ABSTRACT. Hölder's inequality is considered as an estimation of the arithmetic mean to the power mean for positive numbers. The generalized Kantorovich constant $K(h,p)$ is used in a reverse Hölder's inequality where $h$ represents the bound of the ratio for given positive numbers. On the other hand, the Specht ratio $S = S(h)$ was introduced as the ratio of the arithmetic mean to the geometric mean. It is a special case of ratios \( \{S(h,r,s); -1 \leq r < s \leq 1 \} \) among power means. In this note, we give an interpretation to $S(h,r,s)$ for $r < s$ and investigate several useful properties of them, one of which is the inversion formula \( S(h,r,s) = S(h,s,r)^{-1} \). Another is a clear relation: \( S(h,r,s) = K(h^{r}, \frac{\epsilon}{f})^{A} \). By these properties, one can understand the context of a masterly formula \( S = e^{K'(1)} = e^{-K'(0)} \) due to Furuta. Moreover we give the some reverse inequalities by using the Specht ratio $S(h)$ and the generalized Kantorovich constant $K(h,p)$.

1. INTRODUCTION

This note is a short survey related to estimations represented to a reverse Hölder's inequality ([3]).

Let $a_{1}, \ldots, a_{n}$ be positive real numbers and \((w_{1}, \ldots, w_{n})\) be a weight. Then, Hölder's inequality is equivalent to

\[
\left( \sum_{i=1}^{n} w_{i}a_{i}^{p} \right)^{\frac{1}{p}} \leq \sum_{i=1}^{n} w_{i}a_{i} \quad (0 \leq p \leq 1).
\]

The following Kantorovich inequality is studied as one of reverse Hölder's inequalities:

\[
\sum_{i=1}^{n} w_{i}a_{i} \leq \frac{(M+m)^{2}}{4Mm} \left( \sum_{i=1}^{n} w_{i}a_{i}^{-1} \right)^{-1}
\]

where $0 < m \leq a_{i} \leq M$. The estimation $\frac{(M+m)^{2}}{4Mm}$ is called the Kantorovich constant. This constant represents an estimation of the arithmetic mean by the harmonic mean. Furuta continuously generalized Ky Fan's result associated with Hölder-McCarthy and Kantorovich inequalities in [6, Theorem 1.5]: If a positive operator $A$ on a Hilbert space $H$ satisfies $0 < m \leq A \leq M$ for some $m < M$ and $x \in H$ is a unit vector, then

\[
\langle Ax, x \rangle^{p} \leq \langle A^{p}x, x \rangle \leq K_{\pm}(m, M,p)\langle Ax, x \rangle^{p}
\]

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for $p > 1$ and $p < 0$ respectively, where

$$K_{+}(m, M, p) = \frac{(p-1)^{p-1}(M^{p}-m^{p})^{p}}{p^{p}(M-m)(mM^{p}-Mm^{p})^{p-1}} \quad \text{for} \quad p > 1,$$

and

$$K_{-}(m, M, p) = \frac{(mM^{p}-Mm^{p})}{(p-1)(M-m)}\left(\frac{(p-1)(M^{p}-m^{p})}{p(mM^{p}-Mm^{p})}\right)^{p} \quad \text{for} \quad p < 0.$$

Furuta [7] proposed to reformulate the constants $K_{\pm}(m, M, p)$ as follows, cf. [6, Corollary 1.2]: For a given $h > 0$, put

$$K(h, p) = \frac{1}{h-1}\frac{h^{p}-h}{p-1}\left(\frac{p-1h^{p}-1}{h^{p}-hp}\right)^{p} \quad \text{for all} \quad p \in \mathbb{R}. $$

Following him, we call it the generalized Kantorovich constant. It is easily checked that if we take $h = \frac{M}{m}$, then $K(h, p) = K_{+}(m, M, p)$ for $p > 1$ and $K(h, p) = K_{-}(m, M, p)$ for $p < 0$. This formula (4) says that it can be defined for all $p \in \mathbb{R}$, and it has the symmetric property $K(h, p) = K(h, 1-p)$, that is, $K(p) = K(h, p)$ is a symmetric function with respect to $p = \frac{1}{2}$. The inequality (3) implies that

$$\sum_{i=1}^{n}w_{i}a_{i} \leq K(h, p)^{-\frac{1}{p}}\left(\sum_{i=1}^{n}w_{i}a_{1}^{p}\right)^{\frac{1}{p}} \quad (0 \leq p \leq 1).$$

as a reverse inequality of (1).

