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Navier-Stokes 方程式に対する時間2次精度特性曲線有限要素スキームの安定性
Stability of a characteristic finite element scheme of second order in time increment for the Navier-Stokes equations

九州大学大学院数理学研究院
野津 裕史 (Hirofumi NOTSU)* 田端 正久 (Masahisa TABATA)†
Faculty of Mathematics, Kyushu University

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d (d=2,3)$ and T be a positive constant. We consider the problem governed by the nonstationary Navier-Stokes equations subject to the Dirichlet boundary conditions; find $(u,p):\Omega\times(0,T)\to\mathbb{R}^d\times\mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t}+(u\cdot\nabla)u-\nabla(2\nu D(u))+\nabla p=f & \text{in } \Omega\times(0,T), \\ \nabla\cdot u=0 & \text{in } \Omega\times(0,T), \\ u=g & \text{on } \Gamma\times(0,T), \\ u=u^0 & \text{in } \Omega,\ \text{at } t=0, \end{cases} \tag{1}$$

where u is the velocity, p is the pressure, f is an exterior force, g is a boundary velocity, u^0 is an initial velocity, $\nu(>0)$ is a viscosity, $\Gamma\equiv\partial\Omega$, and $D(u)$ is the strain-rate tensor defined by

$$D_{ij}(u)\equiv\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right).$$

A second-order-in-time characteristic finite element scheme for convection-diffusion problems had been developed in [2], where they had proved stability and convergence theorems. Using their idea, we have developed a second order characteristic finite element scheme [1] for (1). Our scheme is the combination of a second-order-in-time characteristic finite element scheme and the first-order-in-time characteristic finite element scheme (see Figure 1). The scheme has such advantages that it is of second order in Δt and that the matrices are symmetric. The combined scheme is stable even for high Reynolds number problems. In this paper we consider the stability. The contents of the paper are as follows.

*E-mail : notsu@math.kyushu-u.ac.jp
†E-mail : tabata@math.kyushu-u.ac.jp

$$\cdots \longrightarrow \begin{pmatrix} u_h^{n-1} \\ p_h^{n-1} \end{pmatrix} \xrightarrow[\Delta t_0 = O(\Delta t)]{\text{Second order}} \begin{pmatrix} u_h^{n-\alpha} \\ p_h^{n-\alpha} \end{pmatrix} \xrightarrow[\Delta t_1 = O(\Delta t^2)]{\text{First order}} \begin{pmatrix} u_h^{n} \\ p_h^{n} \end{pmatrix} \longrightarrow \cdots$$

Figure 1: Time evolution of the scheme

We introduce the scheme in Section 2 and present a sufficient condition for the scheme to be stable in Section 3. In the last section we check the condition numerically in an example and solve a cavity flow problem with high Reynolds numbers.

We use the function spaces $L^2(\Omega)$ and $W^{1,\infty}(\Omega)$. For any normed space X we use the notation $\|\cdot\|_X$ to represent the X-norm. We omit Ω in the notations of norms, and use $\|\cdot\|$ to show the L^2-norms $\|\cdot\|_{L^2}$, $\|\cdot\|_{(L^2)^d}$, $\|\cdot\|_{(L^2)^{d\times d}}$.

2 A second order characteristic finite element scheme

For time increments Δt, $\Delta t'$ and velocities u, $v : \Omega \to \mathbb{R}^d$, we define $X_1(u, \Delta t)$ and $X_2(u, v, \Delta t', \Delta t)$ by

$$X_1(u,\Delta t)(x) \equiv x - u(x)\Delta t,$$

$$X_2(u,v,\Delta t',\Delta t)(x) \equiv x - \left\{(1+\frac{\Delta t'}{2\Delta t})u(x-u(x)\Delta t') - \frac{\Delta t'}{2\Delta t}v(x-u(x)(\Delta t'+\Delta t))\right\}\Delta t',$$

respectively. The X_i $(i = 1, 2)$ are used for the i-th order approximate values of upwind points. For a function ψ over $\Omega \times (0,T)$, we set $\psi^n \equiv \psi(\cdot, t^n)$, where t^n is defined by $t^n \equiv n\Delta t$. The symbol $\circ$ means the composition of functions,

