall the next proposition.

**Proposition 1** ([12], [13]). The implications $\mathfrak{S}(G) \subseteq \text{RO}(G)^{N_2}$ and $\mathfrak{D}\mathfrak{S}(G) \subseteq \text{RO}(G)^{N_2}$ hold.

These facts motivate us to study the following problem.

**Problem A.** Does there exist a finite group $G$ satisfying $\mathfrak{D}\mathfrak{S}(G) \neq \mathfrak{D}\mathfrak{S}(G)^N$?

The notion *gap module* is convenient to study this problem as well as Smith Problem. A real $G$-module $V$ is called a *gap module* if it satisfies the following conditions.

1. $V^L = 0$ for all $L \in \mathcal{L}(G)$.
2. $\dim V^P > \dim V^H$ for all pairs $(P, H)$ of subgroups of $G$ such that $P \in \mathcal{P}(G)$ and $H > P$.

A finite group $G$ is called a *gap group* if $G$ admits a gap real $G$-module. Pawalowski-Solomon showed in [18] that for an arbitrary nonsolvable gap group $G$ with $a_G \geq 2$ and $G \ncong P\Sigma L(2, 27)$,

$$\mathfrak{D}\mathfrak{S}(G) \supseteq \text{RO}(G)^P \neq 0.$$  

Since the appearance of this result, the next problem has been asked.

**Problem B.** Are the sets $\mathfrak{S}(G)$ and $\mathfrak{D}\mathfrak{S}(G)$ nontrivial in the case $G = P\Sigma L(2, 27)$?

The purpose of the present paper is to answer to Problems A and B, and we obtained the following results.

**Theorem 2.** For each odd prime $p$, there exist a gap Oliver group $G$ and real $G$-modules $V$ and $W$ such that $V \sim_{\mathfrak{D}} W$, $\dim V^N > 0$ and $\dim W^N = 0$ for some $N \triangleleft G$ with $|G/N| = p$, hence $\mathfrak{D}\mathfrak{S}(G) \neq \mathfrak{D}\mathfrak{S}(G)^N$.

Let $SG(m, n)$ denote the small group of order $m$ and type $n$ appearing in the computer software GAP [5].

**Theorem 3.** Let $G = P\Sigma L(2, 27)$, $SG(864, 2666)$, or $SG(864, 4666)$. Then $\text{RO}(G)^F = 0$ but

$$\mathfrak{S}(G) = \mathfrak{D}(G) = \mathfrak{D}\mathfrak{S}(G) = \text{RO}(G)^F \cong \mathbb{Z}.$$

2. **Additional Information**

For $g \in G$, let $(g)$ denote the conjugacy class of $g$ in $G$. The *real conjugacy class* $(g)^\pm$ of $g$ is the union of $(g)$ and $(g^{-1})$. Let $a_G$ denote the number of all real conjugacy classes
of elements $g$ of $G$ such that $g$ does not have prime power order. By the representation theory, we have

$$a_G = \text{rank } RO(G)_P.$$

Let $\delta$ denote the homomorphism from $RO(G)_P$ to $\mathbb{Z}$ given by

$$\delta([V] - [W]) = \dim V^G - \dim W^G.$$

Then by definition,

$$RO(G)_P^{\{G\}} = \text{Ker}[\delta : RO(G)_P \to \mathbb{Z}].$$

B. Oliver [17] showed that if $a_G \geq 1$ then

$$\text{Image}[\delta : RO(G)_P \to \mathbb{Z}] \supseteq 2\mathbb{Z}.$$ 

Thus the next proposition immediately follows.

**Proposition** (Laitinen-Pawalowski [8]). *If $a_G \geq 1$ then $\text{rank } RO(G)_P^{\{G\}} = a_G - 1$.***

In addition, B. Oliver [17] implies the next result.

**Theorem** (Oliver). *If $G$ is an Oliver group then $\mathcal{D}(G) = RO(G)_P^{\{G\}}$.*

Viewing these facts, E. Laitinen conjectured the next.

**Laitinen's Conjecture.** If $G$ is an Oliver group with $a_G \geq 2$ then $\mathcal{D}(G) \neq 0$.

This conjecture had been positively expected until 2006. We, however, have a negative example.

