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Kyoto University
Two New Nonexpansive Mappings and
Geometry of Banach Spaces

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Abstract. Our purpose in this article is to discuss new nonlinear operators in a Banach space which are related to nonexpansive mappings and to obtain convergence theorems for the operators. We first deal with a nonlinear operator called a relatively nonexpansive mapping which generalizes a nonexpansive mapping in a Hilbert space. Using this operator, we prove a strong convergence theorem which generalizes Nakajo and Takahashi [29]. We also obtain another theorem for relatively nonexpansive mappings which is connected with Reich's theorem [33]. Next, we define another nonlinear operator in a Banach space called a generalized nonexpansive mapping. This mapping also generalizes a nonexpansive mapping in a Hilbert space. Using this mapping, we also get a strong convergence theorem which is related to Nakajo and Takahashi [29] and is different from the theorem above. Further, we obtain a weak convergence theorem of Reich's type. Finally, we prove a strong convergence theorem for nonexpansive mappings in a Banach space which is closely related to Nakajo and Takahashi [29].

1 Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let $C$ be a nonempty closed convex subset of $H$. Then, a mapping $T$ of $C$ into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of $T$.

Mann [22] introduced the following iterative sequence to approximate a fixed point of a nonexpansive mapping: $x_1 = x$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. Reich [33] proved the following weak convergence theorem for such a sequence. For the proof, see Takahashi [46].

Theorem 1.1 (Reich [33]). Let $C$ be a closed convex subset of a Hilbert space $H$ and let $T$ be a nonexpansive mapping of $C$ into itself such that $F(T)$ is nonempty. Let $P$ be the metric projection of $H$ onto $F(T)$. Let $x \in C$ and let $\{x_n\}$ be a sequence defined by $x_1 = x$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \ldots,$$
where $\{\alpha_n\} \subset [0,1]$ satisfies

$$0 \leq \alpha_n < 1 \text{ and } \sum_{n=1}^{\infty} \alpha_n (1-\alpha_n) = \infty.$$ 

Then, $\{x_n\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n \to \infty} Px_n$.

Reich [33] proved really such a theorem in a uniformly convex Banach space whose norm is a Fréchet differentiable. On the other hand, we know many problems in nonlinear analysis and optimization which are formulated as follows: Find $u \in H$ such that $0 \in Au,$

(1.1)

where $A$ is a maximal monotone operator from $H$ to $H$. Such $u \in H$ is called a zero point (or a zero) of $A$. A well-known method for solving (1.1) in a Hilbert space $H$ is the proximal point algorithm: $x_1 \in H$ and

$$x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, \ldots,$$

(1.2)

where $\{r_n\} \subset (0, \infty)$ and $J_r = (I + rA)^{-1}$ for all $r > 0$. This algorithm was first introduced by Martinet [23]. In [39], Rockafellar proved that if $\lim \inf_{n \to \infty} r_n > 0$ and $A^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ defined by (1.2) converges weakly to a solution of (1.1). Motivated by Rockafellar's result, Kamimura and Takahashi [16] proved the following convergence theorem.

**Theorem 1.2 (Kamimura and Takahashi [16]).** Let $H$ be a Hilbert space and let $A \subset H \times H$ be a maximal monotone operator. Let $J_r = (I + rA)^{-1}$ for all $r > 0$ and let $\{x_n\}$ be a sequence defined as follows: $x_1 \in H$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$\limsup_{n \to \infty} \alpha_n < 1 \text{ and } \liminf_{n \to \infty} r_n > 0.$$ 

If $A^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element $v$ of $A^{-1}0$, where $v = \lim_{n \to \infty} Px_n$ and $P$ is the metric projection of $H$ onto $A^{-1}0$.

Solodov and Svaiter [41] also proved the following strong convergence theorem by the hybrid method in mathematical programming.

**Theorem 1.3 (Solodov and Svaiter [41]).** Let $H$ be a Hilbert space and let $A \subset H \times H$ be a maximal monotone operator. Let $x \in H$ and let $\{x_n\}$ be a sequence defined by

$$\begin{align*}
x_1 &= x \in H, \\
0 &= v_n + \frac{1}{r_n} (y_n - x_n), \quad v_n \in Ay_n, \\
H_n &= \{z \in H : \langle z - y_n, v_n \rangle \leq 0\}, \\
W_n &= \{z \in H : \langle z - x_n, x_1 - x_n \rangle \leq 0\}, \\
x_{n+1} &= P_{H_n \cap W_n} x_1, \quad n = 1, 2, \ldots,
\end{align*}$$

where $\{r_n\}$ is a sequence of positive numbers. If $A^{-1}0 \neq \emptyset$ and $\liminf_{n \to \infty} r_n > 0$, then $\{x_n\}$ converges strongly to $P_{A^{-1}0} x_1$. 

