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A preconditioned method for saddle point problems

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Abstract

A preconditioned method for saddle point problems is presented and analyzed. We show a double preconditioned method for special saddle point problems, and apply the new method to some standard iterative solution methods. The method is illustrated by several examples derived from the surface fitting problems. Preliminary numerical results indicate that the method is efficient.

AMS Subject Classifications : 65F10, 65N25
Keyword : Saddle Point Problem, Double Preconditioned Method, Surface Fitting Problem

1 Introduction

In this paper, we consider the following saddle point problems.

$$\begin{pmatrix} \mathcal{A} & B^T \\ B & 0_m \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{1.1}$$

where $\mathcal{A}$ is an $n \times n$ symmetric and positive semi-definite matrix, B is an $m \times n$ matrix and $f \in R^n$, $g \in R^m$ are given vectors. It is assumed that $n \geq m$.

The problem (1.1) arises in the partial differential equations and optimization problems ([12][14]) and usually becomes large systems. The matrix in (1.1) is known as a saddle point matrix which is an indefinite matrix. In general, a numerical solution method for the indefinite problem is theoretically difficult compared with positive definite problems, and the convergence rate becomes slow. Recently, preconditioned Uzawa algorithms are studied ([3][5][6][8]) and are shown as an effective iterative solution method for the saddle point problem. Under some assumptions, these algorithms will be devised by using the outer iteration. Then we can reformulate the coefficient matrix (1.1) as a symmetric and positive definite problem.

1.1 The unique solvability

Let U be an $m \times m$ symmetric and positive definite matrix. We know that if $(\mathcal{A}, B^T)$ has full-row rank, that is, $\mathrm{rank}(\mathcal{A}, B^T) = n$ then $\mathcal{A}_U \equiv \mathcal{A} + B^T U B$ is positive definite. Hence we now consider the following saddle point problems.

$$\begin{pmatrix} \mathcal{A}_U & B^T \\ B & 0_m \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_U \\ g \end{pmatrix}, \tag{1.2}$$

where $f_U := f + B^T U g$.

Since (1.1) is equivalent to (1.2), we generally know that if $\mathcal{A}_U$ and $\mathcal{S}_U \equiv B\mathcal{A}_U^{-1}B^T$ are positive definite then (1.1) has a unique solution. Then an exact solution (x^*, y^*) of (1.1) is written by

$$\begin{aligned} y^* &= \mathcal{S}_U^{-1}(B\mathcal{A}_U^{-1}f_U - g), \\ x^* &= \mathcal{A}_U^{-1}(f_U - B^T y^*). \end{aligned} \tag{1.3}$$

Notice that for the formula (1.3), we can easily solve the second equation since $n \geq m$. In this paper, we consider the preconditioned method for the first equation (linear system $\mathcal{S}_U$).

1.2 Motivation and Purpose

During the years, much research has been devoted to inner iterative solution methods for symmetric and positive definite problems ([1][11]), then these preconditioned methods are fast. Moreover, the preconditioned method for the saddle point problem is also presented ([2][3][4][9]).

For almost mathematical models of physical and natural phenomena, using the finite element/difference method with a mesh size $h > 0$, we generally know that the cost (time) for numerical computations is increasing when h is decreasing. For example, if we use a preconditioner for discretizations by the incomplete Cholesky decomposition with a fixed tolerance then the iteration number is also increasing. It means that a total computing cost depends on the dimension and iteration numbers. Note that the condition number influences the iteration number. Thus, if we assume that the condition number is a constant independent of the mesh size h then we can easily estimate the total cost of numerical computations for large systems. Note that the saddle point problem is reduced to the linear system $\mathcal{S}_U$ from (1.3). Of course, if we take the preconditioner as the inverse matrix then the condition number is always one. However, we expect that it is difficult for the linear system $\mathcal{S}_U$ to generate a good preconditioner, because $\mathcal{S}_U$ is not give and is generally dense. Hence we consider a double preconditioned method to give a good condition number independent of the dimension without using $\mathcal{A}_U^{-1}$.

Here we only show the preconditioned CG method of the Uzawa-type below.