$$(\psi^n \circ X)(x) \equiv \psi^n(X(x)).$$

Let $\mathcal{T}_h = \{K\}$ be a triangulation of Ω. We define Ω_h by

$$\Omega_h \equiv \text{int} \bigcup \{K :\ K \in \mathcal{T}_h\}$$

and $\Gamma_h \equiv \partial\Omega_h$. Let Δt be a time increment, and N_0 be a positive integer. We set $\alpha \equiv 1/N_0$, $\Delta t_0 \equiv (1-\alpha)\Delta t$ and $\Delta t_1 \equiv \alpha\Delta t$. We define Λ by

$$\Lambda \equiv \{n \in \mathbb{R};\ n = m\alpha,\ m \in \mathbb{N},\ 1 \le m \le N_0\} \cup \{n \in \mathbb{R};\ n \in \mathbb{N} \cup (\mathbb{N} - \alpha),\ 1 \le n \le N_T\}$$

and set $\Lambda_0 \equiv \Lambda \cup \{0\}$, where $N_T \equiv [T/\Delta t]$. In the following definitions we use the same notation $(\cdot,\cdot)$ to represent the $L^2(\Omega_h)$-inner products in the scalar- and vector-valued function spaces. Suppose $\{u_h^n\}_{n\in\Lambda_0} \subset H^1(\Omega_h)^d$, $\{p_h^n\}_{n\in\Lambda} \subset H^1(\Omega_h)$ and $f \in C^0(L^2(\Omega_h)^d)$ be given. We define linear forms $\mathcal{A}_{0h}^{n-\alpha}(u_h, p_h)$, $\mathcal{A}_{1h}^{n}(u_h, p_h)$, $\mathcal{F}_{0h}^{n-\alpha}(f, u_h)$, $\mathcal{F}_{1h}^{n} f$ on $H_0^1(\Omega_h)^d$ and $\mathcal{B}_h^n u_h$ on $L_0^2(\Omega_h)$ as follows:

$$\begin{aligned}
\langle \mathcal{A}_{0h}^{n-\alpha}(u_h,p_h), v_h\rangle \equiv & \left(\frac{u_h^{n-\alpha} - u_h^{n-1}\circ X_2(u_h^{n-1}, u_h^{n-2}, \Delta t_0, \Delta t)}{\Delta t_0}, v_h\right) \\
& + \nu\Big(D(u_h^{n-\alpha}) + D(u_h^{n-1})\circ X_1(u_h^{n-1},\Delta t_0), D(v_h)\Big) + \nu\Delta t_0 \sum_{i,j,k=1}^{d} \Big(D_{ij}(u_h^{n-1})u_{hk,j}^{n-1}, v_{hi,k}\Big) \\
& - \frac{1}{2}\Big\{\Big(\nabla\cdot v_h, p_h^{n-\alpha}\Big) - \Big(\nabla p_h^{n-1}\circ X_1(u_h^{n-1},\Delta t_0), v_h\Big)\Big\},
\end{aligned}$$

$$\langle \mathcal{A}_{1h}^n(u_h,p_h),v_h\rangle \equiv \left(\frac{u_h^n - u_h^{n-\alpha}\circ X_1(u_h^{n-\alpha},\Delta t_1)}{\Delta t_1}, v_h\right) + 2\nu\Big(D(u_h^n),D(v_h)\Big) - \Big(\nabla\cdot v_h, p_h^n\Big),$$
$$\langle \mathcal{F}_{0h}^{n-\alpha}(f,u_h),v_h\rangle \equiv \frac{1}{2}\Big(f^{n-\alpha} + f^{n-1}\circ X_1(u_h^{n-1},\Delta t_0), v_h\Big), \quad \langle \mathcal{F}_{1h}^n f, v_h\rangle \equiv (f^n, v_h),$$
$$\langle \mathcal{B}_h^n u_h, q_h\rangle \equiv (\nabla\cdot u_h^n, q_h).$$

We set finite element spaces

$$\begin{aligned} X_h &\equiv \left\{v_h \in C^0(\overline{\Omega}_h)^d;\ v_h|_K \in P_2(K)^d,\ {}^\forall K\right\},\\ V_h(g) &\equiv \left\{v_h \in X_h;\ v_h(P) = g(P)(\text{node } P\in\Gamma_h)\right\}, \quad V_h \equiv V_h(0),\\ Q_h &\equiv \left\{q_h \in C^0(\overline{\Omega}_h);\ q_h|_K \in P_1(K),\ {}^\forall K,\ \int_{\Omega_h} q_h\, dx = 0\right\}. \end{aligned}$$

For the given function f the notation $f_h(\cdot,t)$ means $\Pi_h f(\cdot,t)$, where $\Pi_h:\ C^0(\overline{\Omega}_h)^d \to X_h$ is the interpolation operator.

