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EXTENSION OF FIBREWISE MAPS FROM DENSE SUBSPACE

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1. INTRODUCTION

In [5], we have studied an alternative definition of fibrewise uniformity and its generalizations with covering types of axioms. Adopting covering uniformity as the starting point, we have studied on fibewise extensions of fibrewise spaces. With this foundation, extendability of fibrewise maps from dense subspace is the main theme of this report. That is, for a fibrewise space X, $A \subset X$ dense in X and a fibrewise continuous map $f : A \to Y$, when f can be extended to whole space X? Some characterization theorems of extendable fibrewise continuous maps are given.

In the next section, we recall definitions and notions on fibrewise topology. In section 3, we recall definitions and notions on fibrewise semi-uniformities from [4] and [5]. Some facts which are used in section 4 are also stated here. Although we use the same terms as used in [4], our definition of fibrewise semi-uniformity is stronger than that of in [4], so we can prove these facts similarly but with more simple methods.

We give some characterizations of extendable fibrewise continuous maps in section 4. Theorem 4.2 is the essential theorem for later characterizations and Theorem 4.7 is the key result for extendability.

2. PRELIMINARIES

In this section, we refer to the notations used in the latter sections, further the notions and notations in Fibrewise Topology.

Let (B, τ) be a fixed topological space with a fixed topology τ. For the base space (B, τ), TOP_B is the fibrewise category over B. (Cf. TOP is the topological category.)

A *fibrewise set* (resp. *space*) over B consists of a set (resp. topological space) X together with a (resp. continuous) function $p : X \to B$ (called the *projection*). Throughout this paper, for fibrewise sets X and Y over B the projections are $p : X \to B$ and $q : Y \to B$, respectively. For each point $b \in B$, the *fibre* over b is the subset $X_b = p^{-1}(b)$ of X. Also for each subset B' of B, we denote $X_{B'} = p^{-1}B'$.

In this report, we assume that the base space B is regular.

Throughout this paper, we will use the abbreviation *nbd(s)* for *neighborhood(s)*. We also use that for $b \in B$, $N(b)$ is the set of all open nbds of b.

Definition 2.1. (1) Let $p : X \to B$ be the continuous projection. The fibrewise space X over B is *fibrewise* T_i, $i = 0, 1, 2$, if for each point $x, x' \in X_b$ such that $x \neq x'$ where $b \in B$, the following condition is respectively satisfied:

$i = 0$: at least one of the points x, x' has a nbd in X not containing the other point.

$i = 1$: each of the points x, x' has a nbd in X not containing the other point.

$i = 2$: the points x and x' have disjoint nbds in X.

(2) ([3] Definition 2.15.) Let $p : X \to B$ be the continuous projection. The fibrewise space X over B is *fibrewise* T_3 if for each point $x \in X_b$, where $b \in B$, and each nbd V of x in X, there exists $W \in N(b)$ such that $X_W \cap \mathrm{Cl}\, U \subset V$, where Cl is the closure operator.

(3) Fibrewise T_3 and fibrewise T_0 space is called *fibrewise regular.*

Note that fibrewise regular space is fibrewise T_2 ([3] Proposition 2.19).

Definition 2.2. For a fibrewise set X over B, by a *b-filter* (resp. *b-filter base*) on X we mean a pair $(b, \mathcal{F})$, where $b \in B$ and $\mathcal{F}$ is a filter (resp. filter base) on X such that b is a limit point of the filter $p_*(\mathcal{F})$ on B.

For the definitions of undefined terms and notions, see [2] and [3].

3. Fibrewise semi-uniformities in the new sense

In this section, we recall the definitions and facts from [4] and [5].

Let X be a fibrewise set over B and $W \in \tau$. Let μ_W be a non-empty family of coverings of X_W and $\{\mu_W\}_{W\in\tau}$ the system of $\mu_W, W \in \tau$. We say that $\{\mu_W\}_{W\in\tau}$ is a *system of coverings* of $\{X_W\}_{W\in\tau}$. (For this, we briefly use the notations $\{\mu_W\}$ and $\{X_W\}$). Let $\mathcal{U}$ and $\mathcal{V}$ be families of subsets of a set X. If $\mathcal{V}$ refines $\mathcal{U}$ in the usual sense, we denote $\mathcal{V} < \mathcal{U}$.

