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Whitney preserving mapについて

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Abstract

In this note we deal with some topics related to Whitney preserving maps.

1 Introduction

In this note, all spaces are separable metrizable spaces and maps are continuous. We denote the interval [0, 1] by I. A compact metric space is called a *compactum* and *continuum* means a connected compactum. If X is a continuum $C(X)$ denotes the space of all subcontinua of X with the topology generated by the Hausdorff metric.

In this note we study maps called *Whitney preserving maps*. If $f : X \to Y$ is a map between continua, then define a map $\hat{f} : C(X) \to C(Y)$ by $\hat{f}(A) = f(A)$ for each $A \in C(X)$. A map $f : X \to Y$ between continua is called a Whitney preserving map if there exist Whitney maps (see p105 of [4]) $\mu : C(X) \to I$ and $\nu : C(Y) \to I$ such that for each $s \in [0, \mu(X)]$, $\hat{f}(\mu^{-1}(s)) = \nu^{-1}(t)$ for some $t \in [0, \nu(Y)]$. In this case, we say that f is μ,ν-*Whitney preserving*. The notion of a Whitney preserving map is introduced by Espinoza (cf. [2] and [3]). In this note we study these maps.

2 Main result

At first we give an example of a Whitney preserving map.

Example 2.1 (Example 2 of [2]) let $f : [0,\pi] \to S^1$ be a map defined by $f(t) = e^{4ti}$. Then f is Whitney preserving. But f is not a homeomorphism.

Let X, Y be continua. If there exists a surjective map from X to Y, then does there always exist a Whitney preserving map f from X to Y? The answer to this quenstion is negative by following results.

Theorem 2.2 (Theorem 16 of [2]) *Let X be a continuum such that X contains a dense arc component. If $f : X \to I$ is a Whitney preserving map, then f is a homeomorphism.*

Recently the author proved the next theorem ([13]).

Theorem 2.3 *Let X be a continuum such that X contains a dense arc component and let D be a dendrite with finite branch points. If $f : X \to D$ is a Whitney preserving map, then f is a homeomorphism.*

Corollary 2.4 *Let X be a continuum such that X contains a dense arc component and let T be a tree. If $f : X \to T$ is a Whitney preserving map, then f is a homeomorphism.*

Generally, Theorem 2.3 does not hold when D is a graph by Example 2.1.

Problem 2.5 *Let X be a continuum such that X contains a dense arc component and let D be a dendrite. Is it true that if $f : X \to D$ is a Whitney preserving map, then f is a homeomorphism ?*

A map $f : X \to Y$ between continua is called an *atomic map* if $f^{-1}(f(A)) = A$ for each $A \in C(X)$ such that $f(A)$ is nondegenerate. A subcontinuum T of a continuum X is *terminal*, if every subcontinuum of X which intersects both T and its complement must contain T. It is known that a map f of a continuum X onto a continuum Y is atomic if and only if every fiber of f is a terminal continuum of X.

A map $f : X \to Y$ between compacta is called a *Krasinkiewicz map* if any continuum in X either contains a component of a fiber of f or is contained in a fiber of f (cf. [11]).

These maps are related to Whitney preserving maps. As the main result of [3] Espinoza proved the next theorem.

Theorem 2.6 (Theorem 3.5 of [3]) *If $f : X \to Y$ is an open atomic map such that each fiber of f is a nondegenerate continuum, then f is Whitney preserving.*

In [12] the author proved the next theorem.

Theorem 2.7 *Let X, Y be continua and let $f : X \to Y$ be a monotone map such that $f^{-1}(y)$ is a nondegenerate continuum in X. Then the following conditions are equivalent.*

(1) *f is an open map and each fiber of f is terminal in X.*
(2) *f is an open Krasinkiewicz map.*
(3) *f is a Whitney preserving map.*

Next we define maps satisfying the following property.

Definition 2.8 A Whitney preserving map $f : X \to Y$ is called a ***dimension raising Whitney preserving map*** if dim $X <$ dim $f(X)$.

It is clear that a dimension raising Whitney preserving map is not a homeomorphism. There does not always exist a dimension raising Whitney preserving map on each continuum X by Proposition 2.10.