A second order characteristic finite element approximation to problem (1) is the following; find $\{(u_h^n,\ p_h^n)\in V_h(g^n)\times Q_h;\ n\in\Lambda\}$ such that

general stage:

$$\begin{cases} \mathcal{A}_{0h}^{n-\alpha}(u_h,p_h) = \mathcal{F}_{0h}^{n-\alpha}(f_h,u_h) & \text{in } V_h',\\ \mathcal{B}_h^{n-\alpha}u_h = 0 & \text{in } Q_h', \end{cases} \tag{2a}$$

$$\begin{cases} \mathcal{A}_{1h}^{n}(u_h,p_h) = \mathcal{F}_{1h}^{n}f_h & \text{in } V_h',\\ \mathcal{B}_h^{n}u_h = 0 & \text{in } Q_h', \end{cases} \tag{2b}$$

$$(n = 2,\cdots,N_T),$$

initial stage:

$$\begin{cases} \mathcal{A}_{1h}^{m\alpha}(u_h,p_h) = \mathcal{F}_{1h}^{m\alpha}f_h & \text{in } V_h',\\ \mathcal{B}_h^{m\alpha}u_h = 0 & \text{in } Q_h',\\ u_h^0 = \Pi_h u^0, \end{cases} \tag{2c}$$

$$(m = 1,\cdots,N_0).$$

We find $(u_h^{n-\alpha},p_h^{n-\alpha})$ from (2a) and find (u_h^n,p_h^n) from (2b). The absence of $(u_h^1,\ p_h^1)$ is covered by (2c). (2a) is a second order scheme in Δt_0, and (2b) and (2c) are first order schemes in Δt_1. If we set $\Delta t_0 = O(\Delta t)$ and $\Delta t_1 = O(\Delta t^2)$, the scheme (2) is of second order in Δt.

3 Stability of the scheme

In this section we present a proposition on the stability of the scheme. Those hypotheses are checked numerically in Section 4.

For a given series $\{w^n\}_{n\in\Lambda_0}$ in a normed space X, we introduce norms $\|\cdot\|_{l^\infty(X)}$, $\|\cdot\|_{l^2(X)}$, $\|\cdot\|_{l^2_\Lambda(X)}$ defined by

$$\|w\|_{l^\infty(X)} \equiv \max_{n\in\Lambda_0}\|w^n\|_X, \quad \|w\|_{l^2(X)} \equiv \left\{\Delta t\sum_{n=1}^{N_T}\|w^n\|_X^2\right\}^{1/2},$$

$$\|w\|_{l^2_\Lambda(X)} \equiv \left\{ \Delta t_1 \sum_{m=1}^{N_0} \|w^{m\alpha}\|_X^2 + \sum_{n=2}^{N_T} \left(\Delta t_0 \|w^{n-\alpha}\|_X^2 + \Delta t_1 \|w^n\|_X^2 \right) \right\}^{1/2}.$$

Hypothesis 1. *There exists constants M_1, M_2, c_1 and c_2 independent of Δt, α and h such that*

$$\begin{cases} \text{(i)} & \|u_h\|_{l^\infty((W^{1,\infty})^d)} \le M_1, \\ \text{(ii)} & X_1(u_h^{(m-1)\alpha}, \Delta t_1)(\Omega) \subset \Omega \ (m = 1, \cdots, N_0),\ X_1(u_h^{n-\alpha}, \Delta t_1)(\Omega) \subset \Omega, \\ & X_1(u_h^{n-1}, \Delta t_0)(\Omega) \subset \Omega,\ X_2(u_h^{n-1}, u_h^{n-2}, \Delta t_0)(\Omega) \subset \Omega \ (n = 2, \cdots, N_T), \\ \text{(iii)} & {}^\forall n \ (n = 2, \cdots, N_T),\ \nu \|D(u_h^n)\|^2 \le \nu(1 + c_1 \Delta t_0) \|D(u_h^{n-\alpha})\|^2 + c_2 \|f_h^n\|^2, \\ \text{(iv)} & \|\nabla p_h\|_{l^2((L^2)^d)} \le M_2. \end{cases}$$

