Expanding Ratios, Box counting Dimension and Hausdorff Dimension

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1 Introduction

All spaces in this note are metric spaces and maps are continuous functions. Let $f : X \to X$ be a map of a compactum $X$. We say that $f$ is positively expansive ([12]) if there is an admissible metric $d$ for $X$ and a positive number $c > 0$ such that if $x, y \in X$ and $x \neq y$, then there is a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) > c$. Note that this property is independent of the choice of metrics for $X$. We say that $f$ is a Ruelle expanding map ([14]) if $f$ is positively expansive and an open onto map. Note that by invariance of domain in $n$–manifolds, if $f : M \to M$ is a positively expansive map, then $f$ is a Ruelle expanding map. We say that $f$ expands small distances if there is an admissible metric $d$ for $X$ and $\epsilon > 0$ and $\lambda > 1$ such that if $0 < d(x, y) \leq \epsilon$, then $d(f(x), f(y)) > \lambda d(x, y)$. In this case, we say that $f : (X, d) \to (X, d)$ expands small distances. A map $f : X \to X$ increases small distances if there is an admissible metric $d$ for $X$ and $\epsilon > 0$ such that if $0 < d(x, y) \leq \epsilon$, then $d(f(x), f(y)) > d(x, y)$. The above two notions are dependent of the choice of metrics for $X$.

In [12], by use of the Frink's metrization theorem ([5]), Reddy proved that the following notions are equivalent:

1. $f : X \to X$ is positively expansive.
2. $f$ expands small distances.
3. $f$ increases small distances.

Hence for any onto open map $f : X \to X$, the following notions are equivalent:

1. $f$ is a Ruelle expanding map.
2. $f$ expands small distances.
3. $f$ increases small distances.

In this note, we are interested in "metrics" related to expandability of maps and we investigate more precise expandability of maps as follows. We say that $f$ expands strictly small distances with an expanding ratio $\lambda > 1$ if there is an admissible metric $d$ for $X$ and a positive number $\epsilon > 0$ such that if $x, y \in X$ and $d(x, y) \leq \epsilon$, then $d(f(x), f(y)) = \lambda d(x, y)$. In this case, we say that $f : (X, d) \to (X, d)$ expands strictly small distances with an expanding ratio $\lambda > 1$. Let $\mathbb{R}$ denote the real line, and let $\mathbb{N}$ be the set of all natural numbers and $\mathbb{Z}$ the set of all integers.
Example 1.1. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map such that $L(\mathbb{Z}^n) \subset \mathbb{Z}^n$ and $|\lambda_i| > 1,$ $|\lambda_i| \neq |\lambda_j|$ $(i \neq j)$ for eigenvalues $\lambda_i$ $(i = 1, 2, \ldots, n)$ of $L$. If $f : T^n \to T^n$ is the map of the $n$-dimensional torus $T^n$ induced by $L$, then for the Euclidean metric $\rho$ for $T^n$, $f : (T^n, \rho) \to (T^n, \rho)$ expands small distances, but it does not expand strictly small distances with a common expanding ratio.

In this note, by use of the Alexandroff-Urysohn's metrization theorem we obtain the following theorem which is a more precise result in case of Ruelle expanding maps: If $f : X \to X$ is a Ruelle expanding map of a compactum $X$ and any positive number $s > 1$, then there exists an admissible metric $d$ for $X$ and positive numbers $\epsilon > 0, \lambda$ $(1 < \lambda < s)$ such that if $x, y \in X$ and $d(x, y) \leq \epsilon$, then $d(f(x), f(y)) = \lambda d(x, y)$. For a case of graphs, we obtain that if $f : X \to X$ is a positively expansive map of a graph $X$ (=1-dimensional compact polyhedron), then the same conclusion holds. In these cases, the metrics $d$ satisfy the following equality:

$$\dim_{H}(X, d) = D_{d}(X) = \underline{D}_{d}(X) = \frac{h(f)}{\log \lambda},$$

