数理解析研究所講究録
1570 巻 2007 年 95-103

SOME REPRESENTATION RESULTS FOR FUNCTION SPACES

FERNANDO COBOS*

ABSTRACT. We recall the representation theorem of Zygmund spaces in term of Lebesgue spaces and we take this as model for introducing logarithmic interpolation spaces between quasi-Banach spaces. Application are given not only to function spaces but also to operator spaces. Part of the results are taken from the joint paper with Fernández-Cabrera, Manzano and Martínez (Z. Anal. Anwendungen **26** (2007), 65-86).

0. INTRODUCTION

Let Ω be a domain in $\mathbb{R}^n$ with finite Lebesgue measure $|\Omega|$. For any p with $0 < p \leq \infty$ we let $L_p(\Omega)$ be the usual quasi-Banach space formed by all Lebesgue-measurable functions $f : \Omega \longrightarrow \mathbb{C}$ which have a finite quasi-norm

$$\|f\|_{L_p(\Omega)} = (\int_\Omega |f(x)|^p \, dx)^{1/p}$$

(with the obvious modification if $p = \infty$). As usual, we identify functions equal almost everywhere. These spaces are Banach spaces for $1 \leq p \leq \infty$ and quasi-Banach if $0 < p < 1$.

Working with the non-increasing rearrangement f^* of the function f defined by

$$f^*(t) = \inf\{s > 0 : \ |\{x \in \Omega : \ |f(x)| > s\}| \leq t\},$$

the quasi-norm of $L_p(\Omega)$ can be written as

$$\|f\|_{L_p(\Omega)} = (\int_0^{|\Omega|} (f^*(t))^p \, dt)^{1/p}.$$

These scale of spaces is enough for dealing with many problems in Analysis but sometimes one need a more refined scale of spaces as the Zygmund spaces $L_p(\log L)_b(\Omega)$ where $0 < p < \infty$ and $b \in \mathbb{R}$, or $p = \infty$ and $b < 0$. A function f belongs to $L_p(\log L)_b(\Omega)$ if and only if it has a finite quasi-norm

$$\|f\|_{L_p(\log L)_b(\Omega)} = \left(\int_0^{|\Omega|} \left[(1 + |\log t|)^b \, f^*(t) \right]^p \, dt \right)^{1/p}$$

(with the usual modification if $p = \infty$).

2000 *Mathematical Subject Classification*: 46B70, 46E30, 47B10.

* The author has been supported in part by the Spanish Ministerio de Educación y Ciencia (MTM2004-01888).

Clearly, $L_p(\Omega) = L_p(\log L)_0(\Omega)$. Zygmund spaces complement the scale of Lebesgue spaces in the sense that if $p < \infty$ and $b > 0$ then

$$L_p(\log L)_b(\Omega) \subset L_p(\Omega) \subset L_p(\log L)_{-b}(\Omega)$$

and for $-\infty < b_1 < b_2 < \infty$

$$L_{p+\varepsilon}(\Omega) \subset L_p(\log L)_{b_2}(\Omega) \subset L_p(\log L)_{b_1}(\Omega) \subset L_{p-\varepsilon}(\Omega).$$

For $p = \infty$ and $-\infty < a_1 < a_2 < 0$ we have

$$L_\infty(\Omega) \subset L_\infty(\log L)_{a_2}(\Omega) \subset L_\infty(\log L)_{a_1}(\Omega).$$

Zygmund spaces are studied in detail in the books by Bennett and Sharpley [2] and Edmunds and Triebel [10]. An important result of their theory allows to reduce Zygmund spaces to the usual Lebesgue spaces (see [10]). We recall this result and some of its applications in Section 1. Taken the representation theorem as model, we define abstract logarithmic interpolation spaces in Section 2. There we also characterize them by using the interpolation method with a function parameter and we apply the results to function spaces. Finally, in Section 3, we show applications to operator spaces defined in terms of approximation numbers.

1. Zygmund spaces

The following result, taken from [10] and [17], characterizes Zygmund spaces in terms of the usual Lebesgue spaces.