Proposition 1. *Let (u_h, p_h) be the solution of (2) with $g = 0$. Suppose that Hypothesis 1 holds. Then there exists a positive constant C independent of Δt, α and h such that*

$$\begin{aligned} &\|u_h\|_{l^\infty((L^2)^d)} + \sqrt{\nu \Delta t_0} \|D(u_h)\|_{l^\infty((L^2)^{d\times d})} \\ &\qquad \le C \left\{ \|u_h^0\| + \sqrt{\nu \Delta t} \|D(u_h^0)\| + \|\nabla p_h\|_{l^2((L^2)^d)} + \|f_h\|_{l^2_\Lambda((L^2)^d)} \right\}. \end{aligned}$$

4 Numerical results

In this section we show numerical results in $d = 2$ with $P2/P1$ element. In [4] it is remarked that much attention should be paid to numerical integration of composite functions. We used a numerical integration formula of degree five on each triangle [3].

For a number $N \in \mathbb{N}$ we set $\Delta t \equiv 1/N$ and $N_0 \equiv N + 1$, i.e.,

$$\Delta t = \frac{1}{N}, \quad \Delta t_0 = \frac{1}{N+1}, \quad \Delta t_1 = \frac{1}{N(N+1)}. \tag{3}$$

Since it holds that

$$\Delta t_0 = O(\Delta t),\ \Delta t_1 = O(\Delta t^2) \text{ as } N \to +\infty,$$

the scheme (2) is of second order in Δt.

To check the assumptions of Proposition 1, we solve the following example.

Example 1. *We take $\Omega = (-0.5, 0.5)^2$, $T = 1$ and five values of ν,*

$$\nu = 1,\ 10^{-1},\ 10^{-2},\ 10^{-3},\ 10^{-4}.$$

The functions f, u^0 and g are given so that the exact solution is

$$\begin{cases} u_1(x_1, x_2, t) = -4\sin^2(2\pi t)\cos^4(\pi x_1)\cos^3(\pi x_2)\sin(\pi x_2), \\ u_2(x_1, x_2, t) = 4\sin^2(2\pi t)\cos^3(\pi x_1)\cos^4(\pi x_2)\sin(\pi x_1), \\ p(x_1, x_2, t) = \sin(2\pi(t + x_1 + x_2)). \end{cases}$$

Let N_Ω be the division number of each side of Ω, and we set $N = N_\Omega$ in (3). We used almost uniform meshes with $N_\Omega = 32,\ 40,\ 48,\ 56$ by FreeFEM [5] (see Figure 2). When $\nu = 10^{-4}$, M_1 decreased as N_Ω increased from $M_1 \approx 40$ to 14. For the other

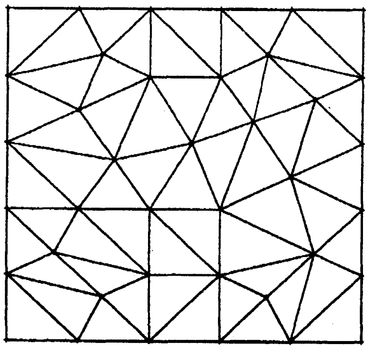

Figure 2: A sample mesh ($N_\Omega = 5$)

values of ν, M_1 remained almost constant, $M_1 \approx 13$. We set $c_* \equiv \max\{c_1,\ c_2\}$. For each $\nu = 1,\ 10^{-1},\ 10^{-2},\ 10^{-3}$ and 10^{-4}, c_* was almost constant,

$$c_* \approx 2 \times 10^{-4},\quad 3 \times 10^{-4},\quad 3 \times 10^{-5},\quad 3 \times 10^{-6},\quad 3 \times 10^{-7},$$

respectively. M_2 remained almost constants,

$$M_2 \approx 6.3,$$

for every ν.