On the other hand, the Specht ratio is introduced in [10] as the ratio of the arithmetic mean to the geometric mean, that is, it is the best constant $S(h)$ satisfying the reverse inequality

$$\frac{a_{1} + \cdots + a_{n}}{n} \leq S(h)(a_{1}\cdots a_{n})^{\frac{1}{n}}$$

for all $0 < a_{1}, \ldots, a_{n} \leq M$, where $h = \frac{M}{m}$ for some $m < M$. Following Specht [10], it is exactly given by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}},$$

see also [2]. It is also expressed as a constant enjoying that if $0 < m \leq a, b \leq M$, then

$$(1 - t)a + tb \leq S(h)a^{1-t}b^{t}$$

for all $t \in [0, 1]$, see also [11].

By the way, we recognize the importance of the family of power means $M_{r,t}$ ($r \in \mathbb{R}$). The mean of 1 and $x \geq 0$ by $M_{r,t}$ with weight \{1 - t, t\} ($t \in [0, 1]$) is defined by

$$M_{r,t}(x) = (1 - t + tx^{r})^{\frac{1}{r}}.$$

From this point of view, one could understand that Specht discussed the ratio among power means in the following general setting: If $-1 \leq r < s \leq 1$, then $M_{r,t}(x) \leq M_{s,t}(x)$ and

$$\frac{M_{s,t}(x)}{M_{r,t}(x)} \leq \left(\frac{s - r}{r - s}\right)^{\frac{1}{2}}\left(\frac{r - h^{s} - h^{r}}{s - r - h^{s} - h^{r}}\right)^{\frac{1}{2}} = S(h, r, s)$$

for $s - r > 0$. For $s - r < 0$, we can derive

$$\frac{M_{s,t}(x)}{M_{r,t}(x)} \geq \left(\frac{s - r}{r - s}\right)^{\frac{1}{2}}\left(\frac{r - h^{s} - h^{r}}{s - r - h^{s} - h^{r}}\right)^{\frac{1}{2}} = S(h, r, s).$$

For $s - r = 0$, we have

$$\frac{M_{s,t}(x)}{M_{r,t}(x)} = 1.$$

Further, it is obvious that

$$\frac{M_{s,t}(x)}{M_{r,t}(x)} \leq \left(\frac{s - r}{r - s}\right)^{\frac{1}{2}}\left(\frac{r - h^{s} - h^{r}}{s - r - h^{s} - h^{r}}\right)^{\frac{1}{2}} = S(h, r, s)$$

for all $r, s \in \mathbb{R}$. Therefore, the equality condition of the above inequality is given by

$$M_{s,t}(x) = M_{r,t}(x) \quad \Leftrightarrow \quad h^{s} = h^{r} \quad \Leftrightarrow \quad r = s = 1.$$
for $\frac{1}{h} \leq x \leq h$. We note that $S(h, r, s)$ is the best constant for upper bounds of $\frac{M_{r,s}}{M_{s,r}}$. Since $M_{0,s}(x) = x^s$, (8) is the special case $r = 0$ and $s = 1$ in (9). In other words, $S(h, 0, 1)$ is the Specht ratio $S(h)$, i.e., $\lim_{r \rightarrow +0} S(h, r, 1) = S(h)$.

The most crucial result on the the generalized Kantorovich constant and Specht ratio is the following formula due to Furuta:

\[(10) \quad S = e^{K'(1)} = e^{-K'(0)},\]

where $S = S(h)$ and $K(p) = K(h, p)$ for a fixed $h > 1$. In the below, this formula (10) is called the Furuta formula (on the generalized Kantorovich constant).

