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Construction of solutions $f'(x)=af(\lambda x)$, $\lambda>1$.

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1. Main result

We denote by $C^\infty(\mathbb{R})$ the set of all infinitely differentiable functions on $\mathbb{R}$. Let

$$C_0^\infty(\mathbb{R})=\left\{f\in C^\infty(\mathbb{R}):\frac{d^\ell f}{dx^\ell}(x)\to 0 \text{ as } |x|\to\infty,\ \ell=0,1,2,\cdots\right\},$$
$$C_{\mathrm{comp}}^\infty(\mathbb{R})=\{f\in C^\infty(\mathbb{R}): f \text{ has a compact support}\}.$$

We denote by $\chi_{[r,s)}$ the characteristic function of the interval $[r,s)$. Let X be the set of all step functions

$$p=\sum_{j=1}^m c_j\chi_{[r_{j-1},r_j)}, \tag{1.1}$$

such that $m=1,2,\cdots$ and

$$c_j\in\mathbb{R}\quad (j=1,2,\cdots,m),\quad c_1=1,$$
$$r_0=0,\quad 0<r_1<r_2<\cdots<r_m<\infty,$$
$$\sum_{j=1}^m c_j(r_j-r_{j-1})=1.$$

We note that $\int_{\mathbb{R}}p(x)\,dx=1$ for $p\in X$.

For $\lambda>1$ and $p\in X$, we define an operator $T=T_{\lambda,p}:L^1(\mathbb{R})\to L^1(\mathbb{R})$ (by [17] (p. 149, Remark 2) and [2], we can say that $T_{\lambda,p}$: $B^s_{1,\infty}(\mathbb{R})\to B^{s+1}_{1,\infty}(\mathbb{R})$ for $s\in\mathbb{R}$, since $\lambda p(\lambda\cdot)\in B^1_{1,\infty}(\mathbb{R})$) as follows:

$$Tf(x)=T_{\lambda,p}f(x)=\lambda(p*f)(\lambda x). \tag{1.2}$$

We note that $Tf\in L^1(\mathbb{R})\cap C(\mathbb{R})$, since $f\in L^1(\mathbb{R})$ and $p\in L^\infty_{\mathrm{comp}}(\mathbb{R})$.

If $f\in B^s_{1,\infty}(\mathbb{R})$ we have $\hat f(1+|\cdot|^2)^{s/2}\in L^\infty(\mathbb{R})$ (see [15]). Therefore we have $T^kf\in L^2(\mathbb{R})\cap C(\mathbb{R})$ for $k>-s-1$ if $f\in B^s_{1,\infty}(\mathbb{R})$.

Theorem 1.1. *For $\lambda > 1$ and $p \in X$, let T be the operator defined by (1.2). Then there exists a function $u = u_{\lambda,p} \in C^{\infty}_{\mathrm{comp}}(\mathbb{R})$ such that, for all $f \in L^1(\mathbb{R})$ (or for all $f \in \cup_{s\in\mathbb{R}} B^s_{1,\infty}(\mathbb{R})$, which function set is strictly bigger than $L^1(\mathbb{R})$),*

$$\lim_{\ell\to\infty} T^{\ell} f(x) = c_f\, u(x) \quad \textit{uniformly with respect to } x \in \mathbb{R},$$
$$\textit{where} \quad c_f = \hat{f}(0).$$

Moreover,

(1) $Tu = u$;
(2) $\int_{\mathbb{R}} u(x)\,dx = 1$;
(3) $\frac{d^k}{dx^k} u(0) = 0$ *for all* $k = 0, 1, 2, \cdots$;
(4) *if* $p \geq 0$, *then* $u \geq 0$; *and,*
(5) *if* $\operatorname{supp} p \subset [0, r]$, *then* $\operatorname{supp} u \subset [0, r/(\lambda - 1)]$.

In the above statement $\operatorname{supp} f$ denotes the support of f.

Remark 1.1. If $p_1 \neq p_2$, then $u_{\lambda,p_1} \neq u_{\lambda,p_2}$.