where $\dim_{H}(X, d)$, $D_{d}(X)$ and $\underline{D}_{d}(X)$ denote the Hausdorff dimension, the lower box-counting dimension and the upper box-counting dimension of the compact metric space $(X, d)$ and $h(f)$ is the topological entropy of $f$. This implies that such a metric $d$ is a "fractal" metric for $X$. In fact, we can consider that the compact metric space $(X, d)$ has some sort of local self-similarity with respect to the inverse $f^{-1}$ of $f$ and the similarity ratio $1/\lambda$. Also, we obtain that if $f : X \to X$ is an expanding homeomorphism of a noncompact metric space $X$, then there exist an admissible metric $d$ for $X$ and a positive number $\lambda > 1$ such that if $x, y \in X$, then $d(f(x), f(y)) = \lambda d(x, y)$.

2 Metrics of Ruelle expanding maps

In this section, we need the following terminology and concepts. Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of a space $X$. We assume that each element of any open cover of a space is not an empty set. If $\mathcal{V}$ refines $\mathcal{U}$, then we denote $\mathcal{V} \leq \mathcal{U}$ (e.g. see [9] and [10]). Suppose that $x \in X$ and $\mathcal{U}$ is an open cover of $X$. Then we denote

$$St(x, \mathcal{U}) = \bigcup \{U \in \mathcal{U} | x \in U\}.$$ 

We put

$$\mathcal{U}^\Delta = \{St(x, \mathcal{U}) | x \in X\}.$$ 

An open cover $\mathcal{V}$ of $X$ is a delta-refinement of an open cover $\mathcal{U}$ of $X$ if $\mathcal{V} \leq \mathcal{U}$. Let $\{\mathcal{U}_i\}_{i=1}^\infty$ be a sequence of open covers of $X$. Then $\{\mathcal{U}_i\}_{i=1}^\infty$ is called a normal delta-sequence (e.g. see [9] and [10]) if $\mathcal{U}_{i+1}$ is a delta-refinement of $\mathcal{U}_i$ $(i = 1, 2, \ldots)$. Also, $\{\mathcal{U}_i\}_{i=1}^\infty$ is called a development of $X$ if $\{St(x, \mathcal{U}_i) | i = 1, 2, \ldots\}$ is a neighborhood base for each point $x$ of $X$. The following theorem is well known as the Alexandroff-Urysohn's metrization theorem (e.g. see [2], [9] and [10]).
Theorem 2.1. (the Alexandroff-Urysohn’s metrization theorem [2]) A $T_1$-space $X$ is metrizable if and only if there exists a sequence $\{U_i\}_{i=1}^{\infty}$ of open covers of $X$ such that $\{U_i\}_{i=1}^{\infty}$ is a normal delta-sequence and a development of $X$.

In this section, by use of the construction of the Alexandroff-Urysohn’s metrics we obtain the theorem which is a more precise result in case of Ruelle expanding maps. For the proof of Theorem 2.5, we need the following propositions.

Proposition 2.2. Let $X$ be a compactum and let $f : X \to X$ be a local embedding. Then there exists $k \in \mathbb{N}$ such that $f$ is at most $k$-to-1 map.

Let $(X, d)$ be a metric space and $x \in X$. Also, let $U_\epsilon(x)$ be the $\epsilon$ neighborhood of $x$ in $X$, i.e., $U_\epsilon(x) = \{y \in X \mid d(y, x) < \epsilon\}$.

Proposition 2.3. Let $f : X \to X$ be a map of a compactum $(X, d)$. Suppose that $\mathcal{W}$ is an open cover of $X$ such that for each $x \in X$, there exists $W \in \mathcal{W}$ such that $f^{-1}(x) \subset W$. Then there is a positive number $r > 0$ such that if $A$ is a subset of $X$ with $\text{diam}(A) \leq r$, then there exists $W \in \mathcal{W}$ with $f^{-1}(A) \subset W$.

