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Kyoto University
CONSTANTS OF REVERSE HÖLDER’S INEQUALITY
(ヘルダーの不等式に繋わる定数)

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ABSTRACT. Hölder’s inequality is considered as an estimation of the arithmetic mean to the power mean for positive numbers. The generalized Kantorovich constant $K(h, p)$ is used in a reverse Hölder’s inequality where $h$ represents the bound of the ratio for given positive numbers. On the other hand, the Specht ratio $S = S(h)$ was introduced as the ratio of the arithmetic mean to the geometric mean. It is a special case of ratios $S(h, r, s)$ for $r < s$. In this note, we give an interpretation to $S(h, r, s)$ for $r < s$ and investigate several useful properties of them, one of which is the inversion formula $S(h, r, s) = S(h, s, r)^{-1}$. Another is a clear relation: $S(h, r, s) = K(h^{r}, \frac{s}{r})^{\frac{1}{r}}$. By these properties, one can understand the context of a masterly formula $S = e^{K'(1)} = e^{-K'(0)}$ due to Furuta. Moreover we give the reverse inequalities by using the Specht ratio $S(h)$ and the generalized Kantorovich constant $K(h, p)$.

1. INTRODUCTION

This note is a short survey related to estimations represented to a reverse Hölder’s inequality ([3]).

Let $a_1, \ldots, a_n$ be positive real numbers and $(w_1, \ldots, w_n)$ be a weight. Then, Hölder’s inequality is equivalent to

$$\left(\sum_{i=1}^{n}w_ia_i^{p}\right)^{\frac{1}{p}} \leq \sum_{i=1}^{n}w_ia_i \quad (0 \leq p \leq 1).$$

The following Kantorovich inequality is studied as one of reverse Hölder’s inequalities:

$$\sum_{i=1}^{n}w_ia_i \leq \frac{(M + m)^2}{4Mm} (\sum_{i=1}^{n}w_ia_i^{-1})^{-1}$$

where $0 < m \leq a_i \leq M$. The estimation $\frac{(M + m)^2}{4Mm}$ is called the Kantorovich constant. This constant represents an estimation of the arithmetic mean by the harmonic mean. Furuta continuously generalized Ky Fan’s result associated with Hölder-McCarthy and Kantorovich inequalities in [6, Theorem 1.5]: If a positive operator $A$ on a Hilbert space $H$ satisfies $0 < m \leq A \leq M$ for some $m < M$ and $x \in H$ is a unit vector, then

$$\langle Ax, x \rangle^{p} \leq \langle A^p x, x \rangle \leq K_{\pm}(m, M, p)\langle Ax, x \rangle^{p}$$

Key words and phrases. Hölder inequality, Specht ratio, Kantorovich constant and power mean Hölder-McCarthy inequality.
for \( p > 1 \) and \( p < 0 \) respectively, where

\[
K_+(m, M, p) = \frac{(p-1)^{p-1}(M^{p}-m^{p})^{p}}{p^{p}(M-m)(mM^{p}-Mm^{p})^{p-1}} \quad \text{for } p > 1,
\]

and

\[
K_-(m, M, p) = \frac{(mM^{p}-Mm^{p})}{(p-1)(M-m)} \left( \frac{(p-1)(M^{p}-m^{p})}{p(mM^{p}-Mm^{p})} \right)^{p} \quad \text{for } p < 0.
\]

Furuta [7] proposed to reformulate the constants \( K_{\pm}(m, M, p) \) as follows, cf. [6, Corollary 1.2]: For a given \( h > 0 \), put

\[
K(h, p) = \frac{1}{h-1} \frac{h^{p}-h}{p-1} \left( \frac{p-1h^{p}-1}{h^{p}-hp} \right)^{p}
\]

for all \( p \in \mathbb{R} \). Following him, we call it the generalized Kantorovich constant. It is easily checked that if we take \( h = \frac{M}{m} \), then \( K(h, p) = K_+(m, M, p) \) for \( p > 1 \) and \( K(h, p) = K_-(m, M, p) \) for \( p < 0 \). This formula (4) says that it can be defined for all \( p \in \mathbb{R} \), and it has the symmetric property \( K(h, p) = K(h, 1-p) \), that is, \( K(p) = K(h, p) \) is a symmetric function with respect to \( p = \frac{1}{2} \). The inequality (3) implies that

\[
\sum_{i=1}^{n} w_{i}a_{i} \leq K(h, p)^{-\frac{1}{p}} \left( \sum_{i=1}^{n} w_{i}a_{1}^{p} \right)^{\frac{1}{p}} \quad (0 \leq p \leq 1).
\]

as a reverse inequality of (1).