For a family $\mathcal{U}$ of subsets of a set X and $A \subset X$, we set

$$\mathcal{U}|_A = \{U \cap A | U \in \mathcal{U}\}.$$

Definition 3.1. (cf. Definition 3.5 of [5]) Let X be a fibrewise set over B, and $\mu = \{\mu_W\}$ be a system of coverings of $\{X_W\}$. We say that the system $\{\mu_W\}$ is a *fibrewise covering uniformity* (and a pair (X, μ) or $(X, \{\mu_W\})$ is a *fibrewise covering uniform space*) if the following conditions are satisfied:

(C1) Let $\mathcal{U}$ be a covering of X_W and for each $b \in W$ there exist $W' \in N(b)$ and $\mathcal{V} \in \mu_{W'}$ such that $W' \subset W$ and $\mathcal{V} < \mathcal{U}$. Then $\mathcal{U} \in \mu_W$.

(C2) For each $\mathcal{U}_i \in \mu_W, i = 1, 2$, there exists $\mathcal{U}_3 \in \mu_W$ such that $\mathcal{U}_3 < \mathcal{U}_i, i = 1, 2$.

(C3) For each $\mathcal{U} \in \mu_W$ and $b \in W$, there exist $W' \in N(b)$ and $\mathcal{V} \in \mu_{W'}$ such that $W' \subset W$ and $\mathcal{V}$ is a star refinement of $\mathcal{U}$.

(C4) For $W' \subset W$, $\mu_{W'} \supset \mu_W|_{X_{W'}}$, where

$$\mu_W|_{X_{W'}} = \{\mathcal{U}|_{X_{W'}} | \mathcal{U} \in \mu_W\} \text{ and } \mathcal{U}|_{X_{W'}} = \{U \cap X_{W'} | U \in \mathcal{U}\}.$$

By weakening the condition (C3) of Definition 3.1, we defined fibrewise g-uniformity (in the new sense) in [5], and studied its properties. In this paper, *fibrewise semi-uniformity* (in the new sense), intermediate concept between fibrewise covering (entourage) uniformity and fibrewise g-uniformity (in the new sense), plays a central role.

Although we use the same term "fibrewise semi-uniformity" as in [4], note that the Definition 3.2 in the below is slightly stronger than that of in [4], because fibrewise covering (entourage) uniformity (in [5]) is slightly stronger than fibrewise uniform structure in [3].

Let $\{\mu_W\}$ be a system of coverings of $\{X_W\}$. For $b \in B$, $W, W' \in N(b)$ with $W' \subset W$, $\mathcal{U} \in \mu_W$ and $\mathcal{V} \in \mu_{W'}$, we define the following:

$\mathcal{V}$ is a *fibrewise local star refinement* of $\mathcal{U}$ at b if for each $V \in \mathcal{V}$ there exist $\mathcal{W} \in \mu_{W'}$ and $U \in \mathcal{U}$ such that $\mathrm{st}(V, \mathcal{W}) \subset U$.

Definition 3.2. (cf. Definition 4.1 of [4]) Let $\mu = \{\mu_W\}$ be a system of coverings of $\{X_W\}$. Then $\mu = \{\mu_W\}$ is a *fibrewise semi-uniformity* if it satisfies (C1), (C2) and (C4) of Definition 3.1 and

(FSU): For each $\mathcal{U} \in \mu_W$ and $b \in W$, there exist $W' \in N(b)$ and $\mathcal{V} \in \mu_{W'}$ such that $W' \subset W$ and $\mathcal{V}$ is a fibrewise local star refinement of $\mathcal{U}$ at b.

The pair (X, μ) (or $(X, \{\mu_W\})$) is called *fibrewise semi-uniform space.*

Clearly a fibrewise covering uniformity is a fibrewise semi-uniformity and a fibrewise semi-uniformity is a fibrewise g-uniformity.