Motivated by the Furuta formula, we investigate several useful properties of $S(h, r, s)$ and $K(h, p)$ in this note. For this, we give an interpretation to $S(h, r, s)$ for $r < s$. Consequently we have the inversion formula $S(h, r, s) = S(h, s, r)^{-1}$. On a relationship of $S(h, s, r)$ to the generalized Kantorovich constant $K(h, p)$, we get

\[\frac{S(h, r, s)}{S(h, s, r)} = K(h^r, \frac{s}{r})^{1}\]

for all $r, s \in \mathbb{R}$ with $rs \neq 0$. By these properties, one can understand the context of the Furuta formula (10). As a consequence, we have the following result:

The Furuta formulas

\[(F0) : S = e^{-K'(0)} \quad \text{and} \quad (F1) : S = e^{K'(1)}\]

are equivalent to the Yamazaki-Yanagida formulas [13]

\[(K0) : \lim_{p \rightarrow +0} K(h^p, \frac{1}{p}) = S \quad \text{and} \quad (K1) : \lim_{p \rightarrow +0} K(h^p, \frac{p + 1}{p}) = S,\]

respectively. From this result we see that (5) implies (6) by $p \rightarrow 0$.

Moreover we give the some reverse inequalities by using $t S(h)$ and $K(h, p)$.

2. **Fundamental Properties of $S(h)$, $S(h, r, s)$ and $K(h, p)$**

Firstly, we mention some properties of this Specht ratio $S(h)$:

**Lemma 1.** Let $h > 0$ be given. Then

1. $S(h) = S(\frac{1}{h})$.
2. $L(1, \frac{1}{h}) \leq S(h) \leq L(1, h)$ for $h \geq 1$ where the logarithmic mean $L(s, t)$ is defined by $L(s, t) := \frac{t^s - s^t}{\log t - \log s}$ for $0 < s, t, s \neq t$.
3. $\lim_{h \rightarrow 1} S(h) = 1$.

Secondary, we state some important properties of $K(h, p)$ and $S(h, r, s)$ which will be needed in the below.

**Lemma 2.** Let $h > 0$ be given. Then

0. $K(h, p)$ is defined for all $p \in \mathbb{R}$.
1. $K(h, p) = K(\frac{1}{h}, p)$ for all $p \in \mathbb{R}$.
2. $K(h, p) = K(h, 1 - p)$ for all $p \in \mathbb{R}$.
3. $K(h, 0) = K(h, 1) = 1$ and $K(1, p) = 1$ for all $p \in \mathbb{R}$, where $K(h, 0) = \lim_{p \rightarrow 0} K(h, p)$, $K(h, 1) = \lim_{p \rightarrow 0} K(h, 1 + p)$ and $K(1, p) = \lim_{h \rightarrow 1} K(h, p)$. 


The property (1) in Lemma 2 is imagined by that in Lemma 1. Related to a result of Mond and Pečarić [9], the following relationship was presented in our seminar talk about five years ago, which is implicitly appeared in [12, Remark 2].

**Lemma 3.** Let $h > 0$ and $r, s \in \mathbb{R}$. Then

$$S(h, r, s) = K(h^r, \frac{s}{r})^\frac{1}{s} \text{ if } rs \neq 0,$$

$$S(h, 0, s) = S(h^s) \text{ and } S(h, r, 0) = S(h^r)^{-1}.$$

By the above lemma, one could recognize that Lemma 2 (0) is quite meaningful. As a corollary, we have the following variant of the Yamazaki-Yanagida formula [13]:

**Corollary 4.** For $h > 0$,

(K0) \[ \lim_{r \to 0} K(h^r, \frac{1}{r}) = S(h). \]

**Proof.** The continuity of $S(h, r, s)$ and Lemma 3 imply that

$$S(h) = \lim_{r \to 0} S(h, r, 1) = \lim_{r \to 0} K(h^r, \frac{1}{r}).$$

\[ \square \]

**Lemma 5.** (Inversion formula) Let $h > 0$ and $r, s \in \mathbb{R}$. Then

$$S(h, r, s) = S(h, s, r)^{-1}.$$

Consequently, if $rs \neq 0$, then

$$K(h^r, \frac{s}{r})^\frac{1}{s} = K(h^s, \frac{r}{s})^{-\frac{1}{s}}.$$

In particular, if $r \neq 0$, then

$$K(h^r, \frac{1}{r}) = K(h, r)^{-\frac{1}{r}}.$$

Incidentally, since $M_{r,t}(x) \leq M_{s,t}(x)$ for $r < s$, $S(h, s, r)$ for $r < s$ might be defined by the lower bound of

$$S(h, s, r)M_{s,t}(x) \leq M_{r,t}(x).$$

It is rephrased by

$$\frac{M_{s,t}(x)}{M_{r,t}(x)} \leq S(h, s, r)^{-1}.$$

Hence the inversion formula could be expected.
3. EQUIVALENT RELATION BETWEEN FURUTA AND YAMAZAKI-YANAGIDA FORMULAS

First of all, we cite the representation of the Specht ratio by the limit of the generalized Kantorovich constant due to Yamazaki and Yanagida [13].

