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Construction of smooth actions on spheres 
for Smith equivalent representations

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1. PROBLEMS AND RESULTS

Throughout this paper, let $G$ be a finite group. A real $G$-representation of finite dimension is meant by a real $G$-module, a smooth manifold is meant by a manifold, and a smooth $G$-manifold is meant by a $G$-manifold. For a $G$-manifold $X$, let $\mathcal{TR}(X)$ denote the set of all isomorphism classes (as real $G$-modules) of tangential representations $T_x(X)$, where $x$ runs over the $G$-fixed point set $X^G$. We are interested in $\mathcal{TR}(X)$ for manifolds $X$ such that $X^G$ consists of exactly two points. In particular, the case where $X$ are homotopy spheres has been studied as Smith Problem.

Smith Problem. Let $\Sigma$ be a homotopy sphere with $G$-action such that the $G$-fixed point set consists of exactly two points $a, b$. Are the tangential representations $T_a(\Sigma)$ and $T_b(\Sigma)$ isomorphic to each other (namely $|\mathcal{TR}(\Sigma)| = 1$)?

We have affirmative answers (e.g. Atiyah-Bott, Milnor, Sanchez) as well as negative answers (e.g. Petrie, Cappell-Shaneson, Petrie-Randall, Petrie-Dovermann, Dovermann-Washington, Dovermann-Suh, Laitinen-Pawalowski, Pawalowski-Solomon), to Smith Problem under various hypotheses. There are surveys relevant to studies on Smith Problem in [24], [18] and [6].

To study the problem, we define the following relations $\sim_D, \sim_E$ and $\sim_{DE}$. In the definition below, $V$ and $W$ are real $G$-modules.

(1) $V \sim_D W$ if there exists a disk $D$ with $G$-action such that $D^G = \{a, b\}$ and $\{[V], [W]\} = \mathcal{TR}(D)$.

(2) $V \sim_E W$ if there exists a homotopy sphere $\Sigma$ with $G$-action such that $\Sigma^G = \{a, b\}$ and $\{[V], [W]\} = \mathcal{TR}(\Sigma)$.

(3) $V \sim_{DE} W$ if $V \sim_D W$ and $V \sim_E W$.

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Here \( \sim_\mathfrak{D} \) and \( \sim_{\mathfrak{DS}} \) may not be equivalence relations, although they stably yield equivalence relations. We have been interested in the relation \( \sim_{\mathfrak{S}} \) (namely the Smith equivalence), but in the present paper we will mainly pay our attention to the relation \( \sim_{\mathfrak{DS}} \).

Let \( \text{RO}(G) \) denote the real representation ring. We define the subsets \( \mathfrak{D}(G) \), \( \mathfrak{S}(G) \) and \( \mathfrak{DS}(G) \) of \( \text{RO}(G) \) by

\[
\mathfrak{D}(G) = \{V - W \in \text{RO}(G) \mid V \sim_\mathfrak{D} W\} \\
\mathfrak{S}(G) = \{V - W \in \text{RO}(G) \mid V \sim_\mathfrak{S} W\} \\
\mathfrak{DS}(G) = \mathfrak{D}(G) \cap \mathfrak{S}(G)
\]

The set \( \mathfrak{S}(G) \) was usually denoted by \( \text{Sm}(G) \). By R. Oliver [16], there exists a disk with \( G \)-action with \( |D^G| = 2 \) if and only if \( G \) is an Oliver group (namely, \( G \) is not a mod \( \mathcal{P} \) hyperelementary group). Thus it is worthwhile to study \( \mathfrak{D}(G) \) and \( \mathfrak{DS}(G) \) only for Oliver groups \( G \).

If \( M \) is a subset of \( \text{RO}(G) \) then for families \( \mathcal{A}, \mathcal{B} \) consisting of subgroups of \( G \) we define

\[
M_\mathcal{A} \overset{\text{def}}{=} \{x \in M \mid \text{res}_H^G x = 0 \forall H \in \mathcal{A}\} \\
M_\mathcal{B} \overset{\text{def}}{=} \{x = V - W \in M \mid V^K = 0 = W^K \forall K \in \mathcal{B}\} \\
M_{\mathcal{A}\mathcal{B}} \overset{\text{def}}{=} \{x = V - W \in M_\mathcal{A} \mid V^K = 0 = W^K \forall K \in \mathcal{B}\}
\]

Using the notation with the families

\[
\mathcal{P} = \mathcal{P}(G) \overset{\text{def}}{=} \{P \leq G \mid |P| = p^a \ (p \text{ a prime})\} \\
\mathcal{N}_2 = \mathcal{N}_2(G) \overset{\text{def}}{=} \{N \leq G \mid |G/N| = 1, 2\} \\
\mathcal{N} = \mathcal{N}(G) \overset{\text{def}}{=} \{N \leq G \mid |G/N| = 1 \text{ or a prime}\} \\
\mathcal{L} = \mathcal{L}(G) \overset{\text{def}}{=} \{L \leq G \mid L \geq G^{(p)} \text{ for some prime } p\}
\]

we study the subsets \( \mathfrak{D}(G) \), \( \mathfrak{S}(G) \) and \( \mathfrak{DS}(G) \) of \( \text{RO}(G) \). Here the group \( G^{(p)} \) is the smallest normal subgroup of \( G \) with prime power index, namely

\[
G^{(p)} = \bigcap_{H \leq G : [G/H] = p^a \text{ for some } a} H.
\]

An element in \( \mathcal{L} \) defined above is called a large subgroup of \( G \).