Definition 3.3. (cf. Definition 4.5 of [4])

(1) Let $\{\mu_W\}$ be a fibrewise fibrewise semi-unifomity and $\{\mu^0_W\}$ be a system of coverings of $\{X_W\}$ satisfying that $\mu^0_W \subset \mu_W$ for all $W \in \tau$, and $\mu^0_{W'} \supset \mu^0_W|_{X_{W'}}$ for every $W' \subset W$.

We say that $\{\mu^0_W\}$ is a *base* for $\{\mu_W\}$ if for each W and $\mathcal{U} \in \mu_W$ there exists $\mathcal{V} \in \mu^0_W$ such that $\mathcal{V} < \mathcal{U}$.

Further, we say that $\{\mu^0_W\}$ is a *subbase* for $\{\mu_W\}$ if for each W and

$$\mu'_W := \{\mathcal{U}_1 \wedge \cdots \wedge \mathcal{U}_n | \mathcal{U}_i \in \mu^0_W, i = 1, \cdots, n, n \in \mathbf{N}\},$$

then $\{\mu'_W\}$ is a base for $\{\mu_W\}$, where we consider that $\mathcal{U}_1 \wedge \cdots \wedge \mathcal{U}_n$ is a coverings of X_W.

(2) Let $\{\mu^0_W\}$ be a system of coverings of $\{X_W\}$. We say that $\{\mu^0_W\}$ is a *fibrewise semi-uniformity base* if $\{\mu^0_W\}$ satisfies (C2), (C4) of Definition 3.1 and (FSU).

Unless otherwise stated, we use the notation $\{\mu^0_W\}$ for a base.

Next, we define various kinds of Cauchy filters for fibrewise semi-uniformity.

Definition 3.4. (cf. Definition 5.1 of [4]) Let $\mathcal{F}$ be a b-filter base.

We say $\mathcal{F}$ is *Cauchy* if for each $W \in N(b)$ and $\mathcal{U} \in \mu_W$ there exist $F \in \mathcal{F}$ and $U \in \mathcal{U}$ such that $F \subset U$.

$\mathcal{F}$ is called *strictly Cauchy* if for each $W \in N(b), \mathcal{U} \in \mu_W$ there exist $W' \in N(b), F \in \mathcal{F}, U \in \mathcal{U}$ and $\mathcal{V} \in \mu_{W'}$ such that $W' \subset W$ and $\mathrm{st}(F, \mathcal{V}) \subset U$.

Definition 3.5. (cf. Definition 5.3 of [4]) Let $\mathcal{F}$ and $\mathcal{F}'$ be strictly Cauchy b-filter bases. We say that $\mathcal{F}$ and $\mathcal{F}'$ are *equivalent*, $\mathcal{F} \sim \mathcal{F}'$ in notation, if for each $W \in N(b), \mathcal{U} \in \mu_W$ and $F \in \mathcal{F}$, there exist $W' \in N(b), \mathcal{V} \in \mu_{W'}$ and $F' \in \mathcal{F}'$ such that $W' \subset W$ and $\mathrm{st}(F', \mathcal{V}) \subset \mathrm{st}(F, \mathcal{U})$.

The relation $\sim$ is an equivalence relation.

Next we cite some facts from [4] and [5]. We can prove these with similar methods as in [4].

Lemma 3.6. (cf. Lemma 5.5 of [4])

(1) If $\mathcal{F}, \mathcal{F}'$ are strictly Cauchy b-filter bases and $\mathcal{F} \sim \mathcal{F}'$, then $\cap\mathrm{Cl}\mathcal{F} = \cap\mathrm{Cl}\mathcal{F}'$.
(2) If $\mathcal{F}$ is a strictly Cauchy b-filter base and converges to x, then $x \in \cap\mathrm{Cl}\mathcal{F}$.
(3) If $\mathcal{F}$ is a strictly Cauchy b-filter base and $x \in \cap\mathrm{Cl}\mathcal{F}$, then $\mathcal{F}$ converges to x.

