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<td>Honda, Aoi; Okazaki, Yoshiaki; Sato, Hiroshi</td>
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Kyoto University
Linear and topological properties of a sequence space defined by an $L_p$-function

Aoi Honda $^a$  Yoshiaki Okazaki $^a$  Hiroshi Sato $^b$

$^a$ Kyushu Institute of Technology, 680-4, Kawazu, Iizuka 820-8502, Japan
$^b$ Kyushu University, Faculty of Mathematics, Fukuoka 812-8581, Japan

Abstract

We introduce a sequence space $\Lambda_p(f)$ defined by an $L_p$-function $f(\neq 0)$ for $1 \leq p < +\infty$ by

$$\Lambda_p(f) := \{a \in \mathbb{R}^\infty : \Psi_p(a : f) < +\infty\},$$

where

$$\Psi_p(a : f) := \left( \sum_n \int_{-\infty}^{+\infty} |f(x-a_n) - f(x)|^p \, dx \right)^{\frac{1}{p}}$$

$$= \left( \sum_n \| f(\cdot - a_n) - f(\cdot) \|_{L_p}^p \right)^{\frac{1}{p}},$$

and discuss the linear and topological properties of $\Lambda_p(f)$, that is, the linearity, the relations with $\ell_p$, the linear topological property of the metric $d_p(a, b) = \Psi_p(a - b : f)$ on $\Lambda_p(f)$, the completeness, and so on.

In the case where $p = 2$, $\Lambda_2(\sqrt{f})$ is studied in the theory of translation equivalence of the infinite product measure $\mu = \otimes_1^\infty f(x) \, dx$ on $\mathbb{R}^\infty$. In fact, if $f(x) > 0$ a.e. (x), then $a \in \Lambda_2(\sqrt{f})$ if and only if the translation $\mu_{a}$ is equivalent to $\mu$, see Kakutani[3], Shepp[4].
1 Introduction

Let $f(\neq 0)$ be an $L_p$-function on the real line $\mathbb{R}$. For $1 \leq p < +\infty$ and for a real sequence $a = \{a_n\} \in \mathbb{R}^\infty$, we set

$$\Psi_p(a : f) := \left( \sum_n \int_{-\infty}^{+\infty} |f(x - a_n) - f(x)|^p \, dx \right)^{\frac{1}{p}} = \left( \sum_n \|f(\cdot - a_n) - f(\cdot)\|_{L_p}^p \right)^{\frac{1}{p}},$$

and define $\Lambda_p(f)$ by

$$\Lambda_p(f) := \{a \in \mathbb{R}^\infty : \Psi_p(a : f) < +\infty\}.$$

By the triangular inequality of $L_p$-norm, we have

$$\Psi_p(a - b : f) \leq \Psi_p(a : f) + \Psi_p(b : f),$$

which implies that $\Lambda_p(f)$ is an additive subgroup of $\mathbb{R}^\infty$.

Define a metric on $\Lambda_p(f)$ by

$$d_p(a, b) := \Psi_p(a - b : f).$$

Then $(\Lambda_p(f), d_p(a, b))$ becomes a topological group.

In this talk, we are concerned with the following problems:

1. the linearity of $\Lambda_p(f)$,
2. the relations between $\Lambda_p(f)$ and $\ell_p$, and
3. the linear topological property of the metric $d_p(a, b)$ on $\Lambda_p(f)$,
4. the completeness of $(\Lambda_p(f), d_p)$.

2 Linearity of $\Lambda_p(f)$

The function $f$ is called unimodal at $\alpha$ if there exists $\alpha \in \mathbb{R}$ such that $f(x)$ is non-decreasing on $(-\infty, \alpha)$ and non-increasing on $(\alpha, +\infty)$.

Theorem 1 ([1]) Assume the $L_p$-function $f(\neq 0)$ is unimodal. Then we have

$$\Psi_p(ta : f) \leq \Psi_p(a : f), \quad 0 < t \leq 1$$

for any $a \in \Lambda_p(f)$. In particular, $\Lambda_p(f)$ is a linear space.
3 Relations between $\Lambda_p(f)$ and $\ell_p$

We say $I_p(f) < +\infty$ if $f(x)$ is absolutely continuous on $\mathbb{R}$ and the $p$-integral defined by

$$I_p(f) := \int_{-\infty}^{+\infty} |f'(x)|^p \, dx$$

is finite. In particular $I_2(\sqrt{f})$, where $f$ is a probability density function on $\mathbb{R}$, coincides with the Shepp’s integral (Shepp[4]).

**Theorem 2** (2) Let $1 \leq p < +\infty$ and let $f(\neq 0)$ be an $L_p$-function on $\mathbb{R}$. Then $\Lambda_p(f) \subseteq \ell_p$

**Theorem 3** (2) Let $1 < p < +\infty$ and $f(\neq 0)$ be an $L_p$-function on $\mathbb{R}$. Then $\Lambda_p(f) = \ell_p$ if and only if $I_p(f) < +\infty$.

4 Linear topological properties of $\Lambda_p(f)$

If $I_p(f) < +\infty$, then $\Lambda_p(f) = \ell_p$ as a sequence space. We shall show in this case the $\ell_p$-norm $\| \cdot \|_p$ is stronger than the metric $d_p$.

**Theorem 4** Assume $I_p(f) < +\infty$. Then the $\ell_p$-norm is stronger than the metric $d_p$ on $\Lambda_p(f) = \ell_p$.