Algorithm 1 *(Q-Preconditioned CG Method of the Uzawa-type)*

For an initial guess y_0,

Solve $x_0 = \mathcal{A}_U^{-1}(f_U - B^T y_0)$;
Compute $r_0 = Bx_0 - g$; $p_0 = Q^{-1}r_0$;
For $k = 0$ to convergence
  Solve $[\mathcal{S}p]_k = B\mathcal{A}_U^{-1}B^T p_k$;
  Compute $y_{k+1} = y_k + \alpha_k p_k$; where $\alpha_k = (r_k, p_k)/(p_k, [\mathcal{S}p]_k)$;
  Compute $r_{k+1} = r_k - \alpha_k [\mathcal{S}p]_k$;
  Solve $u = Q^{-1}r_{k+1}$;
  Compute $p_{k+1} = u - \beta_k p_k$; where $\beta_k = (u, [\mathcal{S}p]_k)/(p_k, [\mathcal{S}p]_k)$;
End
Solve $x_{k+1} = \mathcal{A}_U^{-1}(f_U - B^T y_{k+1})$;

Since many saddle point problems of mathematical models satisfy that there exists a constant $M > 0$ independent of the dimension n such that $\|\mathcal{A}\|_2 \leq M$, if we take preconditioners U and Q satisfying $\mathrm{cond}_2\left(Q^{-\frac{1}{2}}\mathcal{S}_U Q^{-\frac{1}{2}}\right) < 1 + \varepsilon\,\|\mathcal{A}\|_2$, where $1 \gg \varepsilon > 0$ (independent of n) then it becomes a fast solution method for large saddle point systems.

Therefore, our purpose of this paper is to take good preconditioners U and Q, and apply these preconditioners to some standard iterative solution methods (*cf.* Algorithm 1).

We denote the largest and smallest eigenvalues of a matrix by $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$, respectively. Moreover, $\rho(\cdot)$ and $\mathrm{cond}_2(\cdot)$ denote a spectral radius and condition number with respect to 2-norm, respectively. Also I_k denotes the k-th identity matrix and $0_k \equiv 0 \cdot I_k$. For the discussion, we assume that B has full-row rank, that is, $\mathrm{rank}(B) = m$.

2 The preconditioned method

In this section, we discuss a preconditioned method for the saddle point problems (1.1), and we assume that $\mathcal{A}_U \equiv \mathcal{A} + B^T U B$ is positive definite and B has full-row rank, it implies that the matrix $\mathcal{S}_U \equiv B\mathcal{A}_U^{-1}B^T$ is also positive definite. For our problems, it is important for the convergence of iterative solution methods to analyze the property of $\mathcal{S}_U$. Usually, we want to take the preconditioner as approximations of $\mathcal{S}_U^{-1}$, however, $\mathcal{S}_U$ is not a given matrix. Hence we introduce an effective preconditioned method without using $\mathcal{A}_U^{-1}$.

Here we show a standard preconditioned method for the problem (1.2). Let Q be an $m \times m$ symmetric and positive definite matrix. Then (1.2) is equivalent to the following preconditioned problems.

$$\begin{pmatrix} I_n & 0 \\ 0 & Q^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \mathcal{A}_U & B^T \\ B & 0_m \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & Q^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} x \\ Q^{\frac{1}{2}}y \end{pmatrix} = \begin{pmatrix} f_U \\ Q^{-\frac{1}{2}}g \end{pmatrix}. \tag{2.1}$$

Let V and W be $m \times m$ symmetric and positive definite matrices. If $\mathcal{A} + B^T V B$ is positive definite and B has full-row rank then we have the following equality which was presented by Golub et al. in [9].

$$\left(B\left(\mathcal{A}+B^TVB\right)^{-1}B^T\right)^{-1} = \left(B\left(\mathcal{A}+B^T(V+W)B\right)^{-1}B^T\right)^{-1} - W. \tag{2.2}$$