Definition 3.7. (cf. Definition 5.7 of [4]) Let $\mathcal{F}$ be a strictly Cauchy b-filter base. We say that the b-filter generated by $\{\mathrm{st}(F, \mathcal{U}) | F \in \mathcal{F}, \mathcal{U} \in \mu_W^0, W \in N(b)\}$ is the ***star b-filter*** of $\mathcal{F}$ with respect to $\{\mu_W^0\}$ and denote $\mathrm{st}(\mathcal{F}; \{\mu_W^0\})$.

Definition 3.8. (cf. Definition 5.10 of [4])

(1) Let $\mathcal{F}$ be a Cauchy b-filter.
$\mathcal{F}$ is a *weak star b-filter* with respect to $\{\mu_W^0\}$ if for each $F \in \mathcal{F}$ there exist $W \in N(b)$ and $\mathcal{U} \in \mu_W^0$ such that $U \subset F$ for each $U \in \mathcal{U} \cap \mathcal{F}$, that is, $\cup(\mathcal{U} \cap \mathcal{F}) \subset F$.
(2) A Cauchy b-filter is called a *minimal Cauchy b-filter* if it contains no proper subfamily which is a Cauchy b-filter.

Proposition 3.9. (cf. Proposition 6.2 of [4]) Let $\{\mu_W^0\}$ be a fibrewise semi-uniformity base and $\mathcal{F}$ be a b-filter base. Then $\mathcal{F}$ is strictly Cauchy if and only if $\mathcal{F}$ is Cauchy.

Proposition 3.10. (cf. Proposition 6.3 of [4]) Let $\{\mu_W^0\}$ be a fibrewise semi-uniformity base and $\mathcal{F}, \mathcal{F}'$ be strictly Cauchy b-filter bases. Then following statements are equivalent:

(1) $\mathcal{F} \sim \mathcal{F}'$.
(2) For each $W \in N(b), \mathcal{U} \in \mu_W^0$ and $F \in \mathcal{F}$, there exists $F' \in \mathcal{F}'$ such that $F' \subset \mathrm{st}(F, \mathcal{U})$.
(3) For each $W \in N(b)$ and $\mathcal{U} \in \mu_W^0$, there exist $F \in \mathcal{F}, F' \in \mathcal{F}'$ and $U \in \mathcal{U}$ such that $F \cup F' \subset U$.

Theorem 3.11. (cf. Theorem 6.4 of [4]) Let $\{\mu_W^0\}$ be a fibrewise semi-uniformity base. Then every Cauchy b-filter contains a weak star b-filter. And the three types of Cauchy filters — star b-filters, weak star b-filters and minimal Cauchy b-filters — are all coincident.

Definition 3.12. (cf. Definition 5.17 of [4]) $(X, \{\mu_W\})$ is said to be *fibrewise complete* if every weak star b-filter ($b \in B$) with respect to $\{\mu_W^0\}$ converges.

Definition 3.13. (cf. Definition 5.19 of [4]) Let $(X, \{\mu_W\})$ and $(Y, \{\nu_W\})$ be fibrewise semi-uniform spaces and $X \subset Y$. $(Y, \{\nu_W\})$ is a *fibrewise completion* of $(X, \{\mu_W\})$ if

(1) $(Y, \{\nu_W\})$ is fibrewise complete,
(2) $\{\nu_W | x\} = \{\mu_W\}$,

(3) $(X, \tau(\{\mu_W\}))$ is dense in $(Y, \tau(\{\nu_W\}))$.

Theorem 3.14 (cf. Theorem 6.7 of [4]). The fibrewise completion of fibrewise semi-uniform space is also a fibrewise semi-uniform space.

Theorem 3.15 (cf Theorem 4.13 of [5]). Let $p : X \to B$ be a closed map and $b \in B$. Suppose that for every $W \in N(b)$, and open covering $\mathcal{U}$ of X_W there exist $W' \in N(b)$ and $\mathcal{V} \in \mu_{W'}$ such that $W' \subset W$ and $\mathcal{V} < \mathcal{U}$. Then every Cauchy b-filter converges.

Further, under the conditions in this theorem minimal Cauchy b-filters are weak star b-filters.

4. Characterizations of extendable fibrewise maps

Throughout this section A is a dense subspace of a fibrewise space X.