Proposition 2.4. (Reddy [10, p.330, Construction Lemma]) Let $(X, d)$ be a compact metric space and $f : X \to X$ a positively expansive map with an expansive constant $c > 0$. Then for each positive number $r < c$, there exists a natural number $N(r) \in \mathbb{N}$ such that

$$r \leq d(x, y) \leq c \ (x, y \in X) \Rightarrow \max\{d(f^i(x), f^i(y)) \mid 0 \leq i \leq N(r) - 1\} > c.$$

Theorem 2.5. Let $f : X \to X$ be a Ruelle expanding map of a compactum $X$. For any $s > 1$, there exist an admissible metric $\tilde{d}$ for $X$ and a positive number $\lambda$ ($s > \lambda > 1$) such that $f : (X, \tilde{d}) \to (X, \tilde{d})$ expands strictly small distances with the expanding ratio $\lambda$, that is, for some $\epsilon > 0$,

$$\tilde{d}(x, y) \leq \epsilon \ (x, y \in X) \Rightarrow \tilde{d}(f(x), f(y)) = \lambda \tilde{d}(x, y).$$

Generally, we have the following problem.

Problem 2.6. Does Positively expansive maps expand strictly small distances?

In a case of graphs, we obtain the following partial answer to Problem 2.6.

Theorem 2.7. Let $f : X \to X$ be a positively expansive map of a compact connected graph $X = G$ (=1-dimensional compact polyhedron). Then for any $s > 1$, there exist an admissible metric $\tilde{d}$ for $X$ and positive numbers $\epsilon > 0, s > \lambda > 1$ such that

$$\tilde{d}(x, y) \leq \epsilon \ (x, y \in X) \Rightarrow \tilde{d}(f(x), f(y)) = \lambda \tilde{d}(x, y).$$
3 Expanding homeomorphisms of noncompact metric spaces

In this section, we deal with the case of noncompact metric spaces. We obtain the following theorem (cf. Example 1.1).

**Theorem 3.1.** Let \((X, d)\) be a (noncompact) metric space. If \(f : (X, d) \to (X, d)\) is an expanding homeomorphism, that is, there is \(\lambda > 1\) such that \(d(f(x), f(y)) \geq \lambda d(x, y)\) for \(x, y \in X\), then for any \(s > 1\) there is an admissible metric \(\tilde{d}\) for \(X\) and a positive number \(r (s > r > 1)\) such that \(f : (X, \tilde{d}) \to (X, \tilde{d})\) expands strictly distances with the expanding ratio \(r\), that is, for any \(x, y \in X\),

\[
\tilde{d}(f(x), f(y)) = r\tilde{d}(x, y).
\]

**Remark 3.2.** (Alexandroff-Urysohn's metrization theorem [7, Theorem 2.16]) It follows that \(D\) and \(d'\) in the proof of Theorem 3.1 satisfy the following condition: For any \(x, y \in X\),

\[
\frac{1}{4}D(x, y) \leq d'(x, y) \leq D(x, y).
\]

**Remark 3.3.** There is the following relations between the given metric \(d\) of Theorem 3.1 and the metric \(d'\) in the proof of Theorem 3.1:

(a) There are \(A > 0\) and \(\alpha > 0\) such that if \(d(x, y) \geq 1/2\) then

\[
d'(x, y) \leq Ad(x, y)^{\alpha}.
\]

(b) There are \(B > 0\) and \(\beta > 0\) such that if \(d(x, y) < 1/2\) then

\[
d'(x, y) \geq Bd(x, y)^{\beta}.
\]

4 Topological entropy of Ruelle expanding maps and upper box-counting dimension

In this section, we study the dynamical property which is related to Ruelle expanding map, positively expansive map, topological entropy and box-counting dimension. For a map \(f : X \to X\) of a compactum \(X\), we define the topological entropy \(h(f)\) of \(f\) as follows (see [1] and [6]): Let \(n\) be a natural number and \(\epsilon > 0\). A subset \(F\) of \(X\) is an \((n, \epsilon)\)-spanning set for \(f\) if for each \(x \in X\), there is \(y \in F\) such that

\[
\max\{d(f^i(x), f^i(y))| 0 \leq i \leq n - 1\} \leq \epsilon.
\]