Motivated by Solodov and Svaiter [41], Nakajo and Takahashi [29] proved the following strong convergence theorem by using the hybrid method for nonexpansive mappings in a Hilbert space.

**Theorem 1.4 (Nakajo and Takahashi [29]).** Let $C$ be a closed convex subset of a Hilbert space $H$ and let $T$ be a nonexpansive mapping of $C$ into itself such that $F(T)$ is nonempty. Let $P$ be the metric projection of $H$ onto $F(T)$. Let $x_1 = x \in C$ and

$$
\begin{align*}
    y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\
    C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
    Q_n &= \{z \in C : (x_n - z, x_1 - x_n) \geq 0\}, \\
    x_{n+1} &= P_{C_n \cap Q_n}(x_1), \quad n = 1, 2, \ldots,
\end{align*}
$$

where $\{\alpha_n\} \subset [0, 1]$ satisfies $\limsup_{n \to \infty} \alpha_n < 1$ and $P_{C_n \cap Q_n}$ is the metric projection of $H$ onto $C_n \cap Q_n$. Then, $\{x_n\}$ converges strongly to $Px_1 \in F(T)$.

After Nakajo and Takahashi [29], many researchers have studied such theorems by hybrid methods in a Hilbert space; see, for instance, [14, 24, 42, 55]. However, we can not find a theorem for nonexpansive mappings in a Banach space which generalizes Nakajo and Takahashi [29].

Our purpose in this article is to consider new nonlinear operators in a Banach space for extending Nakajo and Takahashi’s result [29] in a Hilbert space to that in a Banach space.

In Section 3, we deal with a nonlinear operator in a Banach space called a relatively nonexpansive mapping which generalizes a nonexpansive mapping in a Hilbert space. We know that a relatively nonexpansive mapping in a Banach space is completely different from a nonexpansive mapping in a Banach space. In this section, we state a strong convergence theorem for relatively nonexpansive mappings which generalizes Nakajo and Takahashi [29]. We also obtain another theorem for relatively nonexpansive mappings which is connected with Reich’s theorem [33].

In Section 4, we define another nonlinear operator in a Banach space which generalizes a nonexpansive mapping in a Hilbert space. We call such a nonlinear operator a generalized nonexpansive mapping. In this section, we obtain a strong convergence theorem which is related to Nakajo and Takahashi [29] and is different from the result in Section 3. Further, we obtain a weak convergence theorem of Reich’s type. Finally, in Section 5, we prove a strong convergence theorem for nonexpansive mappings in a Banach space which is closely related to Nakajo and Takahashi [29].

## 2 Preliminaries

Let $E$ be a real Banach space with norm $\| \cdot \|$ and let $E^*$ denote the dual of $E$. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in $E$, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \rightharpoonup x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus $\delta$ of convexity of $E$ is defined by

$$
\delta(\epsilon) = \inf \left\{ \frac{1 - \|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}
$$
for every $\epsilon$ with $0 \leq \epsilon \leq 2$. A Banach space $E$ is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. If $E$ is uniformly convex, then $\delta$ satisfies that $\delta(\epsilon/r) > 0$ and

$$\left\| \frac{x+y}{2} \right\| \leq r \left( 1 - \delta \left( \frac{\epsilon}{r} \right) \right)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \epsilon$. Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x-z\| \leq \|x-y\|$ for all $y \in C$. Putting $z = PC(x)$, we call $P_C$ the metric projection of $E$ onto $C$. The duality mapping $J$ from $E$ into $2^{E^*}$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of $E$ is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists. In the case, $E$ is called smooth. The norm of $E$ is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space $E$ is called uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of $E$ is uniformly Gâteaux differentiable, then the duality mapping $J$ is single valued and uniformly norm to weak* continuous on each bounded subset of $E$. We know the following result: Let $E$ be a smooth Banach space. Let $C$ be a nonempty closed convex subset of $E$ and $x_1 \in E$. Then, $x_0 = PC(x_1)$ if and only if

$$\langle x_0 - y, J(x_1 - x_0) \rangle \geq 0$$

for all $y \in C$, where $J$ is the duality mapping of $E$.

