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Chern-Simons variation and Hida-Mazur theory

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We would like to discuss the variation of $SL_2(\mathbb{C})$ Chern-Simons invariants over the deformation space of hyperbolic structures on a knot complement in analogy with Hida-Mazur theory on the deformation of Galois representations and modular p-adic L-functions. The motivation and idea are coming from the analogy between knot theory and number theory, and so let us start to recall the basic analogies between knots and primes.

1. Analogies

knot	$\leftrightarrow$	prime
$K : S^1 = K(\mathbb{Z}, 1) \hookrightarrow S^3 = \mathbb{R}^3 \cup \{\infty\}$		$\mathrm{Spec}(\mathbb{F}_p) = K(\hat{\mathbb{Z}}, 1) \hookrightarrow \mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}$
tube neighborhood V_K	$\leftrightarrow$	p-adic integers $\mathrm{Spec}(\mathbb{Z}_p)$
∂V_K		p-adic numbers $\mathrm{Spec}(\mathbb{Q}_p)$
$D_K = \pi_1(\partial V_K)$		$D_p = \pi_1^{\text{ét}}(\mathrm{Spec}(\mathbb{Q}_p))$
$1 \to \langle m_k \rangle \to D_K \to \langle l_K \rangle \to 1$	$\leftrightarrow$	$1 \to I_p \to D_p \to \langle \sigma_p \rangle \to 1$
l_K : longitude of K		σ_p : Frobenius over p
m_K : meridian of K		I_p : inertia gr. over p, $I_p^t = \langle \tau_p \rangle$
$[m_K, l_K] = 1$		τ_p : monodromy over p, $\tau_p^{p-1}[\tau_p, \sigma_p] = 1$

Here, I_p^t denotes the maximal tame quotient of I_p. In general, we call an element of a quotient of I_p a monodromy over p.

$X_K = S^3 \setminus K$	$\leftrightarrow$	$X_p = \mathrm{Spec}(\mathbb{Z}[1/p])$
knot group $G_K = \pi_1(X_K)$		prime group $G_p = \pi_1^{\text{ét}}(X_p)$
infinite cyclic cover	$\leftrightarrow$	$\mathbb{Z}_p$-cover
$X_K^\infty \to X_K$		$X_p^\infty = \mathrm{Spec}(\mathbb{Z}[1/p, \sqrt[p^\infty]{1}]) \to X_p$
$\mathrm{Gal}(X_K^\infty/X_K) = \langle \alpha \rangle$		$\mathrm{Gal}(X_p^\infty/X_p) = \langle \gamma \rangle$
Alexander module/polynomial	$\leftrightarrow$	Iwasawa module/polynomial
$\Delta_K(t) = \det(t - \alpha \mid H_1(X_K^\infty))$		$I_p(T) = \det(T - (\gamma - 1) \mid H_1(X_p^\infty))$
analytic torsion = Reidemeister torsion	$\leftrightarrow$	Iwasawa main conjecture

For more analogies and details, we refer to [Mo]. In fact, there are close analogies between Alexander-Fox theory and Iwasawa theory. From the variational point of view, one sees that:

considering the infinite cyclic cover $X_K^\infty \to X_K$ (resp. $\mathbb{Z}_p$-cover $X_p^\infty \to X_p$)
is equivalent to
considering the deformation of GL_1-representations of the knot group G_K (resp. prime group G_p).

In number theory, there is a non-abelian generalization of the classical Iwasawa theory along this variational viewpoint, due to mainly H. Hida and B. Mazur, namely the deformation theory of GL_n-representations of G_p and the theory of associated arithmetic invariants. Our motivation was to find an analogue of Hida-Mazur theory in the context of knot theory, which would be a natural non-abelian generalization of Alexander-Fox theory:

Alexander-Fox theory	$\leftrightarrow$	Iwasawa theory
$\Vert$		$\Vert$
deformation theory of 1-dim. repr's of G_K and associated topological invariants		deformation theory of 1-dim. repr's of G_p and associated arithmetic invariants
$\downarrow$		$\downarrow$
?	$\leftrightarrow$	Hida-Mazur theory
		$\Vert$
		deformation theory of n-dim. repr's of G_p and associated arithmetic invariants

2. Deformation of hyperbolic structures on a knot complement and of modular Galois representations

We start to recall some general notions in group representations.

