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Linear and topological properties of a sequence space defined by an $L_p$-function (Banach spaces, function spaces, inequalities and their applications)

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Linear and topological properties of a sequence space defined by an $L_p$-function

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Abstract

We introduce a sequence space $\Lambda_p(f)$ defined by an $L_p$-function $f(\neq 0)$ for $1 \leq p < +\infty$ by

$$\Lambda_p(f) := \{ a \in \mathbb{R}^\infty : \Psi_p(a : f) < +\infty \},$$

where

$$\Psi_p(a : f) := \left( \sum_n \int_{-\infty}^{+\infty} |f(x-a_n) - f(x)|^p \, dx \right)^{\frac{1}{p}}$$

and discuss the linear and topological properties of $\Lambda_p(f)$, that is, the linearity, the relations with $\ell_p$, the linear topological property of the metric $d_p(a, b) = \Psi_p(a - b : f)$ on $\Lambda_p(f)$, the completeness, and so on.

In the case where $p = 2$, $\Lambda_2(\sqrt{f})$ is studied in the theory of translation equivalence of the infinite product measure $\mu = \otimes_1^\infty f(x) \, dx$ on $\mathbb{R}^\infty$. In fact, if $f(x) > 0$ a.e.(x), then $a \in \Lambda_2(\sqrt{f})$ if and only if the translation $\mu_a$ is equivalent to $\mu$, see Kakutani[3], Shepp[4].
1 Introduction

Let $f(\neq 0)$ be an $L_p$-function on the real line $\mathbb{R}$. For $1 \leq p < +\infty$ and for a real sequence $a = \{a_n\} \in \mathbb{R}^\infty$, we set

$$\Psi_p(a : f) := \left( \sum_n \int_{-\infty}^{+\infty} |f(x - a_n) - f(x)|^p \, dx \right)^{\frac{1}{p}} = \left( \sum_n \|f(\cdot - a_n) - f(\cdot)\|_{L_p}^p \right)^{\frac{1}{p}},$$

and define $\Lambda_p(f)$ by

$$\Lambda_p(f) := \{a \in \mathbb{R}^\infty : \Psi_p(a : f) < +\infty\}.$$

By the triangular inequality of $L_p$-norm, we have

$$\Psi_p(a - b : f) \leq \Psi_p(a : f) + \Psi_p(b : f),$$

which implies that $\Lambda_p(f)$ is an additive subgroup of $\mathbb{R}^\infty$.

Define a metric on $\Lambda_p(f)$ by

$$d_p(a, b) := \Psi_p(a - b : f).$$

Then $(\Lambda_p(f), d_p(a, b))$ becomes a topological group.

In this talk, we are concerned with the following problems:

1. the linearity of $\Lambda_p(f)$,
2. the relations between $\Lambda_p(f)$ and $\ell_p$, and
3. the linear topological property of the metric $d_p(a, b)$ on $\Lambda_p(f)$,
4. the completeness of $(\Lambda_p(f), d_p)$.

2 Linearity of $\Lambda_p(f)$

The function $f$ is called unimodal at $\alpha$ if there exists $\alpha \in \mathbb{R}$ such that $f(x)$ is non-decreasing on $(-\infty, \alpha)$ and non-increasing on $(\alpha, +\infty)$.

Theorem 1 ([1]) Assume the $L_p$-function $f(\neq 0)$ is unimodal. Then we have

$$\Psi_p(ta : f) \leq \Psi_p(a : f), \quad 0 < t \leq 1$$

for any $a \in \Lambda_p(f)$. In particular, $\Lambda_p(f)$ is a linear space.
3 Relations between $\Lambda_p(f)$ and $\ell_p$

We say $I_p(f) < +\infty$ if $f(x)$ is absolutely continuous on $\mathbb{R}$ and the $p$-integral defined by

$$I_p(f) := \int_{-\infty}^{+\infty} |f'(x)|^p \, dx$$

is finite. In particular $I_2(\sqrt{f})$, where $f$ is a probability density function on $\mathbb{R}$, coincides with the Shepp's integral (Shepp[4]).

**Theorem 2** ([2]) Let $1 \leq p < +\infty$ and let $f(\neq 0)$ be an $L_p$-function on $\mathbb{R}$. Then $\Lambda_p(f) \subset \ell_p$.

**Theorem 3** ([2]) Let $1 < p < +\infty$ and $f(\neq 0)$ be an $L_p$-function on $\mathbb{R}$. Then $\Lambda_p(f) = \ell_p$ if and only if $I_p(f) < +\infty$.

4 Linear topological properties of $\Lambda_p(f)$

If $I_p(f) < +\infty$, then $\Lambda_p(f) = \ell_p$ as a sequence space. We shall show in this case the $\ell_p$-norm $\| \|_p$ is stronger than the metric $d_p$.

**Theorem 4** Assume $I_p(f) < +\infty$. Then the $\ell_p$-norm is stronger than the metric $d_p$ on $\Lambda_p(f) = \ell_p$.