On the other hand, the Specht ratio is introduced in [10] as the ratio of the arithmetic mean to the geometric mean, that is, it is the best constant \( S(h) \) satisfying the reverse inequality

\[
\frac{a_{1} + \cdots + a_{n}}{n} \leq S(h)(a_{1} \cdots a_{n})^{\frac{1}{n}}
\]

for all \( 0 < m \leq a_{1}, \ldots, a_{n} \leq M \), where \( h = \frac{M}{m} \) for some \( m < M \). Following Specht [10], it is exactly given by

\[
S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}},
\]

see also [2]. It is also expressed as a constant enjoying that if \( 0 < m \leq a, b \leq M \), then

\[
(1-t)a + tb \leq S(h)a^{1-t}b^{t}
\]

for all \( t \in [0, 1] \), see also [11].

By the way, we recognize the importance of the family of power means \( M_{r,t}(r \in \mathbb{R}) \). The mean of 1 and \( x \geq 0 \) by \( M_{r,t} \) with weight \( \{1 - t, t\} \) \((t \in [0, 1])\) is defined by

\[
M_{r,t}(x) = (1 - t + tx^{r})^{\frac{1}{r}}.
\]

From this point of view, one could understand that Specht discussed the ratio among power means in the following general setting: If \( -1 \leq r < s \leq 1 \), then \( M_{r,t}(x) \leq M_{s,t}(x) \) and

\[
\frac{M_{s,t}(x)}{M_{r,t}(x)} \leq \left( \frac{s-r}{r} \frac{h^{s}-1}{h^{s-r}} \right)^{\frac{1}{2}} \left( \frac{r}{s-r} \frac{h^{r}-h^{s}}{h^{r}-1} \right)^{\frac{1}{2}} = S(h, r, s)
\]
for $\frac{1}{h} \leq x \leq h$. We note that $S(h, r, s)$ is the best constant for upper bounds of $\frac{M_{l}}{M_{r}}$. Since $M_{0, s}(x) = x^{t}$, (8) is the special case $r = 0$ and $s = 1$ in (9). In other words, $S(h, 0, 1)$ is the Specht ratio $S(h)$, i.e., $\lim_{r \rightarrow +0} S(h, r, 1) = S(h)$.

The most crucial result on the the generalized Kantorovich constant and Specht ratio is the following formula due to Furuta:

(10) \[ S = e^{K'(1)} = e^{-K'(0)}, \]

where $S = S(h)$ and $K(p) = K(h, p)$ for a fixed $h > 1$. In the below, this formula (10) is called the Furuta formula (on the generalized Kantorovich constant).

Motivated by the Furuta formula, we investigate several useful properties of $S(h, r, s)$ and $K(h, p)$ in this note. For this, we give an interpretation to $S(h, s, r)$ for $r < s$. Consequently we have the inversion formula $S(h, r, s) = S(h, s, r)^{-1}$. On a relationship of $S(h, s, r)$ to the generalized Kantorovich constant $K(h, p)$, we get

\[ S(h, r, s) = K(h^{r}, \frac{s}{r})^{1} \]

for all $r, s \in \mathbb{R}$ with $rs \neq 0$. By these properties, one can understand the context of the Furuta formula (10). As a consequence, we have the following result:

The Furuta formulas

(F0) : $S = e^{-K'(0)}$ and (F1) : $S = e^{K'(1)}$

are equivalent to the Yamazaki-Yanagida formulas [13]

(K0) : $\lim_{p \rightarrow +0} K(h^{p}, \frac{1}{p}) = S$ and (K1) : $\lim_{p \rightarrow +0} K(h^{p}, \frac{p + 1}{p}) = S$,

respectively. From this result we see that (5) implies (6) by $p \rightarrow 0$.