Let G be an open set of the subspace A. We define an open set $E_X(G)$ of X with

$$E_X(G) := X - \mathrm{Cl}_X(A - G),$$

where Cl_X is the closure operator in X.

Lemma 4.1. The followings hold for open subsets G, H of A;

(1) $E_X(G) \cap A = G$,
(2) If $G \subset H$, then $E_X(G) \subset E_X(H)$,
(3) $E_X(G \cap H) = E_X(G) \cap E_X(H)$,
(4) $E_X(G) = \cup\{M \subset X | M \text{ is open in } X \text{ and } M \cap A = G\}$.

For a collection $\mathcal{G}$ of open subsets of A, put

$$E_X(\mathcal{G}) := \{E_X(G) | G \in \mathcal{G}\}.$$

Next Theorem is essential for the following results.

Theorem 4.2. Let Y be a fibrewise regular space, $f : A \to Y$ be a fibrewise continuous map. Let $\nu = \{\nu_W\}$ be a fibrewise complete semi-uniformity on Y compatible with the topology of Y and $\nu_0 = \{\nu_W^0\}$ be a base for ν where every ν_W^0 consist of open coverings of Y_W. Let us put

$$H(\nu_0) := \cup_{b \in B} \left[\cap\{\cup E_X(f^{-1}(\mathcal{V})) | \mathcal{V} \in \nu_W^0, W \in N(b)\}\right].$$

Then there exists uniquely a fibrewise continuous map $g : H(\nu_0) \to Y$ which is an extension of f. Moreover, if $\mathcal{V}'$ is a local star refinement of $\mathcal{V}$ at b, then

$$E_X(f^{-1}(\mathcal{V}')) \wedge H(\nu_0) < g^{-1}(\mathcal{V}).$$

We recall the definition of fibrewise uniformly continuous map.

Definition 4.3. Suppose that $(X, \{\mu^0_W\}), (Y, \{\nu^0_W\})$ are fibrewise semi-uniform spaces and $f : X \to Y$ is a fibrewise map.

f is *uniformly continuous* if for every $b \in B, W \in N(b)$ and $\mathcal{V} \in \nu^0_W$, there exists $W' \in N(b)$ such that $W' \subset W$ and $f^{-1}(\mathcal{V})|_{X_{W'}} \in \mu^0_{W'}$, where $f^{-1}(\mathcal{V}) := \{f^{-1}(V)|V \in \mathcal{V}\}$.

f is *uniform isomorphism* or *unimorphism* if f is bijection and f, f^{-1} are uniformly continuous.

We have an application of Theorem 4.2.

Theorem 4.4. Let (X, μ) be an fibrewise semi-uniform space, $(A, \mu|_A)$ a dense subspace of (X, μ) and (Y, ν) a fibrewise complete semi-uniform fibrewise T_2 space. The every fibrewise uniformly continuous map f from $(A, \mu|_A)$ to (Y, ν) can be extended to a fibrewise uniformly continuous map from (X, μ) to (Y, μ).

Corollary 4.5. Let (X, μ) be a fibrewise semi-uniform fibrewise T_2 space. The any two fibrewise complete semi-uniform fibrewise T_2 space as its dense subspace are uniformly isomorphic by a fibrewise uniform isomorphism which leaves invariant each point of X.

Corollary 4.6. Let Y be a fibrewise regular space. Let X be a dense subspace of Y. Then Y is obtained as the fibrewise completion (Y, ν) of fibreiwse semi-uniform space (X, μ), where ν is a fibrewise complete semi-uniformity on Y which is compatible with the topology of Y and $\mu = \nu|_X$.

Next Theorem is the key result for extendability.