A Banach space $E$ is said to satisfy Opial's condition [31] if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup y$ implies

$$\liminf_{n \to \infty} \|x_n - y\| < \liminf_{n \to \infty} \|x_n - z\|$$

for all $z \in E$ with $z \neq y$. A Hilbert space satisfies Opial's condition.

Let $C$ be a closed convex subset of $E$. A mapping $T : C \to E$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote the set of all fixed points of $T$ by $F(T)$. Let $D$ be a subset of $C$ and let $P$ be a mapping of $C$ into $D$. Then $P$ is said to be sunny if

$$P(Px + t(x - Px)) = Px$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A mapping $P$ of $C$ into $C$ is said to be a retraction if $P^2 = P$. We denote the closure of the convex hull of $D$ by $\overline{cD}$.

A multi-valued operator $A : E \to E^*$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \{Az : z \in D(A)\}$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$. A monotone operator $A$ is said to be maximal if its graph $G(A) = \{(x, y) : y \in Az\}$ is not properly contained in the graph of any other monotone operator. The following theorems are well known; for instance, see [46].

**Theorem 2.1.** Let $E$ be a reflexive, strictly convex and smooth Banach space and let $A : E \to 2^{E^*}$ be a monotone operator. Then $A$ is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$. 
Theorem 2.2. Let $E$ be a strictly convex and smooth Banach space and let $x, y \in E$. If $\langle x - y, Jx - Jy \rangle = 0$, then $x = y$.

A duality mapping $J$ of a smooth Banach space is said to be weakly sequentially continuous if $x_n \rightharpoonup x$ implies that $Jx_n \rightharpoonup Jx$, where $\rightharpoonup$ means the weak* convergence.

3 Relatively nonexpansive mappings

In this section, we first deal with a strong convergence theorem in a Banach space which generalizes Nakajo and Takahashi's theorem (Theorem 1.4) in a Hilbert space.

Let $E$ be a reflexive, strictly convex and smooth Banach space. The function $\phi: E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for $x, y \in E$, where $J$ is the duality mapping of $E$; see [1] and [18]. Let $C$ be a nonempty closed convex subset of $E$ and let $x \in E$. Then there exists a unique element $x_0 \in C$ such that

$$\phi(x_0, x) = \inf\{\phi(z, x) : z \in C\}. \quad (3.1)$$

Now, we define the mapping $Q_C$ of $E$ onto $C$ by $Q_Cx = x_0$, where $x_0$ is defined by (3.1). Such $Q_C$ is called the generalized projection of $E$ onto $C$. It is easy to see that in a Hilbert space, the mapping $Q_C$ is coincident with the metric projection.

Lemma 3.1. Let $E$ be a smooth Banach space, let $C$ be a nonempty closed convex subset of $E$, let $x \in E$ and let $x_0 \in C$. Then, the following (1) and (2) are equivalent:

1. $\phi(x_0, x) = \min_{y \in C} \phi(y, x)$;
2. $\langle x_0 - y, Jx - Jx_0 \rangle \geq 0$ for all $y \in C$.

Let $E$ be a smooth Banach space. Let $C$ be a closed convex subset of $E$, and let $T$ be a mapping from $C$ into itself. We denote by $F(T)$ the set of fixed points of $T$. A point $p$ in $C$ is said to be an asymptotic fixed point of $T$ [36] if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ and the strong limit $\lim_{n \to \infty} (x_n - Tx_n) = 0$. The set of asymptotic fixed points of $T$ will be denoted by $\hat{F}(T)$. A mapping $T$ from $C$ into itself is called relatively nonexpansive if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

The following is a strong convergence theorem for relatively nonexpansive mappings in a Banach space which generalizes Nakajo and Takahashi’s theorem [29] in a Hilbert space.

Theorem 3.2 (Matsushita and Takahashi [26]). Let $E$ be a uniformly convex and uniformly smooth Banach space, let $C$ be a nonempty closed convex subset of $E$, let $T$ be a relatively nonexpansive mapping from $C$ into itself with $F(T) \neq \emptyset$ and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n \to \infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by

$$\begin{cases}
x_{n+1} = Q_{\cap_{n=1}^{\infty} W_n}x, \\
y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n x_n), \\
H_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
x_{n+1} = Q_{\cap_{n=1}^{\infty} W_n}x
\end{cases}$$