Moreover we give the some reverse inequalities by using $t S(h)$ and $K(h, p)$.

2. Fundamental Properties of $S(h)$, $S(h, r, s)$ and $K(h, p)$

Firstly, we mention some properties of this Specht ratio $S(h)$:

Lemma 1. Let $h > 0$ be given. Then

(1) $S(h) = S(\frac{1}{h})$.
(2) $L(1, \frac{1}{h}) \leq S(h) \leq L(1, h)$ for $h \geq 1$ where the logarithmic mean $L(s, t)$ is defined by

$L(s, t) := \frac{e^{s} - e^{t}}{e^{t} - e^{s}}$ for $0 < s, t, s \neq t$.

(3) $\lim_{h \rightarrow 1} S(h) = 1$.

Secondary, we state some important properties of $K(h, p)$ and $S(h, r, s)$ which will be needed in the below.

Lemma 2. Let $h > 0$ be given. Then

(0) $K(h, p)$ is defined for all $p \in \mathbb{R}$.
(1) $K(h, p) = K(\frac{1}{h}, p)$ for all $p \in \mathbb{R}$.
(2) $K(h, p) = K(h, 1 - p)$ for all $p \in \mathbb{R}$.
(3) $K(h, 0) = K(h, 1) = 1$ and $K(1, p) = 1$ for all $p \in \mathbb{R}$,

where $K(h, 0) = \lim_{p \rightarrow 0} K(h, p)$, $K(h, 1) = \lim_{p \rightarrow 0} K(h, 1 + p)$ and $K(1, p) = \lim_{h \rightarrow 1} K(h, p)$.
The property (1) in Lemma 2 is imagined by that in Lemma 1. Related to a result of Mond and Pečarić [9], the following relationship was presented in our seminar talk about five years ago, which is implicitly appeared in [12, Remark 2].

**Lemma 3.** Let $h > 0$ and $r, s \in \mathbb{R}$. Then

\[
S(h, r, s) = K(h^r, \frac{s}{r})^{\frac{1}{s}} \text{ if } rs \neq 0,
\]

\[
S(h, 0, s) = S(h^s) \text{ and } S(h, r, 0) = S(h^r)^{-1}.
\]

By the above lemma, one could recognize that Lemma 2 (0) is quite meaningful. As a corollary, we have the following variant of the Yamazaki-Yanagida formula [13]:

**Corollary 4.** For $h > 0$,

(K0) \[ \lim_{r \to 0} K(h^r, \frac{1}{r}) = S(h). \]

**Proof.** The continuity of $S(h, r, s)$ and Lemma 3 imply that

\[
S(h) = \lim_{r \to 0} S(h, r, 1) = \lim_{r \to 0} K(h^r, \frac{1}{r}).
\]

\[ \square \]

**Lemma 5.** (Inversion formula) Let $h > 0$ and $r, s \in \mathbb{R}$. Then

\[
S(h, r, s) = S(h, s, r)^{-1}.
\]

Consequently, if $rs \neq 0$, then

\[
K(h^r, \frac{s}{r})^{\frac{1}{s}} = K(h^s, \frac{r}{s})^{-\frac{1}{r}}.
\]

In particular, if $r \neq 0$, then

\[
K(h^r, \frac{1}{r}) = K(h, r)^{-\frac{1}{r}}.
\]

Incidentally, since $M_{r,t}(x) \leq M_{s,t}(x)$ for $r < s$, $S(h, s, r)$ for $r < s$ might be defined by the lower bound of

\[
S(h, s, r)M_{s,t}(x) \leq M_{r,t}(x).
\]

It is rephrased by

\[
\frac{M_{s,t}(x)}{M_{r,t}(x)} \leq S(h, s, r)^{-1}.
\]

Hence the inversion formula could be expected.
3. Equivalent relation between Furuta and Yamazaki-Yanagida formulas

First of all, we cite the representation of the Specht ratio by the limit of the generalized Kantorovich constant due to Yamazaki and Yanagida [13].