Theorem 1.1. *Let $0 < p < \infty$ and let $j_0 = j_0(p) \in \mathbb{N}$ such that, for all $j \in \mathbb{N}$ with $j \geq j_0$,*

$$\frac{1}{q_j} = \frac{1}{p} - \frac{1}{2^j} > 0. \quad \textit{Put} \quad \frac{1}{r_j} = \frac{1}{p} + \frac{1}{2^j}.$$

(i) Let $b < 0$. Then $L_p(\log L)_b(\Omega)$ consists of all Lebesgue-measurable functions f on Ω such that

$$\Big(\sum_{j=j_0}^{\infty} 2^{jbp} \|f\|^p_{L_{r_j}(\Omega)}\Big)^{1/p} < \infty. \tag{1.1}$$

Moreover, (1.1) defines an equivalent quasi-norm on $L_p(\log L)_b(\Omega)$.

(ii) Let $b > 0$. Then $L_p(\log L)_b(\Omega)$ is the set of all Lebesgue-measurable functions f on Ω which can be represented as

$$f = \sum_{j=j_0}^{\infty} f_j\,, \quad f_j \in L_{q_j}(\Omega) \tag{1.2}$$

such that

$$\Big(\sum_{j=j_0}^{\infty} 2^{jbp} \|f_j\|^p_{L_{q_j}(\Omega)}\Big)^{1/p} < \infty. \tag{1.3}$$

Moreover, the infimum over all expressions (1.3) satifying (1.2) is an equivalent quasi-norm on $L_p(\log L)_b(\Omega)$.

There is also a corresponding result when $p = \infty$.

The advantage of such reduction of the more complicated Zygmund spaces to the simpler Lebesgue spaces is quite clear. For example, in connection with the study of linear continuous (and compact) operators acting in Zygmund spaces (see [10]).

Constructions of type (1.1) to (1.3) have been considered in the framework of extrapolation theory, especially in connection with limiting situations such as $p = 1$ and $p = \infty$. See the monographs by Jawerth and Milman [15], Milman [19] and the article by Karadzhov and Milman [17]. The last paper deals also with methods for $0 < p < \infty$. In particular, they prove (ii) for $0 < p < 1$ complementing the result of [10] based on duality from (i) and so established for $1 \leq p < \infty$.

As an application of Theorem 1.1 one can recover a classical result of Hardy and Littlewood on boundedness of the maximal function from $L(\log L)(\Omega)$ into $L_1(\Omega)$ (see [10]). The representation theorem is also useful in other contexts. For example, Triebel used it in [22] to study the degree of compactness of the embedding from the fractional Sobolev space $H_p^{n/p}(\Omega)$ into $L_\infty(\log L)_b(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $1 < p < \infty$ and $b < 1/p - 1$. The "L_p-counterpart" of this case was studied by Edmunds and Triebel [9], who determined the behaviour of entropy numbers of the embedding from $H^s_{np/(n+sp)}(\Omega) \hookrightarrow L_p(\log L)_b(\Omega)$, where $1 < p < \infty, s > 0$ and $b < 0$. Again Theorem 1.1 was a basic tool in [9].

As it is well-known, L_p spaces can be obtained by complex interpolation. Namely,

$$(L_\infty(\Omega), L_1(\Omega))_{[\theta]} = L_p(\Omega) \quad \text{if} \quad 1/p = \theta.$$

So, taking the representation theorem as a starting point, Edmunds and Triebel studied in [11] the corresponding abstract theory based on complex interpolation. There they introduced interpolation spaces which complement the complex interpolation scale. In particular, they investigated the spaces that come out by replacing in Theorem 1.1 the scale of $L_p(\Omega)$-spaces by the scale of Sobolev spaces $H_p^s(\Omega)$. They called "logarithmic Sobolev spaces" to the resulting spaces.

But $L_p(\Omega)$-spaces can be also obtained by real interpolation. Indeed,

$$(L_\infty(\Omega), L_1(\Omega))_{\theta,p} = L_p(\Omega) \quad \text{if} \quad 1/p = \theta.$$

So, it is also natural to investigate the corresponding abstract theory based on the real interpolation method. For the Banach case, this was done by Cobos, Fernández-Cabrera and Triebel [7]. The quasi-Banach case was studied by Karadzhov and Milman [17] and Cobos, Fernández-Cabrera, Manzano and Martínez [6]. Subsequently, we will describe some results taken from [6].