Many authors (e.g. Petrie-Randall, Petrie-Dovermann, Dovermann-Washington, Dovermann-Suh, Laitinen-Pawalowski, Pawalowski-Solomon) found various pairs \((V, W)\) of nonisomorphic \( \mathfrak{DS} \)-related real \( G \)-modules \( V, W \). But their \((V, W)\) with \( V \sim_{\mathfrak{DS}} W \) satisfy \( V^N = 0 = W^N \) for all \( N \triangleleft G \) with prime index. In other words, they showed

\[
\mathfrak{DS}(G)^N \neq 0
\]
for various \( G \). Now we recall the next proposition.

**Proposition 1** ([12], [13]). The implications \( \mathfrak{S}(G) \subseteq \text{RO}(G)^N \) and \( \mathfrak{D}\mathfrak{S}(G) \subseteq \text{RO}(G)^P \) hold.

These facts motivate us to study the following problem.

**Problem A.** Does there exist a finite group \( G \) satisfying \( \mathfrak{D}\mathfrak{S}(G) \neq \mathfrak{D}\mathfrak{S}(G)^N \)?

The notion **gap module** is convenient to study this problem as well as Smith Problem. A real \( G \)-module \( V \) is called a **gap module** if it satisfies the following conditions.

1. \( V^L = 0 \) for all \( L \in \mathcal{L}(G) \).
2. \( \dim V^P > \dim V^H \) for all pairs \( (P, H) \) of subgroups of \( G \) such that \( P \in \mathcal{P}(G) \) and \( H > P \).

A finite group \( G \) is called a **gap group** if \( G \) admits a gap real \( G \)-module. Pawalowski-Solomon showed in [18] that for an arbitrary nonsolvable gap group \( G \) with \( a_G \geq 2 \) and \( G \not\cong P\Sigma L(2,27) \),

\[
\mathfrak{D}\mathfrak{S}(G) \subseteq \text{RO}(G)^P \neq 0.
\]

Since the appearance of this result, the next problem has been asked.

**Problem B.** Are the sets \( \mathfrak{S}(G) \) and \( \mathfrak{D}\mathfrak{S}(G) \) nontrivial in the case \( G = P\Sigma L(2,27) \)?

The purpose of the present paper is to answer to Problems A and B, and we obtained the following results.

**Theorem 2.** For each odd prime \( p \), there exist a gap Oliver group \( G \) and real \( G \)-modules \( V \) and \( W \) such that \( V \sim \mathfrak{D}\mathfrak{S} W \), \( \dim V^N > 0 \) and \( \dim W^N = 0 \) for some \( N \triangleleft G \) with \( |G/N| = p \), hence \( \mathfrak{D}\mathfrak{S}(G) \neq \mathfrak{D}\mathfrak{S}(G)^N \).

Let \( SG(m, n) \) denote the small group of order \( m \) and type \( n \) appearing in the computer software GAP [5].

**Theorem 3.** Let \( G = P\Sigma L(2,27), SG(864, 2666), \) or \( SG(864, 4666) \). Then \( \text{RO}(G)^P = 0 \) but

\[
\mathfrak{S}(G) = \mathfrak{D}(G) = \mathfrak{D}\mathfrak{S}(G) = \text{RO}(G)^{\{G\}} \cong \mathbb{Z}.
\]

2. Additional Information

For \( g \in G \), let \( (g) \) denote the conjugacy class of \( g \) in \( G \). The real conjugacy class \( (g)^\pm \) of \( g \) is the union of \( (g) \) and \( (g^{-1}) \). Let \( a_G \) denote the number of all real conjugacy classes.
of elements $g$ of $G$ such that $g$ does not have prime power order. By the representation theory, we have

$$a_G = \text{rank } RO(G)_P.$$  

Let $\delta$ denote the homomorphism from $RO(G)_P$ to $\mathbb{Z}$ given by

$$\delta([V] - [W]) = \dim V^G - \dim W^G.$$  

Then by definition,

$$RO(G)^{(G)}_P = \text{Ker}[\delta : RO(G)_P \to \mathbb{Z}].$$  

B. Oliver [17] showed that if $a_G \geq 1$ then

$$\text{Image}[\delta : RO(G)_P \to \mathbb{Z}] \supseteq 2\mathbb{Z}.$$  

Thus the next proposition immediately follows.