2. APPLICATION

To construct solutions for the following equation,

$$\text{(2.1)} \qquad \begin{cases} f'(x) = \lambda^2 f(\lambda x), & x \in \mathbb{R}, \\ f(0) = 0, \end{cases}$$

we use $u = u_{\lambda,p}$ in Theorem 1.1 with $\lambda > 1$ and $p = \sum_{j=1}^{m} c_j \chi_{[r_{j-1}, r_j)} \in X$, and we define a sequence of functions $\{v_\ell\}$ inductively as follows:

$$\begin{cases} v_0(x) = u(x) + \displaystyle\sum_{j=1}^{m} \tilde{c}_j u(x - r_j), \\ v_{\ell+1}(x) = v_\ell(x) + \displaystyle\sum_{j=1}^{m} \tilde{c}_j v_\ell(x - \lambda^{\ell+1} r_j), \quad \ell = 0, 1, 2, \cdots, \end{cases}$$

where $\tilde{c}_j = -(c_j - c_{j+1})$, $j = 1, 2, \cdots, m-1$, $\tilde{c}_m = -c_m$.

Then $v_\ell \in C^{\infty}_{\mathrm{comp}}(\mathbb{R})$, $\ell = 0, 1, 2, \cdots$, and

$$v_\ell(x) = v_{\ell+1}(x), \quad x \in (-\infty, \lambda^{\ell+1} r_1], \quad \ell = 0, 1, 2, \cdots.$$

Therefore we get a function $f = f_{\lambda,p} \in C^{\infty}(\mathbb{R})$ such that

$$f(x) = v_\ell(x), \quad x \in (-\infty, \lambda^{\ell+1} r_1], \quad \ell = 0, 1, 2, \cdots.$$

Since $f(x) = u(x)$ near $x = 0$, $\frac{d^k}{dx^k} f(0) = 0$ for all $k = 0, 1, 2, \cdots$.

Theorem 2.1. *Let $\lambda > 1$. For every $p \in X$, the function $f = f_{\lambda,p}$ given by the above method is a solution for* (2.1).

Remark 2.1. By the property (5) in Theorem 1.1 and the definition of v_ℓ,

$$\begin{aligned} \operatorname{supp} u &\subset [0, r_m/(\lambda-1)], \\ \operatorname{supp}(v_0 - u) &\subset [r_1, r_m\lambda/(\lambda-1)], \\ \operatorname{supp} v_\ell &\subset [0, r_m\lambda^{\ell+1}/(\lambda-1)], \\ \operatorname{supp}(v_{\ell+1} - v_\ell) &\subset [r_1\lambda^{\ell+1}, r_m\lambda^{\ell+2}/(\lambda-1)]. \end{aligned}$$

Therefore, if $r_m/(\lambda-1) < r_1$, then $f(x) = 0$ on $[r_m\lambda^\ell/(\lambda-1), r_1\lambda^\ell]$, $\ell = 0, 1, 2, \ldots$, and f is bounded (See the graph (S12)). Let

$$p_i = \sum_{j=1}^{m_i} c_{i,j}\chi_{[r_{i,j-1}, r_{i,j})} \in X, \quad i = 1, 2.$$

If $p_1 \neq p_2$ and there exists $R > 0$ such taht $r_{i,m_i}/(\lambda-1) < R < r_{i,1}$, $i = 1, 2$, then $f_{\lambda,p_1} \neq f_{\lambda,p_2}$ by Remark 1.1, while $\int_0^R f_{\lambda,p_i}(x)\,dx = \int_0^R u_{\lambda,p_i}(x)\,dx = 1$, $i = 1, 2$.

Remark 2.2. In the definition (1.1), let $m = \infty$ with the condition $\sum_{j=1}^\infty |c_{j+1} - c_j| < \infty$. Then the derivative of p in the sense of distribution is a finite Radon measure and the Fourier transform of the derivative is an almost periodic function. With some conditions we can also construct solutions for (2.1) from such functions. For the Fourier preimage of the space of all finite Radon measures, see, for example, [6, 7].

In the rest of this section, we give graphs of solutions for $f'(x) = 4f(2x)$, $f'(x) = (3/2)^2 f(3x/2)$ and $f'(x) = 9f(3x)$ with $f(0) = 0$.

Let p be as in Table 1. Then, calculating $T^\ell_{\lambda,p}\chi_{[0,1)}$ numerically by computer, we get the graphs of solutions (S1)–(S7) in Figure 1, (S8)–(S11) in Figure 2 and (S12)–(S13) in Figure 3.