2. Logarithmic interpolation spaces

Let A_0, A_1 be quasi-Banach spaces with $A_0 \hookrightarrow A_1$, where $\hookrightarrow$ means continuous inclusion. The Peetre's K-functional is defined by

$$K(t,a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} \ : a = a_0 + a_1,\, a_j \in A_j\}, \quad t > 0, \quad a \in A_1.$$

For $0 < \theta < 1$ and $0 < q \leq \infty$, the real interpolation space $A_{\theta,q} = (A_0, A_1)_{\theta,q}$ consists of all those $a \in A_1$ having a finite quasi-norm

$$\|a\|_{A_{\theta,q}} = \Big(\int_0^\infty \big(t^{-\theta} K(t,a)\big)^q \, dt/t \Big)^{1/q}$$

(with the usual modification if $q = \infty$). Full details on these spaces can be found in the books by Bergh and Löfström [3] or by Triebel [21].

For the couple $A_0 = L_\infty(\Omega) \hookrightarrow A_1 = L_r(\Omega)$, $\quad r > 0$, the K-functional turns out to be

$$K(t,f) \sim t \left(\int_0^{t^{-r}} (f^*(s))^r ds \right)^{1/r}.$$

This yields the interpolation formula

$$(L_\infty(\Omega), L_r(\Omega))_{\theta,p} = L_p(\Omega) \quad \text{if} \quad 1/p = \theta/r.$$

If the value of the parameter q is different from p, then we get Lorentz function spaces

$$(L_\infty(\Omega), L_r(\Omega))_{\theta,q} = L_{p,q}(\Omega) = \left\{ f : \|f\|_{L_{p,q}(\Omega)} = \left(\int_0^{|\Omega|} (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q} < \infty \right\}.$$

Returning to the abstract case, we have for $0 < p, q \leq \infty$ and $0 < \theta < \mu < 1$, that $(A_0, A_1)_{\theta,p} \hookrightarrow (A_0, A_1)_{\mu,q}$. Let $A_{\theta+} = \bigcap_{\theta<\mu<1} A_{\mu,q}$. Note that the space $A_{\theta+}$ is independent of q.

The next definition is modeled on the representation theorem for Zygmund spaces.

Definition 2.1. Let A_0, A_1 be quasi-Banach spaces with $A_0 \hookrightarrow A_1$. Let $0 < \theta < 1$ and let $j_0 = j_0(\theta) \in \mathbb{N}$ such that, for all $j \in \mathbb{N}$ with $j \geq j_0$,

$$\sigma_j = \theta + 2^{-j} < 1 \quad \text{and} \quad \lambda_j = \theta - 2^{-j} > 0.$$

Let $0 < q \leq \infty$.

(i) Assume $b < 0$. We let $A_{\theta,q}(\log A)_b$ denote the space of all $a \in A_{\theta+}$ which have a finite quasi-norm

$$\|a\|_{A_{\theta,q}(\log A)_b} = \Big(\sum_{j=j_0}^\infty 2^{jbq} \, \|a\|^q_{A_{\sigma_j,q}} \Big)^{1/q}.$$

(ii) Let $b > 0$. The space $A_{\theta,q}(\log A)_b$ consists of all $a \in A_1$ which can be represented as $a = \sum_{j=j_0}^\infty a_j$ with $a_j \in A_{\lambda_j,q}$ and

$$\Big(\sum_{j=j_0}^\infty 2^{jbq} \, \|a_j\|^q_{A_{\lambda_j,q}} \Big)^{1/q} < \infty.$$

We put

$$\|a\|_{A_{\theta,q}(\log A)_b} = \inf \left\{ \Big(\sum_{j=j_0}^\infty 2^{jbq} \, \|a_j\|^q_{A_{\lambda_j,q}} \Big)^{1/q} \right\}.$$

(iii) If $b = 0$, then $A_{\theta,q}(\log A)_b = A_{\theta,q}$.

Standard arguments show that $A_{\theta,q}(\log A)_b$ is a quasi-Banach space in all cases of $b \in \mathbb{R}$. It does not depend on j_0 (with equivalence of norms). If A_0 and A_1 are Banach spaces, $1 \le q \le \infty$ and we replace the real method by the complex interpolation method then we obtain the spaces studied in [11]. They are different from the complex interpolation scale and complement it.