First we briefly discuss a single preconditioned method for (1.2). From (2.2), we have

$$\mathrm{cond}_2\left(W^{\frac{1}{2}}\mathcal{S}_{(V+W)}W^{\frac{1}{2}}\right) = \frac{\lambda_{\min}\left(W^{\frac{1}{2}}\mathcal{S}_V W^{\frac{1}{2}}\right)+1}{\lambda_{\max}\left(W^{\frac{1}{2}}\mathcal{S}_V W^{\frac{1}{2}}\right)+1}\mathrm{cond}_2\left(W^{\frac{1}{2}}\mathcal{S}_V W^{\frac{1}{2}}\right), \tag{2.3}$$

where $\mathcal{S}_V \equiv B\mathcal{A}_V^{-1}B^T$. Thus if we take $V = \kappa_1 I_m$ and $W = \kappa_2 I_m$ then it implies that

$$\mathrm{cond}_2\left(\mathcal{S}_{(\kappa_1 I_m+\kappa_2 I_m)}\right) = \frac{\lambda_{\min}\left(\mathcal{S}_{(\kappa_1 I_m)}\right)+\kappa_2^{-1}}{\lambda_{\max}\left(\mathcal{S}_{(\kappa_1 I_m)}\right)+\kappa_2^{-1}}\mathrm{cond}_2\left(\mathcal{S}_{(\kappa_1 I_m)}\right),$$

where κ_1 and κ_2 are positive constants. Hence if we take κ_2 satisfying $\kappa_2 \cdot \lambda_{\min}(\mathcal{S}_{(\kappa_1 I_m)}) \gg 1$ for a fixed κ_1 then $U = (\kappa_1 + \kappa_2)I_m$ becomes a good preconditioner for (1.2).

Next we consider a double preconditioned method. Notice that matrices U and Q in (2.1) become the *first* and *second* preconditioners, respectively. As a similar argument, if we take V and W satisfying $\lambda_{\min}(W^{\frac{1}{2}}\mathcal{S}_V W^{\frac{1}{2}}) \gg 1$ then V and W become good preconditioners . Here we show the following result which was presented by Chen and the author in [7].

Lemma 2 *Let $S \equiv BA^{-1}B^T + C$, where A is an $n \times n$ symmetric and positive definite matrix, C is an $m \times m$ symmetric and positive semi-definite matrix. If B has full-row rank then we have*

$$\left\|L^T S^{-1} L\right\|_2 \le \frac{\|A\|_2}{1+\lambda_{\min}(L^{-1}CL^{-T})\,\|A\|_2},$$

where L is an $m \times m$ nonsingular matrix satisfying $LL^T = BB^T$.

Using above lemma, we show the following main results of this paper.

Theorem 3 *(Double: $U = \kappa Q^{-1}$, $Q = BB^T$) Assume that $(\mathcal{A}, B^T)$ and B have full-row rank. Then we have*

$$\mathrm{cond}_2\left((BB^T)^{-\frac{1}{2}}\mathcal{S}_{(\kappa(BB^T)^{-1})}(BB^T)^{-\frac{1}{2}}\right) < 1 + \kappa^{-1}\|\mathcal{A}\|_2,$$

where κ is a positive constant.

Proof : For (2.3), if we take $W = \kappa_2(BB^T)^{-1}$ then it implies that

$$\begin{aligned}\frac{\mathrm{cond}_2\left(W^{\frac{1}{2}}\mathcal{S}_{(V+W)}W^{\frac{1}{2}}\right)}{\mathrm{cond}_2\left(W^{\frac{1}{2}}\mathcal{S}_V W^{\frac{1}{2}}\right)} &= \frac{\mathrm{cond}_2\left((BB^T)^{-\frac{1}{2}}\mathcal{S}_{(V+\kappa_2(BB^T)^{-1})}(BB^T)^{-\frac{1}{2}}\right)}{\mathrm{cond}_2\left((BB^T)^{-\frac{1}{2}}\mathcal{S}_V(BB^T)^{-\frac{1}{2}}\right)} \\ &= \frac{\lambda_{\min}\left((BB^T)^{-\frac{1}{2}}\mathcal{S}_V(BB^T)^{-\frac{1}{2}}\right) + \kappa_2^{-1}}{\lambda_{\max}\left((BB^T)^{-\frac{1}{2}}\mathcal{S}_V(BB^T)^{-\frac{1}{2}}\right) + \kappa_2^{-1}}.\end{aligned}$$

Then using Lemma 2, we can obtain the following estimation.

$$\lambda_{\min}\left((BB^T)^{-\frac{1}{2}}\mathcal{S}_V(BB^T)^{-\frac{1}{2}}\right) \geq \frac{1}{\|\mathcal{A}_V\|_2}.$$