<u>knot</u>: For a knot $K \subset S^3$ with $G_K := \pi_1(S^3 \setminus K)$ and $n \geq 1$, we set

$$\begin{aligned}\mathfrak{X}_K^n &= \mathrm{Hom}(G_K, GL_n(\mathbb{C}))//GL_n(\mathbb{C}) \\ &:= \mathrm{Hom}_{\mathbb{C}\text{-alg}}((R_K^n)^{G_K}, \mathbb{C}),\end{aligned}$$

where R_K^n denotes the tautological n-dimensional representation ring on which G_K acts by the conjugation via the tautological representation $G_K \to GL_n(R_K^n)$, and $(R_K^n)^{G_K}$ stands for the invariant subring. The set $\mathfrak{X}_K^n$ is a complex affine variety, called the character variety of n-dimensional representations of G_K.

prime: For a prime $\mathrm{Spec}(\mathbb{F}_p)$, the prime group G_p is profinite and hence the naive analogue of the character variety does not provide a good moduli. Thus, following Mazur ([Ma1]), we consider "infinitesimal deformations" of a given residual representation

$$\bar{\rho} : G_p \longrightarrow GL_n(\mathbb{F}_p).$$

Namely, the pair (R, ρ) is called a deformation of $\bar{\rho}$ if

$$\begin{cases} \bullet\ R \text{ is a complete noetherian local ring with residue field } R/m_R = \mathbb{F}_p \\ \bullet\ \rho : G_p \to GL_n(R) \text{ is a continuous representation with } \rho \bmod m_R = \bar{\rho}. \end{cases}$$

In the rest of this note, we assume for simplicity that $p > 2$ and $\bar{\rho}$ is absolutely irreducible. A fundamental theorem by Mazur is:

Theorem 1 ([Ma]). *There is a universal deformation (R_p^n, ρ_p^n) of $\bar{\rho}$ so that any deformation of (R, ρ) is obtained up to a certain conjugacy via a $\mathbb{Z}_p$-algebra homomorphism $R_p^n \to R$.*

We then define the universal deformation space $\mathfrak{X}_p^n(\bar{\rho})$ of $\bar{\rho}$ by

$$\mathfrak{X}_p^n(\bar{\rho}) := \mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(R_p^n, \mathbb{C}_p)$$

where $\mathbb{C}_p$ stands for the p-adic completion of an algebraic closure of $\mathbb{Q}_p$, and $\mathfrak{X}_p^n(\bar{\rho})$ is regarded as a rigid analytic space. We write $\rho_\varphi := \varphi \circ \rho_p^n : G_p \to GL_n(\mathbb{C}_p)$ for a $\varphi \in \mathfrak{X}_p^n(\bar{\rho})$.

We will discuss analogies between $\mathfrak{X}_K^n$ and $\mathfrak{X}_p^n(\bar{\rho})$ and some invariants defined on them for the cases of $n = 1$ and 2.

$\underline{n = 1}$: The GL_1-theory is simply a restatement of the analogy between Alexander-Fox theory and Iwasawa theory:

$$\begin{array}{ccc} R_K^1 = \Lambda_{\mathbb{C}} = \mathbb{C}[t^{\pm 1}] & & R_p^1 = \hat{\Lambda} = \mathbb{Z}_p[[T]] \\ \mathfrak{X}_K^1 \simeq \mathbb{C}^\times & \leftrightarrow & \mathfrak{X}_p^1(\bar{\rho}) \simeq D_p^1 = \{z \in \mathbb{C}_p \mid |z|_p < 1\} \\ \chi \mapsto \chi(\alpha) & & \varphi \mapsto \rho_\varphi(\gamma) - 1 \\ (\mathrm{Gal}(X_K^\infty / X_K) = \langle \alpha \rangle = \mathbb{Z}) & & (\mathrm{Gal}(\mathbb{Q}^\infty/\mathbb{Q}) = \langle \gamma \rangle = \mathbb{Z}_p) \end{array}$$

