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The nonexistence of expansive homeomorphisms on hereditarily indecomposable compacta

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1 Introduction.

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All spaces considered in this note are assumed to be metric spaces. *Maps* are continuous functions. By a *compactum* we mean a nonempty compact metric space. A *continuum* is a connected nondegenerate compactum. A homeomorphism $f : X \to X$ of a compactum X with metric d is called *expansive* if there is $c > 0$ such that for any $x, y \in X$ and $x \neq y$, then there is an integer $n \in \mathbf{Z} = \{0, \pm 1, \pm 2, .., \}$ such that

$$d(f^n(x), f^n(y)) > c.$$

A homeomorphism $f : X \to X$ of a compactum X is *continuum-wise expansive* if there is $c > 0$ such that if A is a nondegenerate subcontinuum of X, then there is an integer $n \in \mathbf{Z}$ such that

$$\operatorname{diam} f^n(A) > c,$$

where $\operatorname{diam} B = \sup\{d(x, y) | \, x, y \in B\}$ for a set B. Such a positive number c is called an *expansive constant* for f. Note that each expansive homeomorphism is continuum-wise expansive, but the converse assertion is not true. There are many continuum-wise expansive homeomorphisms which are not expansive. By the definitions, we see that expansiveness and continuum-wise expansiveness do not depend on the choice of the metric d of X. These notions have been extensively studied in the area of topological dynamics, ergodic theory and continuum theory.

A continuum is *decomposable* if it is the union of two of its proper subcontinua. A continuum is *indecomposable* if it is not decomposable. A continuum is *hereditarily indecomposable* if every subcontinuum is indecomposable. Likewise, a compactum is *hereditarily indecomposable* if every non-degenerate subcontinuum is indecomposable. Hereditarily indecomposable continua are created by "infinite folding" of subcontinua. There are examples of 1-dimensional hereditarily indecomposable continua that admit continuum-wise expansive homeomorphisms. However, the infinite folding also raises the entropy of the homeomorphism, and it is known that expansive homeomorphisms must have finite entropy. In this note, we show that continuum-wise expansive homeomorphisms on finite dimensional hereditarily indecomposable compacta must have infinite entropy. It then will be concluded that hereditarily indecomposable compacta cannot admit expansive homeomorphisms. Mañé has shown that infinite dimensional compacta do not admit expansive homemorphisms [3].

2 Topological entropy and expansive homeomorphisms.

Entropy is a measure of how fast points move apart. Continuum-wise expansive homeomorphisms have positive entropy [2]. The following definition of entropy is due to Bowen [8].

If $h : X \longrightarrow X$ is a map and n a non-negative integer, define

$$d_n^+(x, y) = max_{0 \leq i < n} d(h^i(x), h^i(y)).$$

Let K be a compact subset of X and n be a positive integer. A finite subset E_n of K is said to be (n, ϵ)-*separated* with respect to map h if x and y are distinct elements of E_n implies that $d_n^+(x, y) > \epsilon$. Let $s_n(\epsilon, K, h)$ denote the largest cardinality of any (n, ϵ)-separated subset of K with respect to h. Then

$$s(\epsilon, K, h) = \limsup_{n \to \infty} \frac{\log s_n(\epsilon, K, h)}{n}.$$

The entropy of h on X is then defined as

$$\mathrm{Ent}(h, X) = \sup\{\lim_{\epsilon \to 0} s(\epsilon, K, h) | K \text{ is a compact subset of } X\}.$$

It can be shown that if h is a homeomorphism then $\mathrm{Ent}(h^{-1}, X) = \mathrm{Ent}(h, X)$.

To prove our main results of this note, we need the following results.

Lemma 2.1. *Let $\mathcal{U} = [U_1, ..., U_n]$ be a taut open chain cover of continuum Y where $n \geq 7$ and suppose that $\mathcal{V} = [V_1, ..., V_m]$ is a k-fold refinement of $\mathcal{U}$. Then there exists k subcontinua $\{Y_i\}_{i=1}^k$ of Y such that diam$(Y_i) \geq d(\mathcal{U})$ and $d(Y_i, Y_j) \geq d(\mathcal{V})$ whenever $i \neq j$.*

The next theorem is due to Oversteegen and Tymchatyn:

Theorem 2.2. [6] *Let X be a hereditarily indecomposable compactum and let $\mathcal{U} = [U_1, ..., U_n]$ be an open taut chain cover of X such that there exists a subcontinuum $Z \subset X$ such that $Z \cap core(U_1)$ and $Z \cap core(U_n)$ are both nonempty. Let $f : [1, m] \longrightarrow [1, n]$ be a pattern on $\mathcal{U}$. Then there exists an open taut chain cover $\mathcal{V}$ of X such that $\mathcal{V}$ follows pattern f in $\mathcal{U}$.*

Theorem 2.3. [1] *Let X be a compact metric space. Then $dim(X) \leq m$ if and only if there exists a light map $g : X \longrightarrow I^m$.*

Lemma 2.4. *Let $g : X \longrightarrow Y$ be a light map and X, Y be compact spaces. For each $\delta > 0$, there exists a finite open cover $\mathcal{U}_\delta$ such that if $U \in \mathcal{U}_\delta$, then every component of $g^{-1}(U)$ has diameter less than δ.*

Lemma 2.5. *Suppose that X is a finite dimensional hereditarily indecomposable compactum and $\mathcal{V}_G$ is a taut generalized chain cover for X with $7q$ links. Then for each k, there exists a taut refinement $\mathcal{V}_k$ such that each proper Lucky 7 subchain in $\mathcal{V}$ is refined with a proper k-fold.*

The proofs of the following theorems are exactly the same as the proofs for Corollary 2.4 and Proposition 2.5 in [2].

Theorem 2.6. *Let $h : X \longrightarrow X$ be an continuum-wise expansive homeomorphism on compactum X. Then there exists a $\delta > 0$ such that for every $\gamma > 0$ there exists $N_\gamma > 0$ such that if A is a subcontinuum of X with $diam(A) > \gamma$ then $diam(h^n(A)) > \delta$ for all $n \geq N_\gamma$ or $diam(h^{-n}(A)) > \delta$ for all $n \geq N_\gamma$.*

Theorem 2.7. *Let $h : X \longrightarrow X$ be an continuum-wise expansive homeomorphism on compactum X. Then there exists a non-degenerate subcontinuum A such that either $\lim_{n \longrightarrow -\infty} diam(h^n(A)) = 0$ or $\lim_{n \longrightarrow \infty} diam(h^n(A)) = 0$.*

Corollary 2.8. *Let $h : X \longrightarrow X$ be an continuum-wise expansive homeomorphism on compactum X. Suppose that A is a non-degenerate subcontinuum such that $\lim_{n \longrightarrow -\infty} diam(h^n(A)) = 0$. Then there exists $\delta > 0$ such that for every integer m, subcontinuum $B \subset h^m(A)$ and $\gamma > 0$ there exists $N_\gamma > 0$ such that if $diam(B) > \gamma$, then $diam(h^n(B)) > \delta$ for every $n \geq N_\gamma$.*

Proposition 2.9. [8] *If h is an expansive homeomorphism on a compact space, then $Ent(h, X)$ is finite.*

Finally, by use of the above results, we obtain the main theorems.

Theorem 2.10. *Continuum-wise expansive homeomorphisms on hereditarily indecomposable compacta have infinite entropy.*

Corollary 2.11. *Hereditarily indecomposable compacta do not admit expansive homeomorphisms.*

References

[1] M. L. Curtis and G. S. Young, *A theorem on dimension,* Proc. Amer. Math. Soc. **3** (1952), 159-161.

[2] H. Kato, *Continuum-wise expansive homeomorphisms*, Can. J. Math. **45**(1993), no. 3, 576-598.

[3] R. Mañé, *Expansive homeomorphisms and topological dimension,* Trans. Amer. Math. Soc. **252** (1979), 313-319.

[4] C. Mouron, *Expansive homeomorphisms and indecomposable subcontinua*, Topology Appl. **126** (2002), no. 1-2, 13–28.

[5] C. Mouron, *Tree-like continua do not admit expansive homeomorphisms*, Proc. Amer. Math. Soc. **130** (2002), no. 11, 3409-3413.

[6] L. Oversteegen and E. Tymchatyn, *On hereditarily indecomposable compacta*, Geometric and Algebraic Topology, 407–417, Banach Center Publ., **18**, PWN, Warsaw, 1986.

[7] R.V. Plykin, *On the geometry of hyperbolic attractors of smooth cascades*, Russian Math. Surveys **39**(1974), 85-131.

[8] P. Walters, *A Introduction to Ergodic Theory*, Graduate Texts in Math. **79**, Springer, New York, 1982.