**Theorem A.** The Specht ratio \( S = S(h) \) and the generalized Kantorovich constant \( K(h, p) \) are defined in (7) and (4), respectively, and take \( h > 0 \). Then

\[
\text{(K0)}: \quad \lim_{p \to +0} K(h^p, \frac{1}{p}) = S \quad \text{and} \quad \text{(K1)}: \quad \lim_{p \to +0} K(h^p, \frac{p+1}{p}) = S.
\]

Now we consider the Furuta formulas

\[
\text{(FO)}: \quad S = e^{-K'(0)} \quad \text{and} \quad \text{(F1)}: \quad S = e^{K'(1)}.
\]

Since \( K(0) = K(h, 0) = 1 \) and \( K(1) = K(h, 1) = 1 \) by Lemma 2 (3), they should be understood as

\[
\log S = -\frac{K'(0)}{K(0)} \quad \text{and} \quad \log S = \frac{K'(1)}{K(1)},
\]

respectively, where \( K(p) = K(h, p) \) for a fixed \( h > 0 \). Therefore, if we put \( f(p) = \log K(p) \), then

\[
\text{(FO)}: \quad \log S = -f'(0) \quad \text{and} \quad \text{(F1)}: \quad \log S = f'(1).
\]

By the way, since \( f(0) = 0 \), we have

\[
-f'(0) = -\lim_{p \to 0} \frac{f(p) - f(0)}{p} = -\lim_{p \to 0} \frac{f(p)}{p} = \lim_{p \to 0} \log K(p) = \lim_{p \to 0} \log K(p)^\frac{1}{p}.
\]

Moreover the inversion formula \( K(h^p, \frac{1}{p}) = K(h, p)^{-\frac{1}{p}} \) implies that

\[
-f'(0) = \log \lim_{p \to 0} K(h^p, \frac{1}{p}).
\]

It says that (F0) is equivalent to (K0) in Theorem A.

Next we discuss the equivalence between (F1) and (K1) in Theorem A. Since \( f(1) = 0 \), we have

\[
f'(1) = \lim_{p \to 0} \frac{f(p+1) - f(1)}{p} = \lim_{p \to 0} \frac{f(p+1)}{p} = \lim_{p \to 0} \frac{\log K(p+1)}{p} = \lim_{p \to 0} \log K(p+1)^\frac{1}{p}.
\]

Using the symmetric property \( K(h, p) = K(h, q) \) for \( p + q = 1 \) by Lemma 2 (2) and the inversion formula \( K(h^r, \frac{1}{r}) = K(h, r)^{-\frac{1}{r}} \), we have

\[
K(h^p, \frac{p+1}{p})^p = K(h^{p+1}, \frac{p}{p+1})^{-p+1} = K(h^{p+1}, \frac{1}{p+1}) = K(h, p+1).
\]

Taking the power \( \frac{1}{p} \) on both sides,

\[
K(p+1)^\frac{1}{p} = K(h, p+1)^\frac{1}{p} = K(h^p, \frac{p+1}{p}).
\]

Therefore it follows that

\[
f'(1) = \log \lim_{p \to 0} K(h^p, \frac{p+1}{p}),
\]

which means that (F1) is equivalent to (K1) in Theorem A.

Summing up the above argument, we have the following conclusion:
Theorem 6. The Furuta formulas

(F0) : $S = e^{-K'(0)}$ and (F1) : $S = e^{K'(1)}$

are equivalent to the Yamazaki-Yanagida formulas

(K0) : $\lim_{p \to +0} K(h^p, \frac{1}{p}) = S$ and (K1) : $\lim_{p \to +0} K(h^p, \frac{p + 1}{p}) = S$,

respectively.

4. SOME REVERSE INEQUALITIES BY $S(h)$ AND $K(h, p)$

The generalized Kantorovich constant $K(h, p)$ and the Specht ratio $S(h)$ appear in some reverse inequalities. In this section we note some examples.