for all $n = 1, 2, \ldots$, where $J$ is the duality mapping on $E$. Then $\{x_n\}$ converges strongly to $Q_{F(T)}x$, where $Q_{F(T)}$ is the generalized projection from $C$ onto $F(T)$. 
Using Theorem 3.2, we can prove Nakajo and Takahashi's theorem (Theorem 1.4) as follows: To show Nakajo and Takahashi's theorem, it is sufficient to prove that if $T$ is nonexpansive, then $T$ is relatively nonexpansive. It is obvious that $F(T) \subset \hat{F}(T)$. If $u \in \hat{F}(T)$, then there exists $\{x_n\} \subset C$ such that $x_n \rightarrow u$ and $x_n - Tx_n \rightarrow 0$. Since $T$ is nonexpansive, $T$ is demiclosed. So, we have $u = Tu$. This implies $F(T) = \hat{F}(T)$. Further, in a Hilbert space $H$, we know that

$$\phi(x, y) = \|x - y\|^2$$

for every $x, y \in H$. So, $\|Tx - Ty\| \leq \|x - y\|$ is equivalent to $\phi(Tx, Ty) \leq \phi(x, y)$. Therefore, $T$ is relatively nonexpansive. Using Theorem 3.2, we obtain the desired result.

Using Theorem 3.2, we can prove a strong convergence theorem for maximal monotone operators in a Banach space. Before stating the theorem, we define the following resolvents of maximal monotone operators in a Banach space. Let $E$ be a reflexive, strictly convex and smooth Banach space, and let $A$ be a maximal monotone operator from $E$ to $E^*$. Using Theorem 2.1 and the strict convexity of $E$, we obtain that for every $r > 0$ and $x \in E$, there exists a unique $x_r \in D(A)$ such that

$$Jx \in Jx_r + rAx_r. \tag{3.2}$$

If $Q_r x = x_r$, then we can define a single valued mapping $Q_r : E \rightarrow D(A)$ by $Q_r = (J + rA)^{-1}J$ and such $Q_r$ is called the relative resolvent of $A$. We know that $A^{-1}0 = F(Q_r)$ for all $r > 0$; see [45, 46] for more details.

**Theorem 3.3.** Let $E$ be a uniformly convex and uniformly smooth Banach space, let $A$ be a maximal monotone operator from $E$ to $E^*$, let $Q_r$ be the relative resolvent of $A$, where $r > 0$. If $A^{-1}0$ is nonempty, then $Q_r$ is a relatively nonexpansive mapping on $E$.

Using this result and Theorem 3.2, we prove a strong convergence theorem for relative resolvents of maximal monotone operators in a Banach space.

**Theorem 3.4.** Let $E$ be a uniformly convex and uniformly smooth Banach space, let $A$ be a maximal monotone operator from $E$ to $E^*$, let $Q_r$ be the relative resolvent of $A$, where $r > 0$ and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n \rightarrow \infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by

$$
\begin{cases}
    x_1 = x \in E, \\
    y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JQ_r x_n), \\
    H_n = \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
    W_n = \{z \in E : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
    x_{n+1} = Q_{H_n \cap W_n} x
\end{cases}
$$

for all $n = 1, 2, \ldots$, where $J$ is the duality mapping on $E$. If $A^{-1}0$ is nonempty, then $\{x_n\}$ converges strongly to $Q_{A^{-1}0} x$, where $Q_{A^{-1}0}$ is the generalized projection from $E$ onto $A^{-1}0$.

Next, we obtain a weak convergence theorem for relatively nonexpansive mappings in a Banach space which is connected with Reich [33], Browder and Petryshyn's theorem [6] and Rockafellar's theorem [39]. Before proving it, we need the following proposition.

**Proposition 3.5 (Matsushita and Takahashi [25]).** Let $E$ be a uniformly convex and uniformly smooth Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $T$ be a relatively nonexpansive mapping from $C$ into itself such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a
sequence of real numbers such that \(0 \leq \alpha_n \leq 1\). Let \(x_1 \in C\) and let \(\{x_n\}\) be the sequence defined by
\[
x_{n+1} = Q_CJ^{-1}((\alpha_n Jx_n + (1 - \alpha_n)JTx_n)
\]
for \(n = 1, 2, \ldots\). Then \(\{Q_{F(T)}x_n\}\) converges strongly to a fixed point of \(T\), where \(Q_{F(T)}\) is the generalized projection from \(C\) onto \(F(T)\).

Using Proposition 3.5, we can prove the following weak convergence theorem.