**Proof.** Since $\Psi_p(a : f)$ is lower semi-continuous on $\ell_p$, by the Baire's category theorem, there exists $N$ such that the set $L_N := \{a \in \Lambda_p(f) = \ell_p : \Psi_p(a : f) \leq N\}$ has an interior point with respect to the $\ell_p$-norm. So that there exists $a_0 \in L_N$ and $\delta > 0$ such that $\|a - a_0\|_p \leq \delta$ implies $\Psi_p(a : f) \leq N$, which implies

$$\|a\|_p \leq \delta \Rightarrow \Psi_p(a : f) \leq \Psi_p(a + a_0 : f) + \Psi_p(a_0 : f) \leq 2N.$$ 

and

$$\|a\|_p \leq K \Rightarrow \Psi_p(a : f) \leq 2\left(\left[\frac{K}{\delta}\right] + 1\right)N.$$ 

By Xia[5], Lemma I.2.2, there exists $b_0$ such that $\Psi_p(\cdot : f)$ is $\ell_p$-continuous at $b_0$. So that for every $\epsilon > 0$, there exists $\lambda > 0$ such that

$$\|b - b_0\|_p \leq \lambda \Rightarrow |\Psi_p(b : f)^p - \Psi_p(b_0 : f)^p| \leq \epsilon.$$ 

Now we shall show $\Psi_p(\cdot : f)$ is $\ell_p$-continuous at 0. For every $b$ with $\|b\| \leq \lambda$, and for every natural numbers $n$ and $N$, we set

$$b(m, N) := (b_1^0, \cdots, b_N^0, b_{N+1}^0 + b_1, \cdots, b_{N+m}^0 + b_m, b_{N+m+1}^0, \cdots),$$
where $b_0 = \{b_i^0\}$. Then we have
\[ \|b(m, N) - b_0\|_p = \left(\sum_{i=1}^{m} b_i^p\right)^{\frac{1}{p}} \leq \lambda, \]
which implies
\[ |\Psi_p(b(m, N) : f)^p - \Psi_p(b : f)^p| = \sum_{i=1}^{m} \int_{-\infty}^{+\infty} |f(x - b_{N+i}^0 - b_i) - f(x)|^p dx \leq \varepsilon. \]

Letting $N \to +\infty$, we have
\[ \sum_{i=1}^{m} \int_{-\infty}^{+\infty} |f(x - b_i) - f(x)|^p dx \leq \varepsilon, \]
for every $m$, and
\[ \Psi_p(b : f)^p = \sum_{i=1}^{+\infty} \int_{-\infty}^{+\infty} |f(x - b_i) - f(x)|^p dx \leq \varepsilon, \]
which shows $\Psi_p(\cdot : f)$ is $\ell_p$-continuous at 0.

We can now easily deduce the continuity of $\Psi_p(\cdot : f)$ at any point $c_0$ as follows. If $\|c - c_0\|_p \leq \lambda$, then we have
\[ |\Psi_p(c : f) - \Psi_p(c_0 : f)| \leq \Psi_p(c - c_0 : f) \leq \varepsilon^{\frac{1}{p}}. \]

**Theorem 5** If $f(x)$ is unimodular, then the metric $d_p$ is the vector topology on $\Lambda_p(f)$.

**Proof.** By Theorem 1, the scalar multiplication is continuous.

We consider the largest linear subspace $\Sigma_p(f)$ of $\Lambda_p(f)$ after Yamasaki[6] as follows. Define
\[ \Sigma_p(f) := \{a \in \Lambda_p(f) : ta \in \Lambda_p(f) \text{ for every } t \in \mathbb{R}\}. \]

**Lemma 6** If $a(\neq 0) \in \Sigma_p(f)$, then the real function $\varphi(t : a) = \Psi_p(ta : f)^p$ is continuous on the real line $\mathbb{R}$. Moreover, the metric
\[ \rho(s, t) = \Psi_p((t - s)a : f) \]
gives the equivalent metric with the usual metric $|s - t|$. 
Proof. The continuity of $\varphi(t:a)$ is proved by the similar way to Theorem 5. Since $a \neq 0$, there exists $a_k \neq 0$. If

$$\int_{-\infty}^{+\infty} |f(x - t_n a_k) - f(x)|^p dx \to 0 \text{ as } n \to +\infty,$$

then it follows that $t_n \to 0$, see the proof of Theorem 2. This proves the second assertion.

Let $V_\varepsilon = \{a \in \Sigma_p(f) : \Psi_p(a : f) \leq \varepsilon\}$. Then for every $x \in \Sigma_p(f)$, we can find $\delta > 0$ such that

$$tx \in V_\varepsilon \text{ for every } -\delta < t < \delta.$$  

Consequently we can linearize $d_p$ as follows, see Yamasaki[6], p.185, Xia[5], Lemma I.1.2. The linearization $\sigma_p(a, b)$ of $d_p(a, b)$ is defined by

$$\sigma_p(a, b) := \sup_{|t| \leq 1} d_p(ta, tb)$$

for $a, b \in \Sigma_p(f)$.

Theorem 7 $(\Sigma_p(f), \sigma_p(a, b))$ is a topological vector space.

5 Completeness of $\Lambda_p(f)$

Theorem 8 ([1]) Let $f(\neq 0)$ be an $L_p$-function. Then $\Lambda_p(f)$ is complete with respect to $d_p$ for $1 \leq p < +\infty$.

Theorem 9 $(\Sigma_p(f), \sigma_p(a, b))$ is complete.

6 Examples

Example 10 Define $f(x) := \max\{1 - |x|, 0\}$. Then we have

(1) $\Lambda_p(f) = \ell_p$ for $1 \leq p < 2$,

(2) $\Lambda_2(f) = \left\{a = (a_n) \in R^\infty \mid \sum_n a_n^2 (1 + |\log |a_n||) < +\infty \right\}$, and

(3) $\Lambda_p(f) = \ell_2$ for $p > 2$. 
References