Hence it implies that

$$\lambda_{\min}\left((BB^T)^{-\frac{1}{2}}\mathcal{S}_V(BB^T)^{-\frac{1}{2}}\right) + \kappa_2^{-1} \leq \lambda_{\min}\left((BB^T)^{-\frac{1}{2}}\mathcal{S}_V(BB^T)^{-\frac{1}{2}}\right)\left(1 + \kappa_2^{-1}\|\mathcal{A}_V\|_2\right).$$

Since κ_2 is positive, if we take $V = \kappa_1(BB^T)^{-1}$ then it impies that

$$\mathrm{cond}_2\left((BB^T)^{-\frac{1}{2}}\mathcal{S}_{((\kappa_1+\kappa_2)(BB^T)^{-1})}(BB^T)^{-\frac{1}{2}}\right) < 1 + \kappa_2^{-1}\left\|\mathcal{A}_{(\kappa_1(BB^T)^{-1})}\right\|_2.$$

Moreover, we can obtain

$$\begin{aligned}\left\|\mathcal{A}_{(\kappa_1(BB^T)^{-1})}\right\|_2 &= \left\|\mathcal{A} + \kappa_1 B^T(BB^T)^{-1}B\right\|_2 \\ &\leq \|\mathcal{A}\|_2 + \kappa_1,\end{aligned}$$

which follows that the largest eigenvalue of $B^T(BB^T)^{-1}B$ is equal to 1. Therefore, taking $\kappa \equiv \kappa_1 + \kappa_2$ and $\kappa_1 \to 0$, this proof is completed. ∎

Corollary 4 *(Single: $U = \kappa Q^{-1}$, $Q = I_m$)[9] Assume that $(\mathcal{A}, B^T)$ and B have full-row rank. Then $\mathrm{cond}_2(\mathcal{S}_{(\kappa I_m)})$ is strictly decreasing, that is, $\mathrm{cond}_2(\mathcal{S}_{(\kappa I_m)}) \to 1$ as $\kappa \to \infty$.*

Here we can write matrices $\mathcal{A}$ and B by

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_* & \mathcal{A}_\perp^T \\ \mathcal{A}_\perp & \mathcal{A}_o \end{pmatrix}, \quad B = (B_o, B_*),$$

respectively. It is assumed that $\mathcal{A}_*$, $\mathcal{A}_o$ and B_* are square, and $\dim(\mathcal{A}_o) = \dim(B_*) = m$. If $\mathcal{A}_\perp \neq 0$ then the computing cost of $\mathrm{cond}_2(\mathcal{A})$ and $\mathrm{cond}_2(\mathcal{A}_U)$ in Theorem ?? is almost same. Thus as a special case, we show the following estimate.

Corollary 5 *Assume that $\mathcal{A}_\perp = 0$, that is, $\mathcal{A} = \mathrm{diag}(\mathcal{A}_*, \mathcal{A}_o)$. If $\mathcal{A}_*$ is positive definite and B_* is nonsingular then we have*

$$\left\|\mathcal{A}_U^{-1}\right\|_2 \leq \left\|\begin{pmatrix} I_{(n-m)} & 0 \\ -B_*^{-1}B_o & B_*^{-1} \end{pmatrix}\begin{pmatrix} \mathcal{A}_*^{-1} & 0 \\ 0 & U^{-1} \end{pmatrix}\begin{pmatrix} I_{(n-m)} & -B_o^T B_*^{-T} \\ 0 & B_*^{-T} \end{pmatrix}\right\|_2.$$

Proof : Since $\|\mathcal{A}_U^{-1}\|_2 = \lambda_{\min}(\mathcal{A}_U)^{-1}$ and $(\mathrm{diag}(\mathcal{A}_*, 0_m), B^T)$ has full-row rank, we have

$$\begin{aligned}
\lambda_{\min}(\mathcal{A}_U) &= \lambda_{\min}\left(\begin{pmatrix} \mathcal{A}_* & 0 \\ 0 & \mathcal{A}_o \end{pmatrix} + \begin{pmatrix} B_o^T U B_o & B_o^T U B_* \\ B_*^T U B_o & B_*^T U B_* \end{pmatrix}\right) \\
&\geq \lambda_{\min}\left(\begin{pmatrix} \mathcal{A}_* & 0 \\ 0 & 0_m \end{pmatrix} + \begin{pmatrix} B_o^T U B_o & B_o^T U B_* \\ B_*^T U B_o & B_*^T U B_* \end{pmatrix}\right) \\
&= \lambda_{\min}\left(\begin{pmatrix} I_{(n-m)} & B_o^T \\ 0 & B_*^T \end{pmatrix}\begin{pmatrix} \mathcal{A}_* & 0 \\ 0 & U \end{pmatrix}\begin{pmatrix} I_{(n-m)} & 0 \\ B_o & B_* \end{pmatrix}\right).
\end{aligned}$$