$$\begin{array}{ccc} \text{invariants on } \mathfrak{X}_K^1 : & & \text{invariants on } \mathfrak{X}_p^1(\bar{\rho}) : \\ \text{twisted Alexander poly. (analytic torsion)} & \leftrightarrow & \text{twisted Iwasawa poly. (p-adic L-function)} \\ \text{for a repr. } \rho : G_K \to GL_n(\mathbb{C}) & & \text{for a repr. } \rho : G_p \to GL_n(\mathbb{Z}_p) \\ \Delta_{K,\rho}(t)\ (\tau_{K,\rho}) & & I_{p,\rho}(T)\ (L_p(\rho, s)) \\ \text{describes the variation of} & & \text{describes the variation of} \\ H^1(G_K, \rho \otimes \chi),\ \chi \in \mathfrak{X}_K^1 & & \mathrm{Sel}(G_p, \rho \otimes \rho_\varphi),\ \varphi \in \mathfrak{X}_p^1(\bar{\rho}) \end{array}$$

Here, $\mathrm{Sel}(G_p, M)$ denotes the Selmer group for a G_p-module M (a subgroup of $H^1(G_p, M)$ with a local condition).

$\underline{n=2}$: The GL_2-theory is concerned with hyperbolic geometry and Chern-Simons gauge theory in the knot side and Hida-Mazur theory and p-adic gauge theory in the prime side.

knot: We assume that K is a hyperbolic knot. Since G_K has a trivial center and any representation $G_K \to PGL_2(\mathbb{C}) = PSL_2(\mathbb{C})$ can be lifted to $G_K \to SL_2(\mathbb{C})$, we may consider only $SL_2(\mathbb{C})$-representations without losing generality. So, we set

$$\mathfrak{X}_K^2 := \mathrm{Hom}_{\mathrm{gr}}(G_K, SL_2(\mathbb{C}))//SL_2(\mathbb{C}).$$

Note that the restriction of $[\rho] \in \mathfrak{X}_K^2$ to D_K is conjugate to an upper triangular representation:

$$\rho|_{D_K} \simeq \begin{pmatrix} \chi_\rho & * \\ 0 & \chi_\rho^{-1} \end{pmatrix}.$$

Let ρ^o be a lift of the holonomy representation associated to the hyperbolic structure on $S^3 \setminus K$ and let $\mathfrak{X}_K^{2,o}$ be the irreducible component of $\mathfrak{X}_K^2$ containing $[\rho^o]$. The following theorem was shown by W. Thurston:

Theorem 2 ([T]). *The map $\Phi_K : \mathfrak{X}_K^{2,o} \to \mathbb{C}$ defined by $\Phi_K([\rho]) := \mathrm{tr}(\rho(m_K))$ is bianalytic in a neighborhood of $[\rho^o]$. In particular, $\mathfrak{X}_K^{2,o}$ is a complex algebraic curve.*

Let m and l be functions on $\mathfrak{X}_K^{2,o}$ defined by $m(\rho) = \chi_\rho(m_K)$ and $\chi_\rho(l_K)$ respectively.

Theorem 3 ([NZ]). *Let $x := \log m(\rho)$ ($\log m(\rho^o) = 0$) and suppose $\partial V_K = \mathbb{C}^\times/q^{\mathbb{Z}}$. Then we have*

$$\frac{dl}{dx}|_{x=0} = \frac{1}{2}\frac{\log q}{2\pi\sqrt{-1}}.$$

prime: We assume that $\bar{\rho}$ is a mod p representation associated to an ordinary modular elliptic curve E over $\mathbb{Q}$ which corresponds to an ordinary Hecke eigenform f of weight 2: $\bar{\rho} = \rho_E \bmod p = \rho_f \bmod p$, $\rho_E = \rho_f : G_p \to GL_2(\mathbb{Z}_p)$. Here a representation $\rho : G_p \to GL_2(A)$ is called ordinary if the restriction of ρ to D_p is conjugate to an upper triangular representation:

$$\rho|_{D_p} \simeq \begin{pmatrix} \chi_{\rho,1} & * \\ 0 & \chi_{\rho,2} \end{pmatrix}, \quad \chi_{\rho,1}|_{I_p} = \mathbf{1}.$$