**Theorem 3.6 (Matsushita and Takahashi [25])**. Let \(E\) be a uniformly convex and uniformly smooth Banach space, let \(C\) be a nonempty closed convex subset of \(E\), and let \(T\) be a relatively nonexpansive mapping from \(C\) into itself such that \(F(T) \neq \emptyset\). Let \(\{\alpha_n\}\) be a sequence of real numbers such that
\[
0 \leq \alpha_n \leq 1 \quad \text{and} \quad \lim \inf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0.
\]

Let \(x_1 \in C\) and let \(\{x_n\}\) be the sequence defined by
\[
x_{n+1} = Q_CJ^{-1}((\alpha_n Jx_n + (1 - \alpha_n)JTx_n)
\]
for \(n = 1, 2, \ldots\). If \(J\) is weakly sequentially continuous, then \(\{x_n\}\) converges weakly to \(u\), where \(u = \lim_{n \to \infty} Q_{F(T)}x_n\) and \(Q_{F(T)}\) is the generalized projection from \(C\) onto \(F(T)\).

Using Theorem 3.6, we can prove the following two weak convergence theorems.

**Theorem 3.7 ([6])**. Let \(C\) be a nonempty closed convex subset of a Hilbert space \(H\), let \(T\) be a nonexpansive mapping from \(C\) into itself such that \(F(T) \neq \emptyset\) and let \(\lambda\) be a real number such that \(0 < \lambda < 1\). Let \(x_1 \in C\) and let \(\{x_n\}\) be the sequence defined by
\[
x_{n+1} = \lambda x_n + (1 - \lambda)Tx_n
\]
for \(n = 1, 2, \ldots\). Then \(\{x_n\}\) converges weakly to \(u\), where \(u = \lim_{n \to \infty} P_{F(T)}x_n\) and \(P_{F(T)}\) is the metric projection from \(C\) onto \(F(T)\).

**Theorem 3.8**. Let \(E\) be a uniformly convex and uniformly smooth Banach space, let \(A\) be a maximal monotone operator from \(E\) to \(E^*\) such that \(A^{-1}0 \neq \emptyset\), let \(Q_r\) be the relative resolvent of \(A\) where \(r > 0\), and let \(\{\alpha_n\}\) be a sequence of real numbers such that
\[
0 \leq \alpha_n \leq 1 \quad \text{and} \quad \lim \inf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0.
\]

Let \(x_1 \in E\) and let \(\{x_n\}\) be the sequence defined by
\[
x_{n+1} = J^{-1}((\alpha_n Jx_n + (1 - \alpha_n)JQ_rx_n)
\]
for \(n = 1, 2, \ldots\). If \(J\) is weakly sequentially continuous, then \(\{x_n\}\) converges weakly to \(u\), where \(u = \lim_{n \to \infty} Q_{A^{-1}0}x_n\) and \(Q_{A^{-1}0}\) is the generalized projection from \(E\) onto \(A^{-1}0\).

Kamimura and Takahashi [18] extended Solodov and Svaiter's result [41] to the following theorem by using Lemma 3.1 and the resolvents defined by (3.2).
Theorem 3.9 ([18]). Let $E$ be a uniformly convex and uniformly smooth Banach space and let $A$ be a maximal monotone operator from $E$ into $E^*$ such that $A^{-1}0 \neq \phi$. Let $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and let $\{x_n\}$ be a sequence generated by
\[
\begin{align*}
x_1 & \in E, \\
y_n &= Q_{r_n}x_n, \\
H_n &= \{z \in E : \langle z - y_n, Jx_n - Jy_n \rangle \leq 0\}, \\
W_n &= \{z \in E : \langle z - x_n, Jx_1 - Jx_n \rangle \leq 0\}, \\
x_{n+1} &= Q_{H_n \cap W_n}x_1, \ n = 1, 2, \ldots,
\end{align*}
\]
where $\{r_n\}$ is a sequence of positive numbers such that $\lim \inf_{n \to \infty} r_n > 0$. Then, $\{x_n\}$ converges strongly to $Q_{A^{-1}0}x_1$, where $Q_{A^{-1}0}$ is the generalized projection of $E$ onto $A^{-1}0$.

Kamimura, Kohsaka and Takahashi [15] also proved a weak convergence theorem of Mann's type for maximal monotone operators in a Banach space. Before stating the theorem, we need the following strong convergence theorem.