Theorem A. The Specht ratio $S = S(h)$ and the generalized Kantorovich constant $K(h,p)$ are defined in (7) and (4), respectively, and take $h > 0$. Then

\[ (K0) : \lim_{p \to +0} K(h^p, \frac{1}{p}) = S \quad \text{and} \quad (K1) : \lim_{p \to +0} K(h^p, \frac{p+1}{p}) = S. \]

Now we consider the Furuta formulas

\[ (F0) : S = e^{-K'(0)} \quad \text{and} \quad (F1) : S = e^{K'(1)}. \]

Since $K(0) = K(h, 0) = 1$ and $K(1) = K(h, 1) = 1$ by Lemma 2 (3), they should be understood as

\[ \log S = -\frac{K'(0)}{K(0)} \quad \text{and} \quad \log S = \frac{K'(1)}{K(1)}, \]

respectively, where $K(p) = K(h, p)$ for a fixed $h > 0$. Therefore, if we put $f(p) = \log K(p)$, then

\[ (F0) : \log S = -f'(0) \quad \text{and} \quad (F1) : \log S = f'(1). \]

By the way, since $f(0) = 0$, we have

\[ -f'(0) = -\lim_{p \to 0} \frac{f(p) - f(0)}{p} = -\lim_{p \to 0} \frac{f(p)}{p} = \lim_{p \to 0} \frac{\log K(p)}{-p} = \lim_{p \to 0} \log K(p)^{-\frac{1}{p}}. \]

Moreover the inversion formula $K(h^p, \frac{1}{p}) = K(h, p)^{-\frac{1}{p}} = K(p)^{-\frac{1}{p}}$ implies that

\[ -f'(0) = \log \lim_{p \to 0} K(h^p, \frac{1}{p}). \]

It says that (F0) is equivalent to (K0) in Theorem A.

Next we discuss the equivalence between (F1) and (K1) in Theorem A. Since $f(1) = 0$, we have

\[ f'(1) = \lim_{p \to 0} \frac{f(p + 1) - f(1)}{p} = \lim_{p \to 0} \frac{f(p + 1)}{p} = \lim_{p \to 0} \frac{\log K(p + 1)}{p} = \lim_{p \to 0} \log K(p + 1)^{\frac{1}{p}}. \]

Using the symmetric property $K(h, p) = K(h, q)$ for $p + q = 1$ by Lemma 2 (2) and the inversion formula $K(h^r, \frac{1}{r}) = K(h, r)^{-\frac{1}{r}}$, we have

\[ K(\frac{p+1}{p}) = K(h^p, \frac{p}{p+1})^{-(p+1)} = K(h^{p+1}, \frac{1}{p+1})^{-(p+1)} = K(h, p + 1). \]

Taking the power $\frac{1}{p}$ on both sides,

\[ (p+1)^\frac{1}{p} = K(h, p + 1)^{\frac{1}{p}} = K(h^p, \frac{p+1}{p}). \]

Therefore it follows that

\[ f'(1) = \log \lim_{p \to 0} K(h^p, \frac{p+1}{p}), \]

which means that (F1) is equivalent to (K1) in Theorem A.

Summing up the above argument, we have the following conclusion:
Theorem 6. The Furuta formulas

(F0) : \( S = e^{-K'(0)} \) and (F1) : \( S = e^{K'(1)} \)

are equivalent to the Yamazaki-Yanagida formulas

(K0) : \( \lim_{p \to +0} J(h^p, \frac{1}{p}) = S \) and (K1) : \( \lim_{p \to +0} J(h^p, \frac{p+1}{p}) = S \), respectively.

4. SOME REVERSE INEQUALITIES BY \( S(h) \) AND \( K(h,p) \)

The generalized Kantorovich constant \( K(h,p) \) and the Specht ratio \( S(h) \) appear in some reverse inequalities. In this section we note some examples.