Compared with the knot case, it is natural to impose the ordinary condition to deformations of $\bar{\rho}$.

We have the following fundamental:

Theorem 4 (1) ([H1,2]). *There is a universal ordinary modular deformation* $(R_p^{2,o.m}, \rho_p^{2,o.m})$ *($R_p^{2,o.m}$ is called the p-adic Hecke-Hida ring) of* $\bar{\rho}$ *such that any ordinary modular deformation* (R, ρ) *of* $\bar{\rho}$ *is obtained via a* $\mathbb{Z}_p$*-algebra homomorphism* $R_p^{2,o.m} \to R$.
(2) ([Ma]). *There is a universal ordinary deformation* $(R_p^{2,o}, \rho_p^{2,o})$ *of* $\bar{\rho}$ *such that any ordinary deformation* (R, ρ) *of* $\bar{\rho}$ *is obtained via a* $\mathbb{Z}_p$*-algebra homomorphism* $R_p^{2,o} \to R$.

By the universality of $(R_p^{2,o}, \rho_p^{2,o})$, we have a $\mathbb{Z}_p$-algebra homomorphism

$$R_p^{2,o} \to R_p^{2,o.m}$$

which we assume in the following to be an isomorphism. In fact, this assumption is satisfied under a mild condition owing to the works of A. Wiles etc.

We then define the universal ordinary deformation space of $\bar{\rho}$ by

$$\mathfrak{X}_p^{2,o}(\bar{\rho}) := \mathrm{Hom}_{\mathbb{Z}_p-\mathrm{alg}}(R_p^{2,o}, \mathbb{C}_p)$$

which may be regarded as an (infinitesimal) analog of $\mathfrak{X}_K^{2,o}$. As an analogue of Theorem 2, we have:

Theorem 5. *Take an element* $\gamma \in I_p$ *which is mapped to a generator of* $\mathrm{Gal}(\mathbb{Q}^\infty/\mathbb{Q})$ *where* $\mathbb{Q}^\infty$ *is the unique* $\mathbb{Z}_p$*-extension of* $\mathbb{Q}$. *The map* $\Phi_p : \mathfrak{X}_p^{2,o} \to \mathbb{C}_p$ *defined by* $\Phi_p(\varphi) := \mathrm{tr}(\rho_\varphi(\gamma))$ *is bianalytic in a neighborhood of* φ_f *where* $\varphi_f \circ \rho_p^{2,o} = \rho_f$.

Remark. The analogy between the structures of $\mathfrak{X}_K^{2,o}$ and $\mathfrak{X}_p^{2,o}(\rho)$ was first pointed out by Kazuhiro Fujiwara.

The following theorem by Greenberg and Stevens may be seen as an analog of Neumann-Zagier's theorem 3.

Theorem 6 ([GS]). *Suppose that* E *is split multiplicative at* p. *Write*

$$\rho_p^{2,o}|_{D_p} \simeq \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}, \quad \chi_1|_{I_p} = 1$$

and set $a_p := \chi_1(\sigma_p) : \mathfrak{X}_p^{2,o} \to \mathbb{C}_p$, *and let* $E(\mathbb{C}_p) = \mathbb{C}_p^\times / q^{\mathbb{Z}}$. *Then we have*

$$\frac{da_p}{d\rho}|_{\rho=2} = -\frac{1}{2}\frac{\log_p(q)}{\mathrm{ord}_p(q)} (= \text{Mazur-Tate-Teitelbaum's } \mathcal{L}\text{-invariant}).$$