A continuum X is said to be *continuumwise accessible* if for every subcontinuum $A \subset X$ there exist a nondegenerate subcontinuum $B \subset X$ and a point $x \in A$ such that $A \cap B = \{x\}$ (cf. Definition 4 of [2]).

The next lemma is an immediate consequence of Corollary 6 of [2].

Lemma 2.9 *Let X be a continuum such that X is cik at some point or X is continuum accessible. If $f : X \to Y$ is Whitney preserving, then f is a light map.*

Proposition 2.10 *Let X be a nondegenerate continuum such that*

(1) *X is cik at some point or X is continuum accessible, and*
(2) *each nondegenerate subcontinuum of X contains an arc.*

If $f : X \to f(X)$ is a Whitney preserving map, then dim $f(X) = 1$.

For example, if X is an arc (or a circle, or a $\sin(1/x)$-curve, etc.) and $f : X \to f(X)$ is a Whitney preserving map, Then dim $f(X) = 1$ by Proposition 2.10.

As an application of Theorem 2.7 we obtain the next result.

Theorem 2.11 *For each $n \geq 2$ and a continuum X with* dim $X = n$ *there exists a 1-dimensional subcontinuum T and a monotone Whitney preserving map $q : T \to q(T)$ such that* dim $q(T) \geq n$.

3 applications

Now we consider an applications of Theorem 2.11. A continuum is said to be *indecomposable* if it is not sum of two proper subcontinua. A continuum is called a *hereditarily indecomposable continuum* if each of its subcontinua is indecomposable. In [6] Kelley proved the next result.

Theorem 3.1 (cf. Theorem 8.5 and 8.6 of [6]) *Let X be a hereditarily indecomposable continuum with* dim$X \geq 2$ *and let $\mu : C(X) \to I$ be a Whitney map. Then for each sufficiently small $t > 0$,* dim$\mu^{-1}(t) = \infty$.

If X is a continuum, then for each mutually disjoint closed subsets $B, C \subset X$ there exists a closed partition H between B and C such that each component of H is a hereditarily indecomposable continuum (cf. Theorem 6 of [1]). So if X is a continuum with dim$X \geq 3$, then X contains a hereditarily indecomposable continuum Y such that dim$Y \geq 2$. Hence by Theorem 3.1 we can see that if X is a continuum with dim$X \geq 3$ and $\mu : C(X) \to I$ is a Whitney map, then dim$\mu^{-1}(t) = \infty$ for each sufficiently small $t > 0$.

In [10] Levin and Sternfeld gave a positive answer to the following long-standing open problem: If a continuum X is 2-dimensional, is dim$C(X) = \infty$? Furthermore, they proved the next result.

Theorem 3.2 (Theorem 2.2 of [10]) *Let X be a 2-dimensional continuum and let $\mu : C(X) \to I$ be a Whitney map. Then for all sufficiently small $t > 0$,* dim$\mu^{-1}(\mathrm{t}) = \infty$.

Hence the next result holds.

Theorem 3.3 *Let X be a continuum with* dim$X \geq 2$ *and let $\mu : C(X) \to I$ be a Whitney map. Then for all sufficiently small $t > 0$,* dim$\mu^{-1}(\mathrm{t}) = \infty$.

By Theorem 3.3 if X is a continuum with dim$X \geq 2$ and $\mu : C(X) \to I$ is a Whitney map, then dim$\mu^{-1}([0, t]) = \infty$ for each $t \in (0, \mu(X)]$.

Let T be a continuum and let $\mu : C(T) \to I$ be a Whitney map. If dim$C(T) = \infty$, is dim$\mu^{-1}([0, t]) = \infty$ for all $t \in (0, \mu(T)]$? The answer to this question is negative by the next result.

Theorem 3.4 (cf. Applications (ii) of [8]) *Let X be a 2-dimensional hereditarily indecomposable continuum which is embeddable in I^3. Then there exists a 1-dimensional subcontinuum $T \subset X$ such that*

(1) dim$C(T) = \infty$, *and*

(2) *if $\mu : C(T) \to I$ is a Whitney map, then* dim$\mu^{-1}([0, t]) = 2$ *for all sufficiently small $t > 0$.*

In fact, Levin proved the following : A 2-dimensional hereditarily indecomposable continuum X which is embeddable in I^3 contains a 1-dimensional subcontinuum T such that (1) $\dim C(T) = \infty$, and (2) if $\mu : C(T) \to I$ is a Whitney map, then $\dim \mu^{-1}(t) = 1$ for all sufficiently small $t > 0$.