The reverse Hölder-McCarthy inequality (3) leads for \( 0 \leq p \leq 1 \)

\[
\langle Ax, x \rangle \leq K(h,p)^{-\frac{1}{p}} \langle A^p x, x \rangle^{\frac{1}{p}}.
\]

Moreover since

\[
\lim_{p \downarrow 0} \log \langle A^p x, x \rangle^\frac{1}{p} = \lim_{p \downarrow 0} \frac{d \langle A^p x, x \rangle}{dp} \ln \langle A^p x, x \rangle = \lim_{p \downarrow 0} \frac{d \langle A^p x, x \rangle}{dp} \ln \langle A^p x, x \rangle = \lim_{p \downarrow 0} K(h,p)^{-\frac{1}{p}} \lim_{p \downarrow 0} K(h^{p}, \frac{1}{p}) = S(h)
\]

by Lemma 5 (Inversion formula) and Yamazaki and Yanagida (K0), we have

\[
\langle Ax, x \rangle \leq S(h) \exp((\log A)x, x).
\]

In 2005, Bebiano, Lemos and Providência [1] showed the following norm inequality: For \( A, B \geq 0 \)

\[
\| A^{t} (A^{s} B A^{t}) A^{t} \| \leq \| A^{t} (A^{s} B A^{t}) A^{t} \|
\]

for all \( s \geq t \geq 0 \). In [4], we gave a reverse inequality of (13) by using the generalized Kantorovich constant \( K(h,p) \) as follows:

Corollary 7. Let \( A \) and \( B \) be positive operators such that \( 0 < m \leq B \leq M \) for some scalars \( 0 < m < M \) and \( h := \frac{M}{m} > 1 \). Then

\[
\| A^{\frac{1}{2}} (A^{\frac{1}{2}} B A^{\frac{1}{2}}) A^{\frac{1}{2}} \| \leq K \left( h^{t}, \frac{s}{t} \right)^{\frac{1}{t}} \| A^{\frac{1}{2}} B A^{\frac{1}{2}} \|
\]

for \( s \geq t \geq 0 \).
5. CONCLUDING REMARKS

Concluding this note, we add to two remarks on the Yamazaki-Yanagida formulas (K0), (K1) and a comment on references of Kantorovich type inequalities for readers' convenience.

(i) Though a short proof of (K0) is given as Corollary 4, we cite a direct proof of it. 

\[ K(h^p, \frac{1}{p}) = S(h, p, 1) \]

\[ = \frac{p}{h^p - 1} \left( \frac{1}{(1-p)^{1-p}} \right) \left( \frac{h-1}{h-h^p} \right)^{\frac{1}{p}} \]

\[ \rightarrow \frac{1}{\log h} \frac{1}{e} (h-1) h^{\frac{1}{h-1}} = S(h) \quad \text{as } p \to +0, \]

where the convergence of the final term is assured by l'Hospital theorem as follows:

\[ \lim_{p \to +0} \frac{\log(h-1) - \log(h-h^p)}{p} = \lim_{p \to +0} \frac{h^p \log h}{h-h^p} = \frac{\log h}{h-1} = \log h^{\frac{1}{h-1}}. \]

(ii) The equivalence between (K0) and (K1) is ensured by Theorem 6 because of the symmetric property \( K(p) = K(1-p) \). However, we can show it by a direct computation, in which the symmetric property is used, of course. As a matter of fact, it follows from Lemma 2 (2) that

\[ K(h^p, \frac{p+1}{p}) = K(h^p, 1 - \frac{p+1}{p}) = K(\frac{1}{h}, \frac{1}{-p}). \]

Therefore (K1) holds for \( h \) if and only if so does (K0) for \( \frac{1}{h} \) by noting that \( S(h) = S(\frac{1}{h}) \); thus we have the equivalence between (K0) and (K1). We here want to remark that Lemma 2 (2) played an important role in the above discussion, and that we identified (K0) with

\[ \lim_{p \to 0} K(h^p, \frac{1}{p}) = S \]

by virtue of Corollary 3.

(iii) Finally we mention that the paper [6] by Furuta is quite valuable in this field and that [5] and [8] are a suitable textbook for Kantorovich type inequalities.

REFERENCES