Let \(r_n(f, \epsilon)\) be the smallest cardinality of all \((n, \epsilon)\)-spanning sets for \(f\). A subset \(E\) of \(X\) is an \((n, \epsilon)\)-separated set for \(f\) if for each \(x, y \in E\) with \(x \neq y\), there is \(0 \leq j \leq n - 1\) such that

\[
d(f^j(x), f^j(y)) > \epsilon.
\]
Let $s_n(f, \epsilon)$ be the maximal cardinality of all $(n, \epsilon)$-separated sets for $f$. Put

$$\tau(f, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(f, \epsilon)$$

and

$$s(f, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log s_n(f, \epsilon).$$

Also, put

$$h(f) = \lim_{\epsilon \to 0} r(f, \epsilon).$$

It is well known that $h(f) = \lim_{\epsilon \to 0} s(f, \epsilon)$ and $h(f)$ is equal to the topological entropy of $f$ which was defined by Adler, Konheim and McAndrew (see [1]).

Let $(X, d)$ be a compact metric space and $b(\epsilon)$ the minimum cardinality of a covering of $X$ by $\epsilon$-balls. Put

$$D_d(X) = \limsup_{\epsilon \to 0} \frac{\log b(\epsilon)}{|\log \epsilon|} \in \mathbb{R} \cup \{\infty\}.$$  

Similarly, put

$$D^\mu_d(X) = \liminf_{\epsilon \to 0} \frac{\log b(\epsilon)}{|\log \epsilon|} \in \mathbb{R} \cup \{\infty\}.$$  

$D_d(X)$ is called the upper box-counting dimension of $(X, d)$, and $D^\mu_d(X)$ is called the lower box-counting dimension of $(X, d)$.

Let $p \geq 0$ be any real number. Given $\epsilon > 0$, let

$$m_p^\epsilon(X, d) = \inf \Sigma_{i=1}^{\infty} [\text{diam}(A_i)]^p$$

where $X = \bigcup_{i=1}^{\infty} A_i$ is any decomposition of $X$ in a countable number of subset of diameter less than $\epsilon$. Let

$$m_p(X, d) = \sup_{\epsilon > 0} m_p^\epsilon(X, d).$$

Finally, we denote by the Hausdorff dimension $\dim_H(X, d)$ of $(X, d)$ the supremum of all real numbers $p$ such that $m_p(X, d) > 0$. It is well known that $\dim X \leq \dim_H(X, d) \leq D_d(X) \leq D^\mu_d(X)$.

**Proposition 4.1.** (cf. [7], Theorem 3.2.9) Let $f : X \to X$ be a map of a compactum $X$ with a metric $d$. Suppose that there exist positive numbers $\epsilon > 0$ and $1 \leq \lambda_2 \leq \lambda_1$ such that if $x, y \in X$ and $0 < d(x, y) \leq \epsilon$, then $\lambda_2 d(x, y) \leq d(f(x), f(y)) \leq \lambda_1 d(x, y)$. Then the following inequalities hold

$$D_d(X) \log \lambda_2 \leq h(f) \leq D_d(X) \log \lambda_1.$$  

Dai-Zhou-Geng [4] and Misiurewicz [8] proved that the following interesting result.

**Theorem 4.2.** (Dai-Zhou-Geng [4] and Misiurewicz [8]) If $f : X \to X$ is a Lipshitz continuous map of a compactum $(X, d)$ with Lipshitz constant $\lambda$, then the following equality holds

$$\frac{h(f)}{\log \lambda} \leq \dim_H(X, d).$$
Now, we obtain the following result.