In our case, spaces $A_{\theta,q}(\log A)_b$ are also different from the real interpolation scale, but if we modify the definition of $A_{\theta,q}$ by adding to the function t^θ a logarithmic term

$$t^\theta \rightsquigarrow \varrho_{\theta,b}(t) = t^\theta(1+|\log t|)^{-b},$$

that is, if we consider the spaces

$$(A_0, A_1)_{\varrho_{\theta,b};q} = \left\{ a \in A_1 : \|a\|_{A_{\varrho_{\theta,b};q}} = \Big(\int_0^\infty \big(\frac{K(t,a)}{\varrho_{\theta,b}(t)}\big)^q \frac{dt}{t} \Big)^{1/q} < \infty \right\}$$

then we get the same spaces. Before stating the result, let we mention that spaces $(A_0, A_1)_{\varrho_{\theta,b};q}$ are a special case of the so called real method with a function parameter (see [13] and [14]).

Theorem 2.2. *Let $0 < q \le \infty$, $0 < \theta < 1$ and $b \in \mathbb{R}$. Put $\varrho_{\theta,b}(t) = t^\theta(1+|\log t|)^{-b}$, $t > 0$. Then we have, with equivalent quasi-norms,*

$$A_{\theta,q}(\log A)_b = A_{\varrho_{\theta,b};q}.$$

See [6] for the proof. A similar result holds for $\theta = 0, q = \infty$ and $b < 0$.

Combining the theorem with known results from interpolation theory, we can now derive other representation theorems. For example, let Ω be a domain in $\mathbb{R}^n$ with finite Lebesgue measure, let $0 < p < \infty$, $0 < q \le \infty$ and choose $0 < r < p$. Then interpolating with $\theta = r/p$ and $\varrho_{\theta,b}$, $b \in \mathbb{R}$ we obtain the Lorentz-Zygmund function spaces $L_{p,q}(\log L)_b(\Omega) = (L_\infty(\Omega), L_r(\Omega))_{\varrho_{\theta,b};q}$

$$= \left\{ f : \|f\|_{L_{p,q}(\log L)_b(\Omega)} = \left(\int_0^{|\Omega|} \left[t^{1/p}(1+|\log t|)^b f^*(t) \right]^q \frac{dt}{t} \right)^{1/q} < \infty \right\}.$$

These spaces were introduced by Bennett and Rudnick [1]. For other references and generalization of these spaces see [8].

As a direct application of Theorem 2.2, we obtain the following characterization for Lorentz-Zygmund spaces.

Corollary 2.3. *Let $0 < p < \infty$, $0 < q \le \infty$ and let $j_0 = j_0(p) \in \mathbb{N}$ such that, for all $j \in \mathbb{N}$ with $j \ge j_0$,*

$$\frac{1}{q_j} = \frac{1}{p} - \frac{1}{2^j} > 0. \quad \textit{Put} \quad \frac{1}{r_j} = \frac{1}{p} + \frac{1}{2^j}.$$

(i) Let $b < 0$. Then $L_{p,q}(\log L)_b(\Omega)$ is the set of all measurable functions f on Ω such that

$$\Big(\sum_{j=j_0}^\infty 2^{jbq} \|f\|^q_{L_{q_j,q}(\Omega)} \Big)^{1/q} < \infty.$$

(ii) Let $b > 0$. Then $L_{p,q}(\log L)_b(\Omega)$ consists of all measurable functions f on Ω which can be represented as $f = \sum_{j=j_0}^\infty f_j$ with $f_j \in L_{r_j,q}(\Omega)$ and

$$\Big(\sum_{j=j_0}^\infty 2^{jbq} \|f_j\|^q_{L_{r_j,q}(\Omega)} \Big)^{1/q} < \infty.$$

Since $L_p(\log L)_b = L_{p,p}(\log L)_b$ we have now two representations for Zygmund spaces: The former one in terms of Lorentz spaces $L_{q_j,q}$, $L_{r_j,q}$ and the one given by Theorem 1.1 by using the more simple Lebesgue spaces L_{q_j}, L_{r_j}. In fact it is possible to recover Theorem 1.1 from Theorem 2.2 with the help of the following lemma.