Theorem 3.10 ([15]). Let $E$ be a smooth and uniformly convex Banach space. Let $A \subset E \times E^*$ be a maximal monotone operator such that $A^{-1}0$ is nonempty, let $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and let $Q_{A^{-1}0}$ be the generalized projection of $E$ onto $A^{-1}0$. Let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in E$ and
\[
x_{n+1} = J^{-1}(\alpha_n J(x_n) + (1 - \alpha_n)J(Q_{r_n}x_n)), \quad n = 1, 2, \ldots,
\]
where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$. Then, the sequence $\{Q_{A^{-1}0}(x_n)\}$ converges strongly to an element of $A^{-1}0$, which is a unique element $v \in A^{-1}0$ such that
\[
\lim_{n \to \infty} \phi(v, x_n) = \min_{y \in A^{-1}0} \lim_{n \to \infty} \phi(y, x_n).
\]

Using Theorem 3.10, we can prove the following theorem in a Banach space which generalizes the results of Rockafellar [39] and Kamimura and Takahashi [16] in a Hilbert space.

Theorem 3.11 ([15]). Let $E$ be a smooth and uniformly convex Banach space whose duality mapping $J$ is weakly sequentially continuous. Let $A \subset E \times E^*$ be a maximal monotone operator such that $A^{-1}0$ is nonempty, let $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and let $Q_{A^{-1}0}$ be the generalized projection of $E$ onto $A^{-1}0$. Let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in E$ and
\[
x_{n+1} = J^{-1}(\alpha_n J(x_n) + (1 - \alpha_n)J(Q_{r_n}x_n)), \quad n = 1, 2, \ldots,
\]
where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy
\[
\limsup_{n \to \infty} \alpha_n < 1 \quad \text{and} \quad \liminf_{n \to \infty} r_n > 0.
\]
Then, $\{x_n\}$ converges weakly to an element $v$ of $A^{-1}0$, where $v = \lim_{n \to \infty} Q_{A^{-1}0}(x_n)$.

As a direct consequence of Theorem 3.11, we obtain the following:

Theorem 3.12. Let $E$ be a smooth and uniformly convex Banach space whose duality mapping $J$ is weakly sequentially continuous. Let $A \subset E \times E^*$ be a maximal monotone operator such that $A^{-1}0$ is nonempty, let $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and let $Q_{A^{-1}0}$ be the generalized projection of $E$ onto $A^{-1}0$. Let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in E$ and
\[
x_{n+1} = Q_{r_n}x_n, \quad n = 1, 2, \ldots,
\]
where \( \{r_n\} \subset (0, \infty) \) satisfies \( \liminf_{n \to \infty} r_n > 0 \). Then, the sequence \( \{x_n\} \) converges weakly to an element \( v \) of \( A^{-1}0 \), where \( v = \lim_{n \to \infty} Q_{A^{-1}0}(x_n) \).

**Problem.** If \( E \) and \( E^* \) are uniformly convex Banach spaces, does Theorem 3.12 hold without assuming that \( J \) is weakly sequentially continuous?

### 4 Generalized nonexpansive mappings

Let \( E \) be a smooth Banach space and let \( D \) be a nonempty closed convex subset of \( E \). A mapping \( R: D \to D \) is called generalized nonexpansive if \( F(R) \neq \emptyset \) and

\[
\phi(Rx, y) \leq \phi(x, y), \quad \forall x \in D, \forall y \in F(R),
\]

where \( F(R) \) is the set of fixed points of \( R \). A point \( p \) in \( C \) is said to be a generalized asymptotic fixed point of \( T \) [13] if \( C \) contains a sequence \( \{x_n\} \) such that \( Jx_n \rightharpoonup Jp \) and the strong \( \lim_{n \to \infty} (Jx_n - JTx_n) = 0 \). The set of generalized asymptotic fixed points of \( T \) will be denoted by \( F(T) \).

Let \( E \) be a reflexive and smooth Banach space and let \( B \subset E^* \times E \) be a maximal monotone operator. For each \( \lambda > 0 \) and \( x \in E \), consider the set

\[
R_\lambda x := \{ z \in E : x \in z + \lambda BJ(z) \}.
\]

Then \( R_\lambda x \) consists of one point. We also denote the domain and the range of \( R_\lambda \) by \( D(R_\lambda) = R(I + \lambda BJ) \) and \( R(R_\lambda) = D(BJ) \), respectively. Such \( R_\lambda \) is called the generalized resolvent of \( B \) and is denoted by

\[
R_\lambda = (I + \lambda BJ)^{-1}.
\]

We get some properties of \( R_\lambda \) and \( (BJ)^{-1} \).

