Title: Boundedness of $\gamma$-Cesaro means ($\gamma > 0$) of operators (Banach spaces, function spaces, inequalities and their applications)

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Boundedness of $\gamma$-Cesàro means ($\gamma > 0$) of operators

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In this talk I would like to report some recent results on the boundedness properties of $\gamma$-Cesàro means of operators, where $\gamma > 0$. The results are taken from joint works with Jeng-Chung Chen, Yuan-Chuan Li, and Sen-Yen Shaw (cf. [1], [3]).

1. The discrete case. Let $T : X \to X$ be a bounded linear operator on a Banach space $X$. The Cesàro means of order $\gamma$ (or $\gamma$-Cesàro means) of $T$, where $\gamma \geq 0$, are defined by

$$C_n^{\gamma} = C_n^{\gamma}(T) := \frac{1}{\sigma_{n}^{\gamma}} \sum_{k=0}^{n} \sigma_{n-k}^{\gamma-1} T^k,$$

where $\sigma_{n}^{\beta} = \binom{\beta+n}{n}$ for $n \geq 1$, and $\sigma_{0}^{\beta} = 1$ (see Zygmund [5, Chapter 3]). Among them are the following two particular means: $C_n^{0} = C_n^{0}(T) = T^n$, and $C_n^{1} = C_n^{1}(T) = (n+1)^{-1} \sum_{k=0}^{n} T^k$.

The Abel means of $T$ are the operators $A_{r} = A_{r}(T) := (1-r) \sum_{k=0}^{\infty} r^k T^k$, defined for $0 < r < 1/r(T)$, where $r(T) = \lim_{n \to \infty} \| T^n \|^{1/n}$ denotes the spectral radius of $T$. Clearly, $r(T) \leq 1$ if and only if $A_{r}$ exists for all $0 < r < 1$. (Moreover, in this case, we have $A_{r} = (1-r)(I - rT)^{-1}$ for each $0 < r < 1$.) The following is well-known (cf. [5]):

If $0 < \gamma < \beta < \infty$, then

$$\sup_{0<r<1} \| A_{r} \| \leq \sup_{n \geq 0} \| C_n^{\beta} \| \leq \sup_{n \geq 0} \| C_n^{\gamma} \| \leq \sup_{n \geq 0} \| C_n^{0} \| = \sup_{n \geq 0} \| T^n \|;$$

in particular, if $T$ is a positive linear operator on a Banach lattice $X$, then

$$\sup_{0<r<1} \| A_{r} \| < \infty \iff \sup_{n \geq 0} \| C_n^{1} \| < \infty \quad (\text{cf. Emilion [2]}).$$
In connection with these relations, two questions come up naturally:

(A) *If $T$ is positive, then does the implication* \( \sup_{0<r<1} \| A_r \| < \infty \Rightarrow \sup_{n \geq 1} \| C_n^\gamma \| < \infty \) *hold for a certain constant* $\gamma$, with $0 < \gamma < 1$?

(B) *If $T$ is not assumed to be positive, then does the implication* \( \sup_{0<r<1} \| A_r \| < \infty \Rightarrow \sup_{n \geq 1} \| C_n^\gamma \| < \infty \) *hold for a certain constant* $\gamma$, with $\gamma \geq 1$?

Our answers are as follows.

**Theorem 1.** For any $\gamma$, with $0 < \gamma < 1$, there exists a positive linear operator $T$ on an $L_1$-space such that \( \sup_{n \geq 1} \| C_n^\beta \| < \infty \) for all $\beta > \gamma$, but \( \sup_{n \geq 1} \| C_n^\gamma \| = \infty \).

**Theorem 2.** There exists a positive linear operator $T$ on an $L_1$-space such that \( \sup_{n \geq 1} \| C_n^\beta \| < \infty \) for all $\beta > 0$, but \( \sup_{n \geq 1} \| T^n \| = \infty \).

**Theorem 3.** Let $\dim X = \infty$. Then the following hold:

(i) *For any integer* $k \geq 1$, there exists a bounded linear operator $T$ on $X$ such that \( \sup_{n \geq 1} \| C_n^k \| < \infty \), but \( \sup_{n \geq 1} \| C_n^\beta \| = \infty \) for all $\beta$ with $0 \leq \beta < k$.

(ii) *There exists a bounded linear operator* $T$ on $X$, with $r(T) = 1$, such that \( \sup_{0<r<1} \| A_r \| < \infty \), but \( \sup_{n \geq 1} \| C_n^\beta \| = \infty \) for all $\beta \geq 0$.

2. **The continuous case.** Let $T(\cdot)$ be a $C_0$-semigroup of bounded linear operators on a Banach space $X$. The $\gamma$-th Cesàro means of $T(\cdot)$, where $\gamma \geq 0$, are defined as $C_t^\gamma = C_t^\gamma(T(\cdot)) := T(0)$ and, for $t > 0$,

$$ C_t^\gamma = C_t^\gamma(T(\cdot)) := \left\{ \begin{array}{ll} T(t) & \text{if } \gamma = 0, \\ \gamma t^{-\gamma} \int_0^t (t-u)^{\gamma-1} T(u) du & \text{if } \gamma > 0. \end{array} \right. $$

The Abel means of $T(\cdot)$ are the operators

$$ A_\lambda = A_\lambda(T(\cdot)) := \lambda \int_0^\infty e^{-\lambda u} T(u) du = \lim_{t \to \infty} \lambda \int_0^t e^{-\lambda u} T(u) du, $$

defined for $\lambda > 0$ if the limit exists. As in the discrete case, we have (cf. [4]):

If $0 < \gamma < \beta < \infty$, then

$$(3) \quad \sup_{0<\lambda<\infty} \| A_\lambda \| \leq \sup_{t>0} \| C_t^\beta \| \leq \sup_{t>0} \| C_t^\gamma \| \leq \sup_{t>0} \| C_t^0 \| = \sup_{t>0} \| T(t) \|;$$
in particular, if $T(\cdot)$ is a positive $C_0$-semigroup on a Banach lattice $X$, then

\[(4) \quad \sup_{0<\lambda<\infty} \|A_\lambda\| < \infty \iff \sup_{t>0} \|C^1_t\| < \infty \quad (\text{cf. } [2]).\]

The following are the continuous case results:

**Theorem 1'**. For any $\gamma$, with $0 < \gamma < 1$, there exists a positive $C_0$-semigroup $T(\cdot)$ on an $L_1$-space such that $\sup_{t>0} \|C^\beta_t\| < \infty$ for all $\beta > \gamma$, but $\sup_{t>0} \|C^\beta_t\| = \infty$.

**Theorem 2'**. There exists a positive $C_0$-semigroup $T(\cdot)$ on an $L_1$-space such that $\sup_{t>0} \|C^\beta_t\| < \infty$ for all $\beta > 0$, but $\sup_{t>0} \|T(t)\| = \infty$.

**Theorem 3'**. Let $\dim X = \infty$. Then the following hold:

(i) For any integer $k > 1$, there exists a $C_0$-semigroup $T(\cdot)$ on $X$ such that $\sup_{t>0} \|C^k_t\| < \infty$, but $\sup_{t>0} \|C^\beta_t\| = \infty$ for all $\beta$, with $0 \leq \beta < k$.

(ii) There exists a $C_0$-semigroup $T(\cdot)$ on $X$ such that $\sup_{0<\lambda<\infty} \|A_\lambda\| < \infty$, but $\sup_{t>0} \|C^\beta_t\| = \infty$ for all $\beta \geq 0$.

**References**