Lemma 2.4. *Let A_0, A_1 be quasi-Banach spaces with $A_0 \hookrightarrow A_1$. Let $0 < \theta < 1$ and let $j_0 = j_0(\theta) \in \mathbb{N}$ such that, for all $j \in \mathbb{N}$, $j \geq j_0$,*

$$\sigma_j = \theta + 2^{-j} < 1 \text{ and } \lambda_j = \theta - 2^{-j} > 0.$$

Let $0 < q \leq \infty$ and assume also that for $j \geq j_0$

$$\frac{1}{u_j} = \frac{1}{q} - \frac{1}{2^j} > 0. \quad \text{Put} \quad \frac{1}{s_j} = \frac{1}{q} + \frac{1}{2^j}$$

(i) If $b < 0$, the norm of $A_{\theta,q}(\log A)_b$ is equivalent to

$$\Big(\sum_{j=j_0}^{\infty} 2^{jbq} \|a\|^q_{A_{\sigma_j,s_j}}\Big)^{1/q}.$$

(ii) If $b > 0$, the norm of $A_{\theta,q}(\log A)_b$ is equivalent to

$$\inf\left\{\Big(\sum_{j=j_0}^{\infty} 2^{jbq} \|a_j\|^q_{A_{\lambda_j,u_j}}\Big)^{1/q} : a = \sum_{j=j_0}^{\infty} a_j\,, \Big(\sum_{j=j_0}^{\infty} 2^{jbq} \|a_j\|^q_{A_{\lambda_j,u_j}}\Big)^{1/q} < \infty\right\}.$$

Full details can be found in [6].

3. Operator spaces

In this last section we show applications of the previous results to spaces of operators defined by means of the approximation numbers.

Let E, F be quasi-Banach spaces and let $\mathcal{L}(E,F)$ be the quasi-Banach space of all bounded linear operators acting from E into F. For $k \in \mathbb{N}$, the k-th approximation number $a_k(T)$ of $T \in \mathcal{L}(E,F)$ is defined by

$$a_k(T) = \inf\{\|T - R\| \,:\, R \in \mathcal{L}(E,F) \quad \text{with} \quad \operatorname{rank} R < k\}.$$

For $0 < p < \infty$, $0 < q \leq \infty$ and $b \in \mathbb{R}$, the space $\mathcal{L}_{p,q,b}(E,F)$ consists of all those $T \in \mathcal{L}(E,F)$ having a finite quasi-norm

$$\|T\|_{p,q,b} = \Big(\sum_{k=1}^{\infty} (k^{1/p}(1+\log k)^b a_k(T))^q k^{-1}\Big)^{1/q}$$

(with the usual modification if $q = \infty$). These spaces were studied by Cobos in [4] and [5]. Note that

$$T \in \mathcal{L}_{p,q,b}(E,F) \iff \{a_k(T)\} \in \ell_{p,q}(\log \ell)_b$$

where $\ell_{p,q}(\log \ell)_b$ is the Lorentz-Zygmund sequence space. When $b = 0$, we get the Lorentz operator spaces $(\mathcal{L}_{p,q}(E,F), \|\cdot\|_{p,q})$ which have been investigated in detail in the books by König [18] and by Pietsch [20]. The special case $b = 0$ and $p = q$ gives the spaces $(\mathcal{L}_p(E,F), \|\cdot\|_p)$, which are the analogues of the Schatten p-classes for approximation numbers.

Take $0 < r < \infty$ and let $A_0 = \mathcal{L}_r(E,F) \hookrightarrow A_1 = \mathcal{L}(E,F)$. Then

$$K(t,T) \sim \begin{cases} t\|T\| & \text{if } t \le 1 \\ \left(\sum_{k=1}^{[t^r]} a_k^r(T)\right)^{1/r} & \text{if } t > 1. \end{cases}$$

Here $[\cdot]$ is the greatest integer function. For $0 < r < p < \infty$, $\theta = 1 - r/p$, $0 < q \le \infty$ and $b \in \mathbb{R}$ it follows that

$$(\mathcal{L}_r(E,F), \mathcal{L}(E,F))_{\varrho_{\theta,b;q}} = \mathcal{L}_{p,q,b}(E,F).$$

Hence, applying Theorem 2.2 we obtain the following result.