**Proposition** (Laitinen-Pawalowski [8]). If $a_G \geq 1$ then $\text{rank } RO(G)^{(G)}_P = a_G - 1$.

In addition, B. Oliver [17] implies the next result.

**Theorem** (Oliver). If $G$ is an Oliver group then $\mathfrak{D}(G) = RO(G)^{(G)}_P$.

Viewing these facts, E. Laitinen conjectured the next.

**Laitinen's Conjecture.** If $G$ is an Oliver group with $a_G \geq 2$ then $\mathfrak{D}(G) \neq 0$.

This conjecture had been positively expected until 2006. We, however, have a negative example.

**Theorem 4** ([12], [13]). Let $G = \text{Aut}(A_6)$. Then Laitinen's Conjecture fails, in fact $a_G = 2$ and $\mathfrak{S}(G) = 0 = \mathfrak{D}(G)$.

Most finite Oliver groups are gap groups, but neither $S_5$ nor $\text{Aut}(A_6)$ is a gap group, where $S_5$ is the symmetric group on five letters and $A_6$ is the alternating group on six letters. Pawalowski-Solomon [18] showed the next theorem using a deleting-inserting theorem of $G$-fixed point sets to disks ([10], [15, Appendix]).

**Theorem** (Pawalowski-Solomon [18]). If $G$ is a gap Oliver group then

$$RO(G)^{(G)}_P \subseteq \mathfrak{D}(G).$$  

On the other hand, they also showed the next result using the finite group theory.

**Theorem** (Pawalowski-Solomon [18]). Let $G$ be a nonsolvable gap group with $a_G \geq 2$. If $G \not\cong P\Sigma L(2,27)$ then

$$RO(G)^{(G)}_P \neq 0.$$
Putting these results together, we obtain a corollary.

**Corollary** (Pawalowski-Solomon [18]). Let $G$ be a nonsolvable gap group with $\alpha_G \geq 2$. If $G \not\cong P\Sigma L(2,27)$ then $\mathcal{D}(G) \neq 0$.

Since $S_5 \times C_2$, where $C_2$ is the cyclic group of order 2, is not a gap group, the next result is also interesting.

**Theorem** (X.M. Ju [6]). In the case $G = S_5 \times C_2$, the equalities

$$\mathfrak{S}(G) = \mathfrak{D}(G) = RO(G)^{\mathcal{P}} \cong \mathbb{Z}$$

hold.

We obtained a deleting-inserting theorem [14] of new kind by employing an equivariant interpretation of Cappell-Shaneson's surgery obstruction theory for getting homology (possibly, not homotopy) equivalences as well as employing the induction theory of Wall's surgery obstruction groups. We state here the theorem in a simplified form.

**Theorem 5.** Let $G$ be an Oliver group and $Y$ a disk with $G$-action. Suppose the following conditions are satisfied.

1. $Y^G = \{y_1, \ldots, y_m\}$, where $m \geq 1$.
2. $\partial Y^L = \emptyset$ for all $L \in \mathcal{L}(G)$.
3. $\dim Y^H \geq 5$ for all mod $\mathcal{P}$ cyclic subgroups $H$, i.e. $1 \triangleleft P \triangleleft H$ cyclic.
4. $\dim Y^P > 2(\dim Y^H + 1)$ for all $P \in \mathcal{P}(G)$ and $H > P$.
5. $|\pi_1(Y^P)| < \infty$ and $|\pi_1(Y^P)|, |P| = 1$ for all $P \in \mathcal{P}(G)$.
6. The inclusion induced maps $\pi_1(\partial Y^P) \to \pi_1(Y^P)$ are isomorphisms for all $P \in \mathcal{P}(G)$.

Then there exists a disk $X$ with $G$-action such that $\partial X = \partial Y$ and $X^G = \emptyset$.

Remark that the union $\Sigma = X \cup_\partial Y$ identified along the boundaries of $X$ and $Y$ in the theorem above is a homotopy sphere such that $\mathcal{T}(\Sigma) = \mathcal{T}(Y)$. Since various $G$-actions on disks $Y$ are constructed by Oliver’s theory [17], we would obtain $G$-actions on homotopy spheres $\Sigma$ from those on disks. In fact, the next result is an outcome of Theorem 5.

**Theorem 6.** Let $p$ be an odd prime. Let $G$ be an Oliver group such that $G = G^{(q)}$ for all primes $q \neq p$ and $|G/G^{(p)}| = p$. If $G$ has a dihedral subquotient $D_{2qr}$ (order $2qr$) with distinct primes $q$ and $r$ and further that $G$ contains distinct real $G$-conjugacy classes
$(x)^\pm, (y)^\pm$ of elements $x, y$ not of prime power order, then $\mathfrak{D}\mathfrak{S}(G)$ contains a direct summand of $\text{RO}(G)$ of rank 1.

Theorems 2 and 3 follow from Theorem 6. In addition, we conclude the next.

**Theorem 7.** Laitinen's Conjecture is affirmative for any finite nonsolvable gap group.

**References**