Next, we discuss some analogies between invariants defined on $\mathfrak{X}_K^{2,o}$ and $\mathfrak{X}_p^{2,o}(\rho)$.

prime: A typical invariant on $\mathfrak{X}_p^{2,o}(\bar{\rho})$ is a p-adic modular L-function $L_p(\rho, s)$, $\rho \in \mathfrak{X}_p^{2,o}, s \in \mathbb{Z}_p$ ([GS]). Geometrically, $L_p(\rho, s)$ is given as a section of a rigid analytic line bundle $\mathfrak{L}_p$ of modular symbols:

$$\begin{array}{cl} \mathfrak{L}_p & \\ \downarrow & L_p(\rho, s) \text{ is a section } (s \text{ fixed}) \\ \mathfrak{X}_p^{2,o} & \end{array}$$

We note that the value $L_p(\rho, 0) = L_p(f_\rho, 0)$ at $s = 0$ (f_ρ being a modular form corresponding to ρ) is given by $r_p(\{u, v\})(\omega) \cdot c$, where $r_p : K_2(C_\rho) \to H^1_{DR}(C_\rho/\mathbb{Q}_p)$ (C_ρ being a modular curve) is the p-adic regulator.

knot: Take a small affine open $\mathfrak{X} \subset \mathfrak{X}_K^{2,o}$ containing ρ^o if necassary, and let

$$L_K(\rho) := -2\pi^2 \mathrm{CS}(\rho) + \sqrt{-1}\mathrm{Vol}(\rho)$$

be the $SL_2(\mathbb{C})$ Chern-Simons invariant. Our theorem is

Theorem 7. *There is a holomorphic line bundle $\mathfrak{L}_K$ with holomorphic connection on $\mathfrak{X}$ such that $L_K(\rho)$ is given by a flat section:*

$$\begin{array}{cl} \mathfrak{L}_K & \\ \downarrow & L_K(\rho) \text{ is a flat section} \\ \mathfrak{X} & \end{array}$$

For the construction of $\mathfrak{L}_K$, we apply S. Bloch's geometric construction of a tame symbol ([Bl]). Let H be the 3×3 Heisenberg group:

$$H(R) := \{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} | a, b, c \in R\} \ (R: \text{ a commutative ring})$$

The complex manifold $P := H(\mathbb{Z}) \setminus H(\mathbb{C})$ is a principal $\mathbb{C}^\times$-bundle over $\mathbb{C}^\times \times \mathbb{C}^\times$ by the map

$$P \to \mathbb{C}^\times \times \mathbb{C}^\times; \ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto (\exp(2\pi\sqrt{-1}a), \exp(2\pi\sqrt{-1}b))$$

and 1-form $\theta = dc - adb$ gives a connection on P. Let $T(l, m^2) : \mathfrak{X} \to \mathbb{C}^\times \times \mathbb{C}^\times$ be a holomorphic map defined by $T(l, m^2)(\rho) := (l(\rho), m^2(\rho))$ and define

$$\mathfrak{L}_K := T(l, m^2)^*(P, \theta).$$

Then the flat section is given by

$$S(\rho^o) + \log l(\rho^o) \int_{\rho^o}^{\rho} d \log m^2 + \int_{\rho^o}^{\rho} d \log l d \log m^2 = L_K(\rho).$$

To see why $L_K(\rho)$ is seen as an analog of $L_p(\rho, 0)$, we give a cohomological interpretation of the above construction. Let $r_\infty : K_2(\mathfrak{X}) \to H^1(\mathfrak{X}, \mathbb{R})$ be the Beilinson regulator. We consider the natural map $\iota : H^2_D(\mathfrak{X}, \mathbb{Z}(2)) \to H^2_D(\mathfrak{X}, \mathbb{R}(2)) \hookrightarrow H^1(\mathfrak{X}, \mathbb{R})$ where H^*_D stands for the Deligne cohomology ([Br]). It is known that $H^2(\mathfrak{X}, \mathbb{Z}(2))$ is interpreted as the group of isomorphism classes of holomorphic line bundlee on $\mathfrak{X}$ with holomorphic connection ([ibid]), and so $\mathfrak{L}_K$ is regarded as an element of $H^2(\mathfrak{X}, \mathbb{Z}(2))$. Then we can show that

$$\iota(\mathfrak{L}_K) = r_\infty(\{l, m^2\})$$

which is reminiscient of the connection between $L_p(\rho, 0)$ and the p-adic regulator.