Corollary 3.1. *Let $0 < p < \infty$, $0 < q \le \infty$ and let $j_0 = j_0(p) \in \mathbb{N}$ such that, for all $j \in \mathbb{N}$ with $j \ge j_0$,*

$$\frac{1}{q_j} = \frac{1}{p} - \frac{1}{2^j} > 0. \quad \textit{Put} \quad \frac{1}{r_j} = \frac{1}{p} + \frac{1}{2^j}.$$

(i) Let $b < 0$. Then $\mathcal{L}_{p,q,b}(E,F)$ is the set of all $T \in \mathcal{L}(E,F)$ such that

$$\left(\sum_{j=j_0}^{\infty} 2^{jbq} \|T\|_{q_j,q}^q\right)^{1/q} < \infty.$$

(ii) Let $b > 0$. Then $\mathcal{L}_{p,q,b}(E,F)$ consists of all $T \in \mathcal{L}(E,F)$ which can be represented as $T = \sum_{j=j_0}^{\infty} T_j$ with $T_j \in \mathcal{L}_{r_j,q}(E,F)$ such that $\left(\sum_{j=j_0}^{\infty} 2^{jbq} \|T_j\|_{r_j,q}^q\right)^{1/q} < \infty$.

We finish the paper by describing the operator analogue of the classical result of Hardy and Littlewood on the boundedness of the maximal function from $L(\log L)$ into L_1. For this we need to introduce the space that plays the role of $L(\log L)$ in the scale $\{\mathcal{L}_p\}$. This is

$$\mathcal{L}_{\mathcal{M}}(E,F) = \left\{ T \in \mathcal{L}(E,F) : \|T\|_{\mathcal{M}} = \sup_{m \ge 1} \left\{ \frac{\sum_{j=1}^m a_j(T)}{1 + \log m} \right\} < \infty \right\}.$$

One can check that

$$\mathcal{L}_1(E,F) \subseteq \mathcal{L}_{\mathcal{M}}(E,F) \subseteq \mathcal{L}_p(E,F) \quad \text{for any} \quad 1 < p < \infty.$$

For $E = F = H$, a Hilbert space, the space $\mathcal{L}_{\mathcal{M}}(H)$ is referred in the literature as one of the Macaev ideals (see [12]).

Corollary 3.2. *Let E and F be quasi-Banach spaces and let $\mathfrak{F}$ be a bounded linear operator from $\mathcal{L}_p(E,F)$ into $\mathcal{L}_p(E,F)$ for $1 < p \le 2$. If*

$$\|\mathfrak{F}\|_{\mathcal{L}_p(E,F),\mathcal{L}_p(E,F)} \le \frac{c}{p-1} \quad \textit{as} \quad p \downarrow 1,$$

then

$$\mathfrak{F} : \mathcal{L}_1(E,F) \longrightarrow \mathcal{L}_{\mathcal{M}}(E,F)$$

is bounded.

Sketch of the proof. Using the expression for the K-functional between operator spaces one can show that

$$\mathcal{L}_{\mathcal{M}}(E,F) = (\mathcal{L}_1(E,F), \mathcal{L}(E,F))_{\varrho_{0,-1};\infty} \text{ with } \varrho_{0,-1}(t) = 1 + |\log t|.$$

By the version of Theorem 2.2 for $\theta = 0, q = \infty$ and $b = -1$, we obtain that

$$\|T\|_{\mathcal{M}} \sim \sup_{j\geq 1} \left\{ 2^{-j} \|T\|^*_{q_j,\infty} \right\}.$$

Here $q_j = (1-2^{-j})^{-1}$ and $\|\cdot\|^*_{q_j,\infty} = \|\cdot\|_{(\mathcal{L}_1(E,F),\mathcal{L}(E,F))_{2^{-j},\infty}} \leq \|\cdot\|_{q_j}$. Hence, if $T \in \mathcal{L}_1(E,F)$, we derive

$$\begin{aligned} \|\mathfrak{F}T\|_{\mathcal{M}} &\sim \sup_{j\geq 1} \left\{ 2^{-j} \|\mathfrak{F}T\|^*_{q_j,\infty} \right\} \leq \sup_{j\geq 1} \left\{ 2^{-j} \|\mathfrak{F}T\|_{q_j} \right\} \\ &\leq c \sup_{j\geq 1} \left\{ 2^{-j} (q_j - 1)^{-1} \|T\|_{q_j} \right\} \leq c' \sup_{j\geq 1} \left\{ \|T\|_{q_j} \right\} = c' \|T\|_1. \end{aligned}$$

□

The result is taken from [6] where full details can be found. In the Hilbert case, that is, when $E = F = K$, the result is due to Jawerth and Milman [16]

Acknowledgements. The author would like to thank Professors Mikio Kato and Kichi-Suke Saito for their hospitality.