**Theorem 4.3.** Let $f : X \to X$ be a map of a compactum $X$ with a metric $d$. Suppose that there exist positive numbers $\epsilon > 0$ and $\lambda > 1$ such that if $x, y \in X$ and $d(x, y) \leq \epsilon$, then $d(f(x), f(y)) = \lambda d(x, y)$. Then the following equality holds

$$h(f) = D_d(X) \log \lambda.$$ 

In particular, the followings hold.

1. If $f : X \to X$ is a Ruelle expanding map of a compactum $X$ and $s > 1$, then there exist an admissible metric $d$ for $X$ and a positive number $1 < \lambda \leq s$ such that $f : (X, d) \to (X, d)$ expands strictly small distances with the expanding ratio $\lambda$, and hence

$$\dim_H(X, d) = D_d(X) = D_d(X) = \frac{h(f)}{\log \lambda}.$$ 

2. If $f : G \to G$ is a positively expansive map of a graph $G$ and $s > 1$, then there exist an admissible metric $d$ for $G$ and a positive number $1 < \lambda \leq s$ such that $f : (G, d) \to (G, d)$ expands strictly small distances with the expanding ratio $\lambda$, and hence

$$\dim_H(G, d) = D_d(G) = D_d(G) = \frac{h(f)}{\log \lambda}.$$ 

**Remark 4.4.** In [9], Pontrjagin and Schnirelmann proved that for any compactum $X$,

$$\dim X = \min\{D_d(X)\mid d \text{ is a metric for } X\},$$

where $\dim X$ denotes the topological dimension of $X$. Suppose that $\dim X \geq 1$ and a map $f : (X, d) \to (X, d)$ expands strictly small distances with an expanding ratio $\lambda > 1$. Then $0 < \log \lambda \leq h(f)/\dim X$, which implies that the set of expanding ratios of $f$ are bounded. Note that there exist a sequence $\{\lambda_i\}_{i=1}^\infty$ of metrics for $X$ such that $f : (X, d_i) \to (X, d_i)$ expands strictly small distances with an expanding ratio $\lambda_i$ satisfying $\lambda_i > \lambda_{i+1}$ and $\lim_{i \to \infty} \lambda_i = 1$. Then $\lim_{i \to \infty} D_d(X) = \infty$, which implies that $d_i$ is a "fractal" metric on $X$. In fact, we can consider that the space $(X, d_i)$ has some sort of local self-similarity with respect to the inverse $f^{-1}$ of $f$ and the similarity ratio $1/\lambda_i$. In [5], we investigated the relation between metrics $d$, box-counting dimensions $D_d(X)$ and $D_d(X)$ of a separable metric space $(X, d)$.

The topological entropy of endomorphisms of the $n$-dimensional torus $T^n$ is well known and hence we have the following.

**Corollary 4.5.** Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map such that $L(\mathbb{Z}^n) \subset \mathbb{Z}^n$ and $|\lambda_i| > 1$ for each eigenvalue $\lambda_i$ ($i = 1, 2, \ldots, n$) of $L$. Then the followings hold.

1. For any $s > 1$, there exists an admissible metric $d$ for $\mathbb{R}^n$ and a positive number $\lambda$ with $s > \lambda > 1$ such that if $x, y \in \mathbb{R}^n$, then $d(L(x), L(y)) = \lambda d(x, y)$. 

2. Let $T^n$ be the $n$-dimensional torus and let $f : T^n \to T^n$ be the map induced by the linear map $L$. Then for any $s > 1$, there exists an admissible metric $d$ for $T^n$ and positive numbers $\epsilon > 0$ and $1 < \lambda < s$ such that if $x, y \in T^n$ and $d(x, y) \leq \epsilon$, then $d(f(x), f(y)) = \lambda d(x, y)$. Also,

$$\sum_{i=1}^{n} \log |\lambda_i| = \sum_{|\lambda_i| > 1} \log |\lambda_i| = h(f) = D_d(X) \log \lambda$$

and hence

$$\dim_H(X, d) = D_d(X) = D_d(X) = \frac{\sum_{i=1}^{n} \log |\lambda_i|}{\log \lambda}.$$ 

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