Example 2 (a regularized cavity-flow). *We take* $\Omega = (0,1)^2$, $T = 500$, $f = 0$, $u^0 = 0$,

$$g_1(x_1, x_2, t) = \begin{cases} 16x_1^2(1-x_1)^2 & (x_2 = 1) \\ 0 & (x_2 \neq 1) \end{cases},$$

$g_2 = 0$ *and three values of* ν,

$$\nu = 10^{-2},\ 10^{-3},\ 2 \times 10^{-4}.$$

Reynolds number Re ($\equiv 1/\nu$) equals to 100, 1,000 and 5,000, respectively. Considering the boundary layers, we used nonuniform meshes refined near the boundary. Figure 3 shows the meshes ($N_\Omega = 50$ for $Re = 100,\ 1{,}000$, $N_\Omega = 100$ for $Re = 5{,}000$).

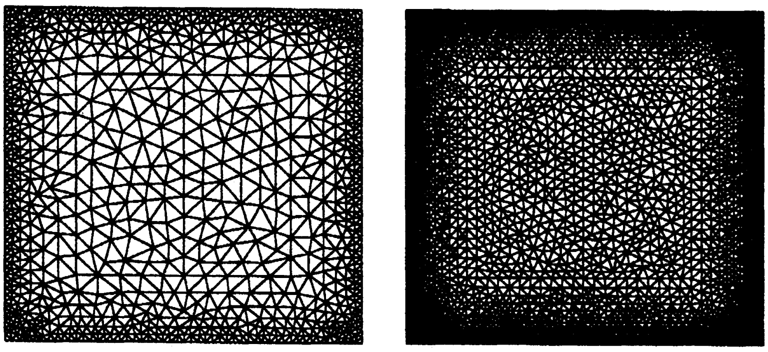

Figure 3: Meshes of $Re = 100,\ 1{,}000$ (left) and $Re = 5{,}000$ (right)

We set $N = 10$ ($Re = 100,\ 1{,}000$), 20 ($Re = 5{,}000$) in (3), and total step numbers are 5,000 and 10,000, respectively. We computed (minimum) values, which satisfied (i)-(iv) of Hypothesis 1. For $Re = 100$, they were

$$M_1 \approx 92,\quad c_* \approx 0.17,\quad M_2 \approx 5.7.$$

For $Re = 1000$, they were

$$M_1 \approx 173, \quad c_* \approx 0.22, \quad M_2 \approx 4.1.$$

For $Re = 5000$, they were

$$M_1 \approx 398, \quad c_* \approx 0.24, \quad M_2 \approx 3.6.$$

At $t = 500$ every solution was almost stationary. Figure 4 exhibits the streamlines, which show the flow patterns well of this problem.

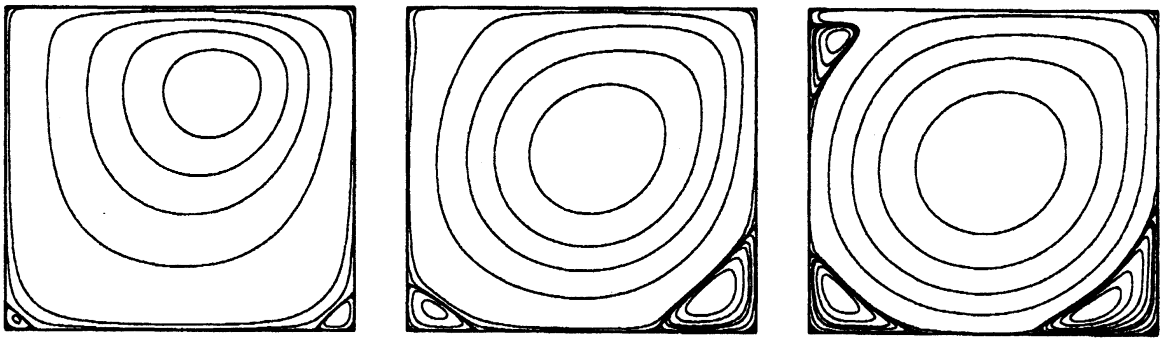

Figure 4: The streamlines of $Re = 100$ (left), $Re = 1,000$ (center) and $Re = 5,000$ (right)

5 Conclusions

We have presented a sufficient condition for the scheme (2) to be stable. In a numerical example we have seen the condition is reasonable. In another numerical example we have solved successfully a cavity flow problem with Reynolds numbers up to 5,000.

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