A continuum T in this result is not embeddable in I^2 since T is hereditarily indecomposable and $\dim C(T) = \infty$ (cf. Corollary 1 of [7]). In [13] as an application of Theorem 2.11 the author proved Theorem 3.6. In the proof we use a *Bing-Krasinkiewicz-Lelek maps* effectively.

A map between compacta is called a *Bing map* if each of its fibers is a Bing compactum.

Let $f : X \to Y$ be a map between compacta. For each $a > 0$, let $F(f, a)$ be the union of components A of fibers with diam $A > a$, and put

$$F(f) = \bigcup_{i=1}^{\infty} F(f, 1/i).$$

For each $n \geq 1$, $f : X \to Y$ is called an *n-dimensional Lelek map* if dim $F(f) \leq n$. In case $n \leq 0$, for convenience sake, a map $f : X \to Y$ is an n-dimensional Lelek map if and only if f is a 0-dimensional map. Note that an n-dimensional Lelek map is an n-dimensional map.

A map $f : X \to Y$ is called a *Bing-Krasinkiewicz map* if f has properties of a Bing map and a Krasinkiewicz map. A map $g : X \to Y$ is called an *n-dimensional Bing-Krasinkiewicz-Lelek map* if g has properties of a Bing map, a Krasinkiewicz map and an n-dimensional Lelek map.

Theorem 3.5 (cf. [5], [11] and [16]) *Let X be an $(n+1)$-dimensional compactum and P a connected polyhedron. Then the set of all n-dimensional Bing-Krasinkiewicz-Lelek maps is a dense G_δ-subset of the space of all maps from X to P.*

Theorem 3.6 *There exists a 1-dimensional continuum $T \subset I^2$, a Whitney map $\mu : C(T) \to I$ and $s_0, s_1 \in I$ such that*

(1) $0 < s_0 < s_1 < \mu(T)$,

(2) $\dim \mu^{-1}(s) = 1$ *for each* $s \in [0, s_0)$,

(3) $\dim \mu^{-1}(s_0) = 2$, *and*

(4) $\dim \mu^{-1}(s) = \infty$ *for each* $s \in (s_0, s_1]$.

Theorem 3.7 *There exists a 1-dimensional continuum $T \subset I^2$ such that*

(1) $\dim C(T) = \infty$, *and*

(2) *for each Whitney map $w : C(T) \to I$ there exists $a_0 \in (0, w(T))$ such that* $\dim w^{-1}(s) = 1$ *for each* $s \in [0, a_0]$.

At last we give some results related to Whitney preserving maps.

Proposition 3.8 *Let $f : X \to Y$ be a monotone μ, ν-Whitney preserving map and let $s_0 = \max \{s \in I | \hat{f}(\mu^{-1}(s)) = \nu^{-1}(0)\}$. Then $\hat{f}|_{\mu^{-1}([s_0, \mu(X)])} : \mu^{-1}([s_0, \mu(X)]) \to C(Y)$ is a homeomorphism. Hence $\mu^{-1}(s)$ is homeomorphic to $\hat{f}(\mu^{-1}(s))$ for each $s \in [s_0, \mu(X)]$.*

A topological property P is said to be a *Whitney property* provided that if a continuum X has property P, so does $\mu^{-1}(t)$ for each Whitney map μ for $C(X)$ and for each $t \in [0, \mu(X)]$. As a corollary of Proposition 3.8 we get the next result.

Corollary 3.9 *Let $f : X \to Y$ be a monotone Whitney preserving map. If X has a topological property P which is a Whitney property, then so does Y.*

Also we give an application of Proposition 3.8.

Theorem 3.10 *Let X, Y be continua and let $f : X \to Y$ be a map. Let $f = h \circ g$ be the monotone-light decomposition of f with g monotone and h light. Then f is Whitney preserving if and only if g and h are Whitney preserving.*

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