Theorem 4.7. Let $f : A \to Y$ be a fibrewise continuous map where Y is a fibrewise regular space. Let $\nu = \{\nu_W\}$ be a fibrewise complete semi-uniformity on Y compatible with the topology, and $\nu_0 = \{\nu^0_W\}$ a subbase for ν such that ν^0_W consists of open coverings of Y_W for every $W \in \tau$. Let us put

$$H(\nu_0) := \cup_{b\in B} \left[\cap\{\cup E_X(f^{-1}(\mathcal{V}))|\mathcal{V} \in \nu^0_W, W \in N(b)\}\right].$$

Then the followings hold:

(a) f is extended to a fibrewise continuous map $g : H(\nu_0) \to Y$.
(b) $H(\nu_0)$ is the largest subspace of X which contains A and over which f is extendable.
(c) $H(\nu_0) = \{x \in X|f(\mathcal{N}(x) \wedge A)$ converges to a point of $Y_{p(x)}\}$, where $\mathcal{N}(x)$ is the nbd filter of x in X.

Following theorems are easily proved by Theorem 4.7 and Theorem 4.2.

Theorem 4.8. Let (Y, ν) be a fibrewise complete semi-uniform fibrewise T_2 space, $f : A \to Y$ a fibrewise continuous map, and $\nu_0 = \{\nu^0_W\}$ a subbase for ν such that ν^0_W consists of open coverings of Y_W for every $W \in \tau$. Then f is extendable over X if and only if $\cup E_X(f^{-1}(\mathcal{V})) \supset X_b$ for every $b \in B, W \in N(b)$ and $\mathcal{V} \in \nu^0_W$.

Theorem 4.9. Let $f : A \to Y$ be a fibrewise continuous map, where Y is a fibrewise regular space. Then f is extendable over X if and only if $f(\mathcal{N}(x) \wedge A)$ converges for each $x \in X$.

Theorem 4.10. Let Y be a fibrewise regular space, projection $q : Y \to B$ be closed and $\mathcal{B}$ be a base for the open sets of Y. Then fibrewise continuous map $f : A \to Y$ is extendable over X if and only if $\cup E_X(f^{-1}(\mathcal{V})) \supset X_b$ for every $b \in B$ and $\mathcal{G} \subset \mathcal{B}$ with $\cup\mathcal{G} \supset Y_b$.

If the range space Y is fibrewise compact and fibrewise T_2, we can deduce more precise result.

Proposition 4.11. Let Y be a fibrewise compact and fibrewise T_2 space, $f : A \to Y$ fibrewise continuous map and projection $p : X \to B$ be closed. Then f is extendable over X if and only if for every $b \in B, W \in N(b)$ and open cover $\mathcal{V}$ of Y_W there exist $W' \in N(b)$ and finite open cover $\mathcal{U}$ of $X_{W'}$ such that $W' \subset W$ and $\mathcal{U} \wedge A < f^{-1}(\mathcal{V})$.

Theorem 4.12. Let Y be a fibrewise compact and fibrewise T_2 space. Then fibrewise continuous map $f : A \to Y$ is extendable over X if and only if $\mathrm{Cl}_X f^{-1}(C) \cap \mathrm{Cl}_X f^{-1}(D) = \emptyset$ for any $W \in \tau$ and closed subsets C and D of Y_W with $C \cap D = \emptyset$.

At last, we can prove a dual form of Theorem 4.8.

Theorem 4.13. Let (Y, ν) be a fibrewise complete semi-uniform fibrewise T_2 space, $f : A \to Y$ a fibrewise continuous map, and $\nu_0 = \{\nu_W^0\}$ a subbase for ν such that ν_W^0 consists of open coverings of Y_W for every $W \in \tau$.

Then fibrewise continuous map $f : A \to Y$ is extendable over X if and only if for every $b \in B, W \in N(b)$ and $\mathcal{V} \in \nu_W^0$,

$$\left[\cap\{\mathrm{Cl}_X f^{-1}(Y - V) | V \in \mathcal{V}\}\right] \cap X_b = \emptyset.$$

Next theorem is dual form of Theorem 4.10 and easy to prove.

Theorem 4.14. Let Y be a fibrewise regular space, projection $q : Y \to B$ be closed and $\mathcal{A}$ be a base for the closed sets of Y. Then f is extendable over X if and only if

$$\left[\cap \mathrm{Cl}_X f^{-1}(F) | F \in \mathcal{F}\right] \cap X_b = \emptyset$$

for every $b \in B$ and $\mathcal{F} \subset \mathcal{A}$ with $(\cap\mathcal{F}) \cap Y_b = \emptyset$.

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