**Proposition 4.1** ([12]). Let \( E \) be a reflexive and strictly convex Banach space with a Fréchet differentiable norm and let \( B \subset E^* \times E \) be a maximal monotone operator with \( B^{-1}0 \neq \emptyset \). Then the following hold:

1. \( D(R_\lambda) = E \) for each \( \lambda > 0 \);
2. \( (BJ)^{-1}0 = F(R_\lambda) \) for each \( \lambda > 0 \), where \( F(R_\lambda) \) is the set of fixed points of \( R_\lambda \);
3. \( (BJ)^{-1}0 \) is closed;
4. \( R_\lambda \) is generalized nonexpansive for each \( \lambda > 0 \).

**Proposition 4.2** ([13]). Let \( E \) be a smooth and uniformly convex Banach space, let \( B \subset E^* \times E \) be a maximal monotone operator with \( B^{-1}0 \neq \emptyset \), and let \( R_\lambda \) be the generalized resolvent of \( B \) for \( \lambda > 0 \). Then \( F(R_\lambda) = F(R_\lambda) \).

Next, we get the following result for generalized nonexpansive mappings.

**Proposition 4.3.** Let \( C \) be a nonempty closed subset of a smooth and strictly convex Banach space \( E \). Let \( R_C \) be a retraction of \( E \) onto \( C \). Then \( R_C \) is sunny and generalized nonexpansive if and only if

\[
\langle x - R_Cx, J(R_Cx) - J(y) \rangle \geq 0
\]

for each \( x \in E \) and \( y \in C \).
Let $E$ be a smooth and strictly convex Banach space and let $C$ be a nonempty closed subset of $E$. Then, a sunny generalized nonexpansive retraction of $E$ onto $C$ is unique. In fact, let $R, S$ be two sunny generalized nonexpansive retractions of $E$ onto $C$. Then, by Proposition 4.3, for each $x \in E$, we have

$$(x - Rx, J(Rx) - J(y)) \geq 0, \quad (x - Sx, J(Sx) - J(y)) \geq 0, \quad \forall y \in C.$$  

From $Rx, Sx \in C$, we get

$$(x - Rx, J(Rx) - J(Sx)) \geq 0, \quad (x - Sx, J(Sx) - J(Rx)) \geq 0.$$  

From these inequalities, we have

$$\langle Sx - Rx, J(Rx) - J(Sx) \rangle \geq 0.$$  

Since $E$ is strictly convex, we get $Sx = Rx$.

Before showing an example of sunny generalized nonexpansive retractions, we recall the following theorem.

**Theorem 4.4 ([34]).** Let $E$ be a Banach space and let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$. If $E^*$ is strictly convex and has a Fréchet differentiable norm. Then, for each $x \in E$, $\lim_{\lambda \rightarrow \infty} (J + \lambda A)^{-1} J(x)$ exists and belongs to $A^{-1}0$.

Using Theorem 4.4, we get the following result.

**Theorem 4.5 ([12]).** Let $E$ be a uniformly convex Banach space with a Fréchet differentiable norm and let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then the following hold:

1. For each $x \in E$, $\lim_{\lambda \rightarrow \infty} R_{\lambda} x$ exists and belongs to $(BJ)^{-1}0$;
2. If $Rx := \lim_{\lambda \rightarrow \infty} R_{\lambda} x$ for each $x \in E$, then $R$ is a sunny generalized nonexpansive retraction of $E$ onto $(BJ)^{-1}0$.

Next, we discuss proximal point algorithms for generalized resolvents of a maximal monotone operator $B \subset E^* \times E$. We start with the following lemma. Compare this lemma with the results in Kamimura and Takahashi [18], and Kohsaka and Takahashi [20].

**Lemma 4.6.** Let $E$ be a reflexive, strictly convex, and smooth Banach space, let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$, and $R_r = (I + rBJ)^{-1}$ for all $r > 0$. Then

$$\phi(x, R_r x) + \phi(R_r x, u) \leq \phi(x, u)$$  

for all $r > 0$, $u \in (BJ)^{-1}0$, and $x \in E$.

The following is a strong convergence theorem for generalized nonexpansive mappings in a Banach space which is related to Nakajo and Takahashi’s theorem [29] in a Hilbert space.