**Theorem 4** ([12], [13]). *Let $G = \text{Aut}(A_6)$. Then Laitinen's Conjecture fails, in fact $a_G = 2$ and $\mathcal{D}(G) = 0 = \mathcal{D}(G)$.*

Most finite Oliver groups are gap groups, but neither $S_5$ nor $\text{Aut}(A_6)$ is a gap group, where $S_5$ is the symmetric group on five letters and $A_6$ is the alternating group on six letters. Pawalowski-Solomon [18] showed the next theorem using a deleting-inserting theorem of $G$-fixed point sets to disks ([10], [15, Appendix]).

**Theorem** (Pawalowski-Solomon [18]). *If $G$ is a gap Oliver group then $RO(G)_P^{\mathcal{L}} \subseteq \mathcal{D}(G)$.*

On the other hand, they also showed the next result using the finite group theory.

**Theorem** (Pawalowski-Solomon [18]). *Let $G$ be a nonsolvable gap group with $a_G \geq 2$. If $G \not\cong P\Sigma L(2, 27)$ then $RO(G)_P^{\mathcal{L}} \neq 0$.***
Putting these results together, we obtain a corollary.

**Corollary** (Pawalowski-Solomon [18]). Let $G$ be a nonsolvable gap group with $a_G \geq 2$. If $G \not\cong P\Sigma L(2,27)$ then $\mathcal{D}\mathcal{G}(G) \neq 0$.

Since $S_5 \times C_2$, where $C_2$ is the cyclic group of order 2, is not a gap group, the next result is also interesting.

**Theorem** (X.M. Ju [6]). In the case $G = S_5 \times C_2$, the equalities

$$\mathcal{S}(G) = \mathcal{D}\mathcal{G}(G) = RO(G)_{\mathcal{P}}^\mathcal{L} \cong \mathbb{Z}$$

hold.

We obtained a deleting-inserting theorem [14] of new kind by employing an equivariant interpretation of Cappell-Shaneson's surgery obstruction theory for getting homology (possibly, not homotopy) equivalences as well as employing the induction theory of Wall's surgery obstruction groups. We state here the theorem in a simplified form.

**Theorem 5.** Let $G$ be an Oliver group and $Y$ a disk with $G$-action. Suppose the following conditions are satisfied.

1. $Y^G = \{y_1, \ldots, y_m\}$, where $m \geq 1$.
2. $\partial Y^L = \emptyset$ for all $L \in \mathcal{L}(G)$.
3. $\dim Y^H \geq 5$ for all mod $\mathcal{P}$ cyclic subgroups $H$, i.e. $1 \triangleleft P \triangleleft H$ cyclic.
4. $\dim Y^P > 2(\dim Y^H + 1)$ for all $P \in \mathcal{P}(G)$ and $H > P$.
5. $|\pi_1(Y^P)| < \infty$ and $|\pi_1(Y^P)|, |P| = 1$ for all $P \in \mathcal{P}(G)$.
6. The inclusion induced maps $\pi_1(\partial Y^P) \rightarrow \pi_1(Y^P)$ are isomorphisms for all $P \in \mathcal{P}(G)$.

Then there exists a disk $X$ with $G$-action such that $\partial X = \partial Y$ and $X^G = \emptyset$.

Remark that the union $\Sigma = X \cup_\emptyset Y$ identified along the boundaries of $X$ and $Y$ in the theorem above is a homotopy sphere such that $\mathcal{T}\mathcal{R}(\Sigma) = \mathcal{T}\mathcal{R}(Y)$. Since various $G$-actions on disks $Y$ are constructed by Oliver's theory [17], we would obtain $G$-actions on homotopy spheres $\Sigma$ from those on disks. In fact, the next result is an outcome of Theorem 5.

**Theorem 6.** Let $p$ be an odd prime. Let $G$ be an Oliver group such that $G = G^{(q)}$ for all primes $q \neq p$ and $|G/G^{(p)}| = p$. If $G$ has a dihedral subquotient $D_{2qr}$ (order $2qr$) with distinct primes $q$ and $r$ and further that $G$ contains distinct real $G$-conjugacy classes.
$(x)\pm, (y)\pm$ of elements $x, y$ not of prime power order, then $\mathfrak{D}\mathfrak{S}(G)$ contains a direct summand of $\text{RO}(G)$ of rank 1.

Theorems 2 and 3 follow from Theorem 6. In addition, we conclude the next.

**Theorem 7.** Laitinen's Conjecture is affirmative for any finite nonsolvable gap group.

**References**