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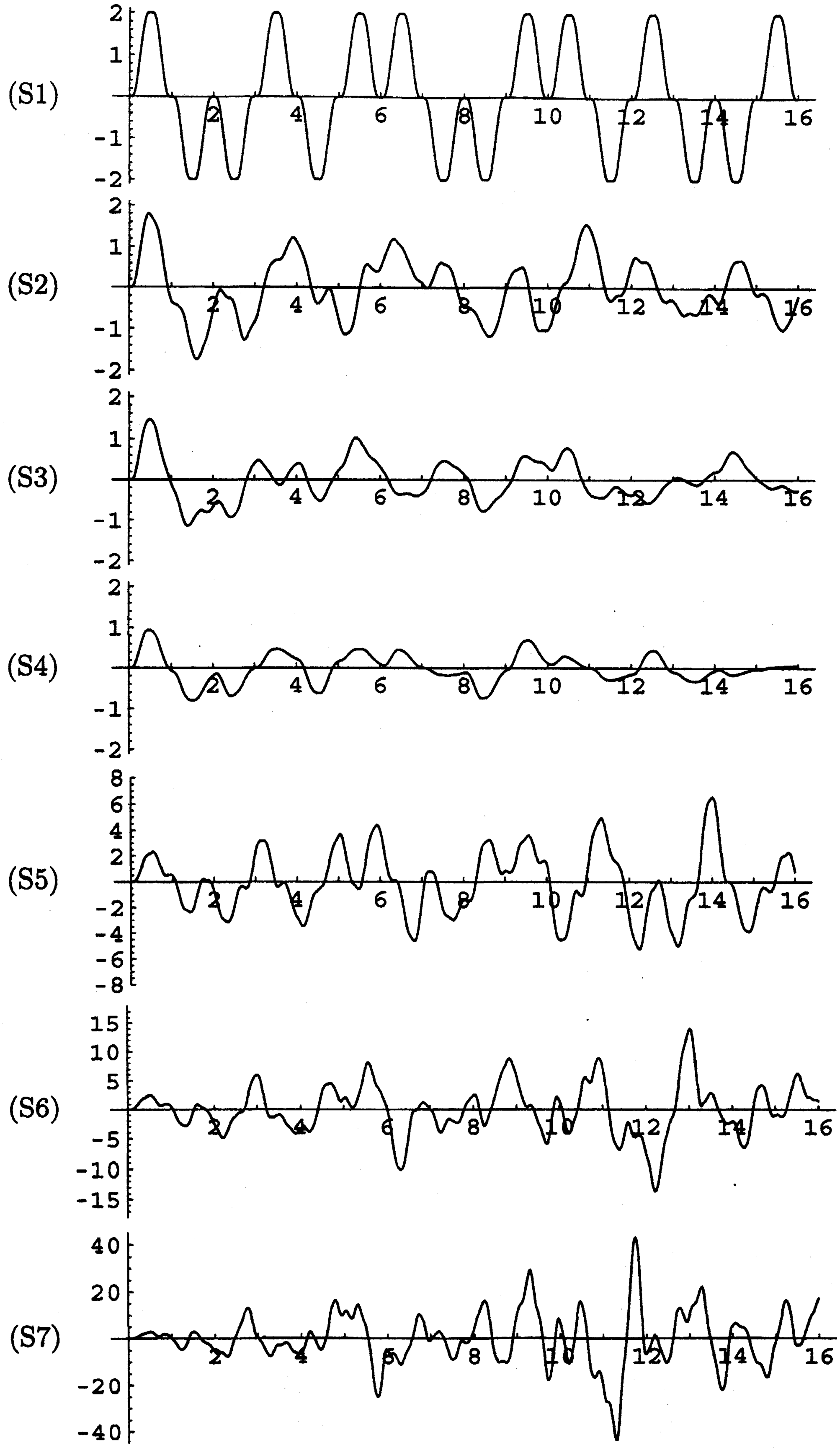

FIGURE 1. Solutions of $f'(x) = 4f(2x)$.