**Theorem 4.7 (Ibaraki and Takahashi [13]).** Let $E$ be a uniformly convex and uniformly smooth Banach space, let $T$ be a generalized nonexpansive mapping from $E$ into itself with $F(T) \neq \emptyset$ and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n \rightarrow \infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by

$$
\begin{align*}
x_1 &= x \in E, \\
y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\
H_n &= \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
W_n &= \{z \in E : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
x_{n+1} &= R_{H_n \cap W_n} x
\end{align*}
$$
for all \( n = 1, 2, \ldots \), where \( J \) is the duality mapping on \( E \). If \( \bar{F}(T) = F(T) \), then \( \{x_n\} \) converges strongly to \( R_{F(T)}x \), where \( R_{F(T)} \) is the sunny generalized nonexpansive retraction from \( C \) onto \( F(T) \).

We can also prove the following weak convergence theorem, which is a generalization of Kamimura and Takahashi's weak convergence theorem (Theorem 1.2).

**Theorem 4.8.** Let \( E \) be a smooth and uniformly convex Banach space whose duality mapping \( J \) is weakly sequentially continuous. Let \( B \subset E^* \times E \) be a maximal monotone operator, let \( R_r = (I + rBJ)^{-1} \) for all \( r > 0 \) and let \( \{x_n\} \) be a sequence defined as follows: \( x_1 = x \in E \) and

\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)R_{r_n}x_n, \quad n = 1, 2, \ldots,
\]

where \( \{\alpha_n\} \subset [0, 1) \) and \( \{r_n\} \subset (0, \infty) \) satisfy

\[
\limsup_{n \to \infty} \alpha_n < 1 \quad \text{and} \quad \liminf_{n \to \infty} r_n > 0.
\]

If \( B^{-1} \neq \emptyset \), then the sequence \( \{x_n\} \) converges weakly to an element of \( (BJ)^{-1}0 \).

## 5 Concluding remarks

Recently, Matsushita and Takahashi [27] proved the following strong convergence theorem for nonexpansive mappings in a Banach space which is related to Nakajo and Takahashi's theorem [29].

**Theorem 5.1 (Matsushita and Takahashi [27]).** Let \( E \) be a uniformly convex and smooth Banach space, let \( C \) be a nonempty bounded closed convex subset of \( E \) and let \( T \) be a nonexpansive mapping from \( C \) into itself. Let \( \{x_n\} \) be a sequence in \( C \) defined by

\[
\begin{cases}
x_1 = x \in C, \\
C_n = \overline{co}\{z \in C : \|z - y_n\| \leq \|z - x_n\|\}, \\
D_n = \{z \in C : (x_n - z, Jx - Jx_n) \leq 0\}, \\
x_{n+1} = P_{C_n \cap D_n}x
\end{cases}
\]

for all \( n = 1, 2, \ldots \), where \( P_{C_n \cap D_n} \) is the metric projection from \( E \) onto \( C_n \cap D_n \) and \( \{t_n\} \) is a sequence in \((0, 1)\) with \( t_n \to 0 \). Then \( \{x_n\} \) converges strongly to the element \( P_{F(T)}x \), where \( P_{F(T)} \) is the the metric projection from \( E \) onto \( F(T) \).

For the proof of Theorem 5.1, Matsushita and Takahashi [27] used essentially the following Bruck's theorem [7]:

**Theorem 5.2 (Bruck [7]).** Let \( C \) be a closed convex subset of a uniformly convex Banach space \( E \). Then for each \( r > 0 \), there exists a strictly increasing convex continuous function \( \lambda : [0, \infty) \to [0, \infty) \) such that \( \lambda(0) = 0 \) and

\[
\lambda\left(\left\| T\left( \sum_{j=0}^{n} \lambda_j x_j \right) - \sum_{j=0}^{n} \lambda_j Tx_j \right\| \right) \leq \max_{0 \leq j < k \leq n} (\|x_j - x_k\| - \|Tx_j - Tx_k\|)
\]

for all \( n \in \mathbb{N} \), \( \{\lambda_j\} \in \Delta^n \), \( \{x_j\} \subset C \cap B_r \) and \( T \in \text{Lip}(C, 1) \), where \( \Delta^n = \{\{\lambda_0, \lambda_1, \ldots, \lambda_n\} : 0 \leq \lambda_j \quad \text{and} \quad \sum_{j=0}^{n} \lambda_j = 1\}, B_r = \{z \in E : \|z\| \leq r\} \) and \( \text{Lip}(C, 1) \) is the set of all nonexpansive mappings of \( C \) into \( E \).
Problem. Can we prove Theorem 5.1 under assuming that $C$ is a closed and convex subset of $E$ and $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$?

References


[42] A. Tada and W. Takahashi, **Weak and Strong convergence theorems for a nonexpansive