Remark. Kirk and Klassen ([KK]) also constructed a line bundle E_K over $\mathfrak{X}_K^{2,o}$ so that $\mathfrak{L}_K(\rho)$ is regarded as a section. Though we have not seen the connection between E_K and $\mathfrak{L}_K$ yet, our construction using Deligne cohomology seems to be natural conceptually.

Now, compared with the prime side, we may expect that

there should be a 2-variable L-function $L_K(\rho, s)$, $\rho \in \mathfrak{X}, s \in \mathbb{C}$ such that $L_K(\rho)$ would be a dominant term (special value) of $L_K(\rho, s)$ at $s = 0$.

Here is a candidate for such a L-function. Let M_ρ be the hyperbolic deformation of $M = S^3 \setminus K$ with holonomy ρ. Then M_ρ is a spin manifold with $Spin(3) = SU(2)$-principal bundle $Spin(M_\rho) \to M_\rho$. Let D_ρ be the corresponding Dirac operator acting on $C^\infty(Spin(M_\rho) \otimes (\mathbb{C}^2)_\rho)$ and we define the spectral zeta function by

$$L_K(\rho, s) := \sum_\lambda \pm(\pm\lambda)^s, \ \pm = sign(\mathrm{Re}(\lambda)), \ \mathrm{Re}(s) >> 0$$

where λ's run over eigenvalues of D_ρ. Note that D_ρ may not be self-adjoint (though its symbol is self-adjoint) and so λ may be imaginary. For a closed hyperbolic 3-manifold, Jones-Westbury ([JW],[Y]) showed that $L_K(\rho, s)$ is continued as a meromorphic function to $\mathbb{C}$ and the equality

$$L_K(\rho) = 2\pi^2 L_K(\rho, 0).$$

It is desirable to extend this equality for a non-closed hyperbolic 3-manifold.

Finally, we note the following

Theorem 8 (cf. [MT]). *$L_K(\rho)$ gives a variation of mixed Hodge structure (V, W_*, F^*) on $\mathfrak{X}$ defined by:*

$V = \mathbb{Z}^3$,

$V = W_0 \supset W_{-1} = \mathbb{Z}v_2 \oplus \mathbb{Z}v_3 \supset W_{-2} = \mathbb{Z}v_3$ *where*

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} := \begin{pmatrix} 1 & \log l(\rho^o) & S(\rho^o) \\ 0 & 1 & \log m^2(\rho^o) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \int_{\rho^o}^{\rho} d\log l & \int_{\rho^o}^{\rho} d\log l d\log m^2 \\ 0 & 1 & \int_{\rho^o}^{\rho} d\log m^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ (2\pi\sqrt{-1})e_2 \\ (2\pi\sqrt{-1})^2 e_3 \end{pmatrix}.$$

Here $\{e_1, e_2, e_3\}$ *is a standard basis of* V, *and* $W_0/W_{-1} = \mathbb{Z}(0), W_{-1}/W_{-2} = \mathbb{Z}(1)$, $W_{-2} = \mathbb{Z}(2)$.

$V = F^{-2} \supset F^{-1} = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \supset F^0 = \mathbb{Z}e_1$ *with* $\nabla F^{i-1} \subset \Omega^1 \otimes F^i$ *so that*

$$\nabla v := dv - v \begin{pmatrix} 1 & d\log l & 0 \\ 0 & 1 & d\log m^2 \\ 0 & 0 & 1 \end{pmatrix}.$$

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