Therefore, this proof is completed. ∎

Corollary 6 *If* $U = \kappa(BB^T)^{-1}$ *then we have* $\|\mathcal{A}_{(\kappa(BB^T)^{-1})}\|_2 \leq \|\mathcal{A}\|_2 + \kappa$.

Proof : Since the largest eigenvalue of $B^T(BB^T)^{-1}B$ is equal to 1, this proof is completed. ∎

3 Numerical examples

In this section, we report some numerical results for the preconditioned method to solve the saddle point problems. Let $\Omega \subset R^2$ be a bounded and open domain. We denote the usual k-th order L^2 Sobolev space on Ω (note that $(\mathbf{x}, \mathbf{y}) \in R^2$) by $H^k(\Omega)$ and define $(\cdot,\cdot)_{L^2}$ as the L^2 inner product. We also consider the following some Sobolev spaces.

$$\begin{aligned}
H_0^1(\Omega) &\equiv \{v \in H^1(\Omega)\ ;\ v = 0 \text{ on } \partial\Omega\}, \\
H_0^2(\Omega) &\equiv \{v \in H^2(\Omega)\ ;\ v = \nabla_{\mathbf{x}} v = \nabla_{\mathbf{y}} v = 0 \text{ on } \partial\Omega\}.
\end{aligned}$$

Let $X_h \subset H_0^1(\Omega)$ be a finite element subspace which depends on a parameter h, and let $\{\varphi_i\}_{i=1}^m$ be the basis of X_h.

3.1 The surface fitting problems

Consider the following surface fitting problems.

$$\min_{u \in H_0^2(\Omega)} \nu \|\Delta u\|_{L^2}^2 + \|u(\mathbf{z}) - \mathbf{f}\|_2^2, \tag{3.1}$$

where $\mathbf{z} = (\mathbf{z}_i^{(1)}, \mathbf{z}_i^{(2)}) \in R^{c\times 2}$, $\mathbf{f} = (\mathbf{f}_i) \in R^c$ are given vectors and $\nu > 0$ is a relaxation parameter.

Since $u \in H_0^2(\Omega)$, $\nabla_{\mathbf{x}} u$, $\nabla_{\mathbf{y}} u$ and u belong to $H_0^1(\Omega)$. Thus using the H^1-method ([12]), an approximate problem of (3.1) is given by

$$\begin{aligned}
\min_{u_h^{(1)}, u_h^{(2)}, u_h^{(3)} \in X_h} \quad & \nu \left\|\nabla u_h^{(1)}\right\|_{L^2}^2 + \nu \left\|\nabla u_h^{(2)}\right\|_{L^2}^2 + \left\|u_h^{(3)}(\mathbf{z}) - \mathbf{f}\right\|_2^2, \\
\text{subject to} \quad & (\nabla u_h^{(3)}, \nabla v_h)_{L^2} = (-\mathrm{div}(u_h^{(1)}, u_h^{(2)}), v_h)_{L^2} \quad \forall v_h \in X_h.
\end{aligned} \tag{3.2}$$

Then using the solutions $u_h^{(1)}, u_h^{(2)}, u_h^{(3)} \in X_h$ and $u_h^{(4)} \in X_h$, we can take an approximate solution $u_h \in H_0^2(\Omega)$ of (3.1) by the Hermite spline functions ([13]), where $u_h^{(4)} \in X_h$ satisfies

$$(u_h^{(4)}, v_h)_{L^2} = \frac{1}{2}(\mathrm{div}(u_h^{(2)}, u_h^{(1)}), v_h)_{L^2} \quad \forall v_h \in X_h.$$