λ	$p \in X$	Solution
$\lambda = 2$	$\chi_{[0,1)}$	(S1)
$\lambda = 2$	$\chi_{[0,1/2)} + (3/4)\chi_{[1/2,7/6)}$	(S2)
$\lambda = 2$	$\chi_{[0,1/2)} + (1/2)\chi_{[1/2,3/2)}$	(S3)
$\lambda = 2$	$\chi_{[0,1/2)} + (1/4)\chi_{[1/2,5/2)}$	(S4)
$\lambda = 2$	$\chi_{[0,1/2)} + (5/4)\chi_{[1/2,9/10)}$	(S5)
$\lambda = 2$	$\chi_{[0,1/2)} + (3/2)\chi_{[1/2,5/6)}$	(S6)
$\lambda = 2$	$\chi_{[0,1/2)} + 2\chi_{[1/2,3/4)}$	(S7)
$\lambda = 3/2$	$\chi_{[0,1)}$	(S8)
$\lambda = 3/2$	$\chi_{[0,1/2)} + (5/4)\chi_{[1/2,9/10)}$	(S9)
$\lambda = 3/2$	$\chi_{[0,1/2)} + (3/2)\chi_{[1/2,5/6)}$	(S10)
$\lambda = 3/2$	$\chi_{[0,1/2)} + 2\chi_{[1/2,3/4)}$	(S11)
$\lambda = 3$	$\chi_{[0,1)}$	(S12)
$\lambda = 3$	$\chi_{[0,1/2)} + (1/4)\chi_{[1/2,5/2)}$	(S13)

TABLE 1. λ, p and the solution $f_{p,\lambda}$.

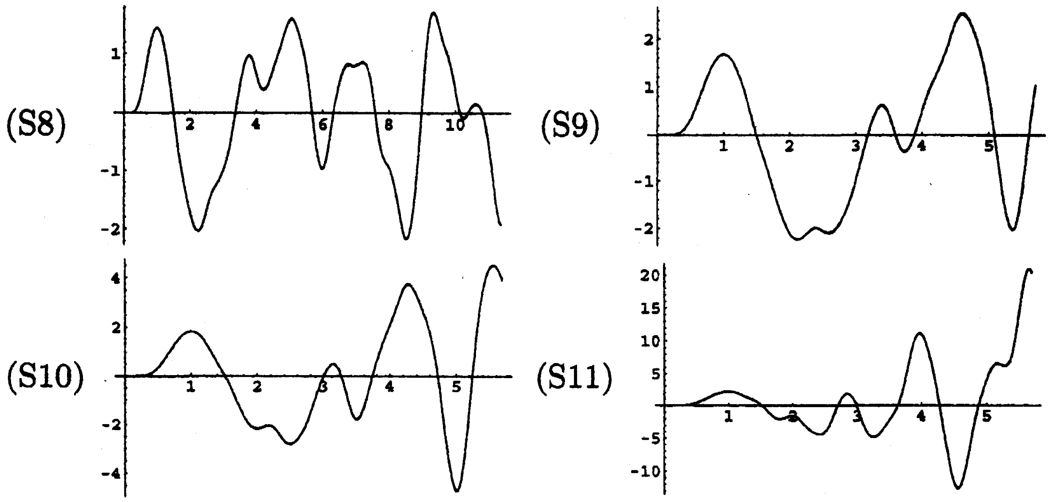

FIGURE 2. Solutions of $f'(x) = (3/2)^2 f(3x/2)$.

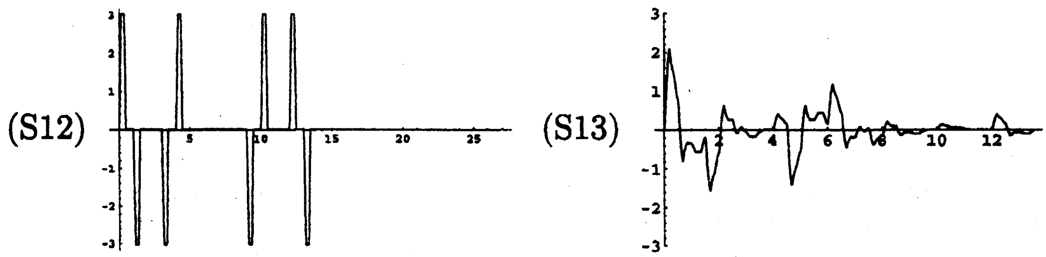

FIGURE 3. Solutions of $f'(x) = 9f(3x)$.

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