**Proof.** Since $\Psi_p(a : f)$ is lower semi-continuous on $\ell_p$, by the Baire’s category theorem, there exists $N$ such that the set $L_N := \{a \in \Lambda_p(f) = \ell_p : \Psi_p(a : f) \leq N\}$ has an interior point with respect to the $\ell_p$-norm. So that there exists $a_0 \in L_N$ and $\delta > 0$ such that $\|a - a_0\|_p \leq \delta$ implies $\Psi_p(a : f) \leq N$, which implies

$$\|a\|_p \leq \delta \Rightarrow \Psi_p(a : f) \leq \Psi_p(a + a_0 : f) + \Psi_p(a_0 : f) \leq 2N.$$  

and

$$\|a\|_p \leq K \Rightarrow \Psi_p(a : f) \leq 2\left(\left\lceil \frac{K}{\delta} \right\rceil + 1\right)N.$$  

By Xia[5], Lemma I.2.2, there exists $b_0$ such that $\Psi_p(\cdot : f)$ is $\ell_p$-continuous at $b_0$. So that for every $\varepsilon > 0$, there exists $\lambda > 0$ such that

$$\|b - b_0\|_p \leq \lambda \Rightarrow |\Psi_p(b : f)^p - \Psi_p(b_0 : f)^p| \leq \varepsilon.$$  

Now we shall show $\Psi_p(\cdot : f)$ is $\ell_p$-continuous at 0. For every $b$ with $\|b\| \leq \lambda$, and for every natural numbers $n$ and $N$, we set

$$b(m, N) := (b_0^1, \cdots, b_0^N, b_{N+1}^0 + b_1, \cdots, b_{N+m}^0 + b_m, b_{N+m+1}^0, \cdots),$$
where \( b_0 = \{b_i^0\} \). Then we have

\[
\|b(m, N) - b_0\|_p = \left(\sum_{i=1}^{m} b_i^p\right)^{\frac{1}{p}} \leq \lambda,
\]

which implies

\[
|\Psi_p(b(m, N) : f)^p - \Psi_p(b : f)^p| = \sum_{i=1}^{m} \int_{-\infty}^{+\infty} |f(x - b^0_{N+i} - b_i) - f(x)|^p dx \leq \epsilon.
\]

Letting \( N \to +\infty \), we have

\[
\sum_{i=1}^{m} \int_{-\infty}^{+\infty} |f(x - b_i) - f(x)|^p dx \leq \epsilon,
\]

for every \( m \), and

\[
\Psi_p(b : f)^p = \sum_{i=1}^{+\infty} \int_{-\infty}^{+\infty} |f(x - b_i) - f(x)|^p dx \leq \epsilon,
\]

which shows \( \Psi_p(\cdot : f) \) is \( \ell_p \)-continuous at 0.

We can now easily deduce the continuity of \( \Psi_p(\cdot : f) \) at any point \( c_0 \) as follows. If \( \|c - c_0\|_p \leq \lambda \), then we have

\[
|\Psi_p(c : f) - \Psi_p(c_0 : f)| \leq \Psi_p(c - c_0 : f) \leq \epsilon^{\frac{1}{p}}.
\]

**Theorem 5** If \( f(x) \) is unimodular, then the metric \( d_p \) is the vector topology on \( \Lambda_p(f) \).

**Proof.** By Theorem 1, the scalar multiplication is continuous.

We consider the largest linear subspace \( \Sigma_p(f) \) of \( \Lambda_p(f) \) after Yamasaki[6] as follows. Define

\[
\Sigma_p(f) := \{a \in \Lambda_p(f) : ta \in \Lambda_p(f) \text{ for every } t \in \mathbb{R}\}.
\]

**Lemma 6** If \( a(\neq 0) \in \Sigma_p(f) \), then the real function \( \varphi(t : a) = \Psi_p(ta : f)^p \) is continuous on the real line \( \mathbb{R} \). Moreover, the metric

\[
\rho(s, t) = \Psi_p((t - s)a : f)
\]

gives the equivalent metric with the usual metric \( |s - t| \).
Proof. The continuity of $\varphi(t : a)$ is proved by the similar way to Theorem 5. Since $a \neq 0$, there exists $a_k \neq 0$. If
\[
\int_{-\infty}^{+\infty} |f(x - t_n a_k) - f(x)|^p dx \to 0 \text{ as } n \to +\infty,
\]
then it follows that $t_n \to 0$, see the proof of Theorem 2. This proves the second assertion.

Let $V_\varepsilon = \{a \in \Sigma_p(f) : \Psi_p(a : f) \leq \varepsilon\}$. Then for every $x \in \Sigma_p(f)$, we can find $\delta > 0$ such that
\[
tx \in V_\varepsilon \text{ for every } -\delta < t < \delta.
\]
Consequently we can linearize $d_p$ as follows, see Yamasaki[6], p.185, Xia[5], Lemma I.1.2. The linearization $\sigma_p(a, b)$ of $d_p(a, b)$ is defined by
\[
\sigma_p(a, b) := \sup_{|t| \leq 1} d_p(ta, tb)
\]
for $a, b \in \Sigma_p(f)$.

Theorem 7 $(\Sigma_p(f), \sigma_p(a, b))$ is a topological vector space.

5 Completeness of $\Lambda_p(f)$

Theorem 8 ([1]) Let $f(\neq 0)$ be an $L_p$-function. Then $\Lambda_p(f)$ is complete with respect to $d_p$ for $1 \leq p < +\infty$.

Theorem 9 $(\Sigma_p(f), \sigma_p(a, b))$ is complete.

6 Examples

Example 10 Define $f(x) := \max\{1 - |x|, 0\}$. Then we have
(1) for $1 \leq p < 2$, $\Lambda_p(f) = \ell_p$,
(2) $\Lambda_2(f) = \{a = (a_n) \in \ell^\infty \mid \sum_n a_n^2 (1 + |\log |a_n||) < +\infty\}$, and
(3) for $p > 2$, $\Lambda_p(f) = \ell_2$. 
References