The reverse Hölder-McCarthy inequality (3) leads for $0 \leq p \leq 1$

\[(11) \quad \langle Ax, x \rangle \leq K(h, p)^{-\frac{1}{p}} \langle A^p x, x \rangle^{\frac{1}{p}}.\]

Moreover since

\[
\lim_{p \downarrow 0} \log \langle A^p x, x \rangle^{\frac{1}{p}} = \lim_{p \downarrow 0} \frac{d \langle A^p x, x \rangle}{dp} \langle A^p x, x \rangle = \lim_{p \downarrow 0} \frac{(\log A)x, x)}{\langle A^p x, x \rangle} = \langle (\log A)x, x \rangle
\]

and

\[
\lim_{p \downarrow 0} K(h, p)^{-\frac{1}{p}} = \lim_{p \downarrow 0} K(h^p, \frac{1}{p}) = S(h)
\]

by Lemma 5 (Inversion formula) and Yamazaki and Yanagida (K0), we have

\[(12) \quad \langle Ax, x \rangle \leq S(h) \exp(\langle (\log A)x, x \rangle).\]

In 2005, Bebiano, Lemos and Providência [1] showed the following norm inequality: For $A, B \geq 0$

\[(13) \quad \| A^\frac{s}{2} B^\frac{s}{2} A^\frac{t}{2} \| \leq \| A^\frac{s}{2} (A^\frac{t}{2} B^\frac{t}{2} A^\frac{t}{2})^\frac{s}{t} A^\frac{t}{2} \|
\]

for all $s \geq t \geq 0$. In [4], we gave a reverse inequality of (13) by using the generalized Kantorovich constant $K(h, p)$ as follows:

Corollary 7. Let $A$ and $B$ be positive operators such that $0 < m \leq B \leq M$ for some scalars $0 < m < M$ and $h := \frac{M}{m} > 1$. Then

\[(14) \quad \| A^\frac{t}{2} (A^\frac{s}{2} B^\frac{s}{2} A^\frac{t}{2})^\frac{s}{t} A^\frac{t}{2} \| \leq K \left( h^\frac{s}{t}, \frac{s}{t} \right)^{\frac{t}{2}} \| A^\frac{s}{2} B^t A^\frac{t}{2} \|
\]

for $s \geq t \geq 0$. 

5. CONCLUDING REMARKS

Concluding this note, we add to two remarks on the Yamazaki-Yanagida formulas (K0), (K1) and a comment on references of Kantorovich type inequalities for readers’ convenience.

(i) Though a short proof of (K0) is given as Corollary 4, we cite a direct proof of it.

\[
K(h^p, \frac{1}{p}) = S(h, p, 1) = \frac{p}{h^p - 1} \left( h - \frac{1}{p} \right) \frac{h - 1}{h^p - 1} \to \frac{1}{\log h} \frac{1}{e} (h - 1) h^\frac{1}{h-1} = S(h) \text{ as } p \to +0,
\]

where the convergence of the final term is assured by l’Hospital theorem as follows:

\[
\lim_{p \to +0} \frac{\log(h - 1) - \log(h - h^p)}{p} = \lim_{p \to +0} \frac{h^p \log h}{h - h^p} = \frac{\log h}{h - 1} = \log h^{\frac{1}{h-1}}.
\]

(ii) The equivalence between (K0) and (K1) is ensured by Theorem 6 because of the symmetric property $K(p) = K(1 - p)$. However, we can show it by a direct computation, in which the symmetric property is used, of course. As a matter of fact, it follows from Lemma 2 (2) that

\[
K(h^p, \frac{p+1}{p}) = K(h^{1 - p}, \frac{1}{p}) = S(h),
\]

Thus we have the equivalence between (K0) and (K1). We here want to remark that Lemma 2 (1) played an important role in the above discussion, and that we identified (K0) with

\[
\lim_{p \to 0} K(h^p, \frac{1}{p}) = S
\]

by virtue of Corollary 3.

(iii) Finally we mention that the paper [6] by Furuta is quite valuable in this field and that [5] and [8] are a suitable textbook for Kantorovich type inequalities.

REFERENCES