Let matrices $A_1 = (a_{ij}^{(1)}) \in R^{m\times m}$, $A_3 = (a_{ij}^{(3)}) \in R^{c\times m}$ and $B_1 = (b_{ij}^{(1)})$, $B_2 = (b_{ij}^{(2)}) \in R^{m\times m}$ have the entries

$$a_{ij}^{(1)} = (\nabla\varphi_j, \nabla\varphi_i)_{L^2}, \quad b_{ij}^{(1)} = (\nabla_{\mathbf{x}}\varphi_j, \varphi_i)_{L^2},$$
$$a_{ij}^{(3)} = \varphi_j(\mathbf{z}_i^{(1)}, \mathbf{z}_i^{(2)}), \quad b_{ij}^{(2)} = (\nabla_{\mathbf{y}}\varphi_j, \varphi_i)_{L^2},$$

respectively. ($n = 3m$) In this example, we choose the basis for the finite element subspace X_h as piecewise bi-linear functions for the uniform and square mesh. Then letting $A_2 = B_3 = A_1$, the problem (3.2) has the following saddle point form.

$$\left(\begin{array}{ccc|c} \nu A_1 & 0 & 0 & B_1^T \\ 0 & \nu A_2 & 0 & B_2^T \\ 0 & 0 & A_3^T A_3 & B_3^T \\ \hline B_1 & B_2 & B_3 & 0_m \end{array}\right)\left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \hline y \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \\ A_3^T\mathbf{f} \\ \hline 0 \end{array}\right). \tag{3.3}$$

Notice that if $c < m$ then $A_3^T A_3$ is always positive semi-definite. Thus we cannot assume that $A_3^T A_3$ is positive definite. However, this example satisfies that $(\mathcal{A}, B^T)$ and B have full-row rank since $B_3 = A_1$, where $\mathcal{A} := \mathrm{diag}(\nu A_1, \nu A_2, A_3^T A_3)$ and $B := (B_1, B_2, B_3)$. Hence using techniques of Section 1.1, we reformulate (3.3) as

$$\left(\begin{array}{c|c} \mathcal{A} + B^T U B & B^T \\ \hline B & 0_m \end{array}\right)\left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \hline y \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \\ A_3^T\mathbf{f} \\ \hline 0 \end{array}\right).$$

Here we briefly consider a linear system $\mathcal{A}_U \equiv \mathcal{A} + B^T U B$ for this example. For the surface fitting problems, we expect that if $h \to 0$ then $A_3^T A_3$ becomes sparse, it implies that $\mathcal{A}_U \approx \mathrm{diag}\,(\mathcal{A}_*, 0_m) + B^T U B$, where $\mathcal{A}_* := \mathrm{diag}\,(\nu A_1, \nu A_2)$. Thus setting $B_o := (B_1, B_2)$, the following matrix will become a good preconditioner for the linear system $\mathcal{A}_U$ since the matrices satisfy the assumption in Corollary 5.

$$\begin{pmatrix} \mathcal{A}_* & 0 \\ 0 & 0_m \end{pmatrix} + B^T U B = \begin{pmatrix} I_{(n-m)} & B_o^T \\ 0 & B_3^T \end{pmatrix}\begin{pmatrix} \mathcal{A}_* & 0 \\ 0 & U \end{pmatrix}\begin{pmatrix} I_{(n-m)} & 0 \\ B_o & B_3 \end{pmatrix}.$$

3.2 Numerical results

In this section, we consider the following two examples.

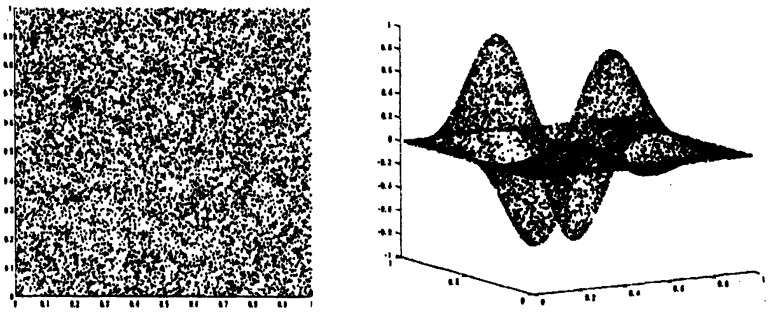

Figure 1: Data points for Example 1

Example 1 *We set the exact solution* $u^* \equiv u^*(\mathbf{x}, \mathbf{y})$ *as*

$$u^*(\mathbf{x}, \mathbf{y}) = \pi^4(\mathbf{xy})^2(1-\mathbf{x})^2(1-\mathbf{y})^2 \sin(2\pi\mathbf{x}) \sin(3\pi\mathbf{y}),$$

for $\Omega = (0,1)^2$. *Moreover, we take* 10^4 *random data points* ($c = 10^4$) *in* Ω.