References

[1] C. Bennett and K. Rudnick, On Lorentz-Zygmund spaces, *Dissertationes Math.* **175** (1980), 1-72.

[2] C. Bennett and R. Sharpley. *Interpolation of Operators* Academic Press, Boston, 1988.

[3] J. Bergh and J. Löfström. *Interpolation Spaces. An introduction* Springer, Berlin, 1976.

[4] F. Cobos, On the Lorentz-Marcikiewicz operator ideal, *Math. Nachr.* **126** (1986), 281-300.

[5] F. Cobos, Entropy and Lorentz-Marcikiewicz operator ideals, *Arkiv Mat.* **25** (1987), 211-219.

[6] F. Cobos, L.M. Fernández-Cabrera, A. Manzano and A. Martínez, Logarithmic interpolation spaces between quasi-Banach spaces, *Z. Anal. Anwendungen* **26** (2007), 65-86.

[7] F. Cobos, L.M. Fernández-Cabrera and H. Triebel, Abstract and concrete logarithmic interpolation spaces, *J. London Math. Soc.* **70** (2004), 231-243.

[8] D.E. Edmunds and W.D. Evans. *Hardy Operators, Function Spaces and Embeddings*, Springer, Berlin, 2004.

[9] D.E. Edmunds and H. Triebel, Logarithmic Sobolev spaces and their applications to spectral theory, *Proc. London Math. Soc.* **71** (1995), 333-371.

[10] D.E. Edmunds and H. Triebel. *Function Spaces, Entropy Numbers, Differential Operators* Cambridge Univ. Press, Cambridge, 1996.

[11] D.E. Edmunds and H. Triebel, Logarithmic spaces and related trace problems, *Funct. Approx. Comment. Math.* **26** (1998), 189-204.

[12] I.C. Gohberg and M.G. Krein. *Theory and Applications of Volterra Operators in Hilbert Space* Amer. Math. Soc., Providence, R.I., 1970.

[13] J. Gustavsson, A function parameter in connection with interpolation of Banach spaces, *Math. Scand.* **42** (1978), 289-305.

[14] S. Janson, Minimal and maximal methods of interpolation, *J. Funct. Anal.* **44** (1981), 50-73.

[15] B. Jawerth and M. Milman, Extrapolation theory with applications, *Mem. Amer. Math. Soc.* **89** (1991) no. 440.

[16] B. JAWERTH AND M. MILMAN, New results and applications of extrapolation theory, in *Interpolation Spaces and Related Topics*, Israel Math. Confer. Proc. **5** (1992).

[17] G.E. KARADZHOV AND M. MILMAN, Extrapolation theory: new results and applications, *J. Approx. Theory* **133** (2005), 38-99.

[18] H. KÖNIG. *Eigenvalue Distribution of Compact Operators*, Birkhäuser, Basel, 1986.

[19] M. MILMAN. *Extrapolation and Optimal Decompositions*, Lect. Notes in Math. **1580**, Springer, Berlin, 1994.

[20] A. PIETSCH. *Eigenvalues and s-Numbers*, Cambridge Univ. Press, Cambridge, 1987.

[21] H. TRIEBEL. *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.

[22] H. TRIEBEL, Approximation numbers and entropy numbers of embeddings of fractional Besov-Sobolev spaces in Orlicz spaces, *Proc. London Math. Soc.* **66** (1993), 589-618.

Fernando Cobos, DEPARTAMENTO DE ANÁLISIS MATEMATICO, FACULTAD DE MATEMÁTICAS, UNIVERSIDAD COMPLUTENSE DE MADRID, 28040 MADRID. SPAIN

E-mail address: cobos@mat.ucm.es