Example 2 *We use real data ($c = 327$) of Niigata Prefecture Chuetsu Earthquake which was happened at 17:56 in October 23, 2004 (37.291N, 138.867E, 13Km, M6.8), and take* $\Omega = (135, 34) \times (142, 41)$. *Note that we take data* $\mathbf{z} = (\mathbf{z}_i^{(1)}, \mathbf{z}_i^{(2)})$ *and* $\mathbf{f} = (\mathbf{f}_i)$ *as (East longitude, North latitude) and the maximum acceleration (gal) at the station point, respectively. Moreover, the maximum value of data* $\mathbf{f}$ *is* 1307.911. *This data is supported by K-Net in National Research Institute for Earth Science and Disaster Prevention.*

First we show some norms for Examples 1 and 2 in Tables 1 and 2, respectively. Note that computed minimum eigenvalues of $A_3^T A_3$ for $h^{-1} = 80, 100$ in Table 1 are less than 2^{-52}.

Table 1: The eigenvalues for Example 1

h^{-1}	m	$\|A_1\|_\infty$	$\lambda_{\max}(A_1)$	$\lambda_{\max}(A_3^T A_3)$	$\lambda_{\min}(A_3^T A_3)$
60	3481	5.3333	3.9963	5.6031	4.7695e-4
80	6241	5.3333	3.9979	4.1814	2^{-52}
100	9801	5.3333	3.9986	2.9200	2^{-52}

Table 2: The eigenvalues for Example 2

h^{-1}	m	$\|A_1\|_\infty$	$\lambda_{\max}(A_1)$	$\lambda_{\max}(A_3^T A_3)$	$\lambda_{\min}(A_3^T A_3)$
100	9801	5.3333	3.9986	3.4708	0
200	39601	5.3333	3.9996	1.8753	0
300	89401	5.3333	3.9998	1.5538	0

From above tables, we can assume that $\|\mathcal{A}\|_2 = \max\{\nu \cdot \lambda_{\max}(A_1), \lambda_{\max}(A_3^T A_3)\}$ is the constant independent of the dimension. Hence we expect that our preconditioned methods lead good (effective) condition numbers for both examples.

Therefore, we next show the condition number $\mathrm{cond}_2(Q^{-\frac{1}{2}} S_U Q^{-\frac{1}{2}})$ for Example 1 in Figure 2. For $\nu = 10^{-2}$, the left-hand side and right-hand side in Figure 2 show several results for $U = \kappa Q^{-1}$ and $U \neq \kappa Q^{-1}$, respectively. Note that $U \neq \kappa Q^{-1}$ means if $Q = BB^T$ and $Q = I_m$ then $U = \kappa I_m$ and $U = \kappa (BB^T)^{-1}$, respectively.

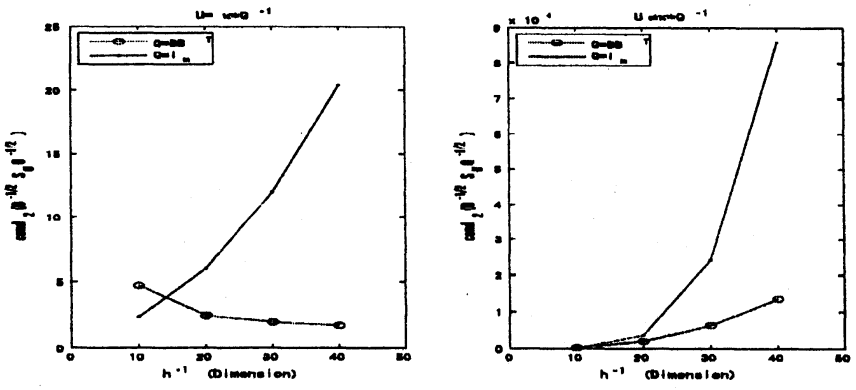

Figure 2: $\mathrm{cond}_2(Q^{-\frac{1}{2}} S_U Q^{-\frac{1}{2}})$ for Example 1 for $\kappa = 10$

Finally we show numerical results for Examples 1 and 2 by the preconditioned CG method (CG: Algorithm 1) in Tables 3 and 4, respectively. We apply the single and double preconditioned methods to the cases $h^{-1} = 60, 80, 100$ for Example 1 and $h^{-1} = 100, 200, 300$ for Example 2.

Table 3: Outer iteration numbers (total time) for Example 1

Preconditioners		$h^{-1}=60$		$h^{-1}=80$		$h^{-1}=100$		
$U=\kappa Q^{-1}$	κ	CG	(sec)	CG	(sec)	CG	(sec)	ν
$Q=I_m$	10	27	(458.8)	36	(1577.7)	44	(3938.7)	10^{-2}
$Q=BB^T$	10	8	(132.3)	7	(255.5)	7	(422.4)	10^{-2}
$Q=I_m$	100	10	(101.0)	12	(280.1)	15	(642.3)	10^{-2}
$Q=BB^T$	100	4	(77.4)	4	(167.1)	3	(235.2)	10^{-2}

Table 4: Outer iteration numbers (total time) for Example 2

Preconditioners		$h^{-1}=100$		$h^{-1}=200$		$h^{-1}=300$		
$U=\kappa Q^{-1}$	κ	CG	(sec)	CG	(sec)	CG	(sec)	ν
$Q=I_m$	10	17	(455.8)	31	(6620.7)	45	(33812.6)	10^{-3}
$Q=BB^T$	10	7	(303.5)	7	(1750.4)	7	(4803.0)	10^{-3}
$Q=I_m$	10	8	(453.0)	12	(4121.7)	16	(16356.4)	10^{-4}
$Q=BB^T$	10	7	(793.4)	7	(4608.1)	6	(11341.1)	10^{-4}

Note that we use the direct method (Complete Cholesky Decomposition and Gaussian Elimination) for linear systems A_1 ($A_3^TA_3$) and BB^T. Moreover, for the linear system $\mathcal{A}+B^TUB$, we use the preconditioned CG method with respect to the criterion δ of the relative residual error, that is, $\delta < 10^{-12}$. We only show an approximation of Example 2 for $h^{-1}=300$ in Figure 3.

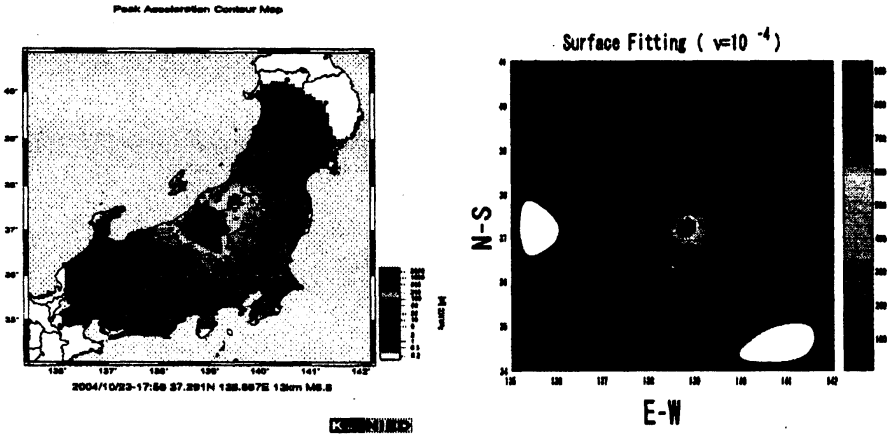

Figure 3: An approximation for Example 2 for $\nu = 10^{-4}$

All computations in tables and figures are carried out on the Dell Precision 650 Workstation Intel Xeon CPU 3.20GHz by MATLAB.

Conclusion

We have proposed a good preconditioner and double preconditioned method for saddle point problems, and shown the actual effectiveness to numerical computations. Comparing other case, our methods lead a few iteration numbers independent of the dimension by numerical results. Of course, when we take $U=\kappa BB$ then a cost for one time iteration is more expensive than the case $U=\kappa I_m$. However, total computing cost for the former is much less than the latter, because only a few iteration numbers are required.

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