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SOME COMPLETE-TYPE MAPS

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1. Introduction

As well known, in the topological category TOP uniform spaces are studied as the generalization of metric spaces, compact spaces and topological groups. In the fibrewise category TOP_B with the base space B, the study of fibrewise uniform space in TOP_B is found in James [5] Ch.3 and Konami-Miwa [6], [7]. Especially in [6] and [7], they studied the fibrewise uniform spaces by using coverings, and proved in [7] the equivalence of fibrewise uniform spaces by using entourages (in [5]) and their one (in [7]). The study of metrizable maps in TOP_B is found in [11], [9], [2], [8] and [3]. But for a metrizable map $p : X \to B$, the study of fibrewise uniformity on X has not been done.

In this paper, we announce the existence of fibrewise uniformities on some metrizable maps, and study the relations between the completeness induced by a trivial metric and the one defined by fibrewise uniformities. Further, we discuss the relations between completely metrizable maps and Čech-complete maps.

2. Preliminaries

In this section, we refer to the notions and notations in Fibrewise Topology. For the definitions of undefined terms and notations, see [4], [3], [7] and [5].

Throughout this paper, we will use the abbreviation *nbd(s)* for *neighborhood(s)*. Let B be a topological space with a fixed topology τ. For each $b \in B$, $N(b)$ is the family of all open nbds of b, and $\mathbf{N}$, $\mathbf{Q}$, $\mathbf{R}$ and I are the sets of all natural numbers, all rational numbers, all real numbers and the unit interval, respectively. In this paper, we assume that (B, τ) is a regular space, all spaces are topological spaces and all maps are continuous.

For a map $p : X \to B$ and each $b \in B$, the *fibre* over b is the subset $X_b = p^{-1}(b)$ of X. Also for each subset B' of B, we denote $X_{B'} = p^{-1}B'$. For a filter $\mathcal{F}$ on X, by a *b-filter* on X we mean a pair $(b, \mathcal{F})$ such that b is a limit point of the filter $p_*(\mathcal{F})$ on B, where $p_*(\mathcal{F})$ is the filter generated by the family $\{p(F) | F \in \mathcal{F}\}$. By an *adherence point* of a b-filter $\mathcal{F}$ $(b \in B)$ on X, we mean a point of the fibre X_b

which is an adherence point of $\mathcal{F}$ as a filter on X. For a projection $p : X \to B$ and $W \subset B$, we use the notation $X_W \times X_W = X_W^2$ and $X \times X = X^2$. For $D, E \subset X^2$, $D \circ E = \{(x,z) | \exists y \in X \text{ such that } (x,y) \in D, (y,z) \in E\}$ and $D(x) = \{y | (x,y) \in D\}$. For a family $\mathcal{U}$ of subsets of a set X and a subset A of X, $\mathcal{U}|_A = \{U \cap A | U \in \mathcal{U}\}$.

Next, according to [11] let us refer to (completely) trivially metrizable maps. For a map $p : X \to B$ with a pseudometric ρ on X is called a *trivial metric* (*T-metric*, for short) on p if the restriction of ρ to every fibre $p^{-1}(b)$, $b \in B$, is a metric and $p^{-1}\tau \cup \tau_\rho$, where τ_ρ is the topology on X generated by ρ, is a subbase of the topology of X. A map $p : X \to B$ is called *trivially metrizable* (a *TM-map*, for short) if there exists a T-metric on p. A T-metric on a map $p : X \to B$ is called *complete* (a *CT-metric*, or short) if

(*) For any b-filter $\mathcal{F}$, $b \in B$, on X containing elements of arbitrary small diameter, $\mathcal{F}$ has adherence points.

A map $p : X \to B$ is called *completely trivially metrizable* (a *complete TM-map*, for short) if there exists a CT-metric on it.

A map $p : X \to B$ is called (resp. *closedly*) *parallel* to a space Z if there exists an embedding $e : X \to B \times Z$ such that (resp. $e(X)$ is closed in $B \times Z$ and) $p = \pi \circ e$, where $\pi : B \times Z \to B$ is the projection (see [10]).

The following are proved in [11]: A map $p : X \to B$ is a TM-map if and only if p is parallel to a metrizable map, and p is a complete TM-map if and only if it is closedly parallel to a completely metrizable (i.e., metrizable by complete metric) space.

Remark: By these, for a TM-map $p : X \to B$ there exists a metric space (M, ρ) and an embedding $e : X \to B \times M$ such that $p = \pi \circ e$. Then it is easy to see that we can define a T-metric (pseudometric) ρ' on X by $\rho'(x,y) = \rho(\pi \circ e(x), \pi \circ e(y))$, and vice versa. So, we can identify ρ on M and ρ' on X in the above meaning. In latter sections, we use the same notation ρ on M and on X.

We shall conclude this section by referring to fibrewise uniformities according to [7]. First, we recall the following definition.

Definition 2.1. Let $p : X \to B$ be a projection, and Δ be the diagonal of $X \times X$. A *fibrewise entourage uniformity* on X is a filter Ω on $X \times X$ satisfying the following four conditions:

(J1) $\Delta \subset D$ for every $D \in \Omega$.

(J2) Let $D \in \Omega$. Then for each $b \in B$ there exist $W \in N(b)$ and $E \in \Omega$ such that $E \cap X_W^2 \subset D^{-1}$.

(J3) Let $D \in \Omega$. Then for each $b \in B$ there exist $W \in N(b)$ and $E \in \Omega$ such that

$$(E \cap X_W^2) \circ (E \cap X_W^2) \subset D$$

(J4) If $E \subset X \times X$ satisfies that for each $b \in B$ there exist $W \in N(b)$ and $D \in \Omega$ such that $D \cap X_W^2 \subset E$, then $E \in \Omega$.

Note that in [5] Section 12, a filter Ω on $X \times X$ satisfying (J1),(J2) and (J3) is called a *fibrewise uniform structure* on X. So, the notion of a fibrewise entourage uniformity is slightly stronger than one of a fibrewise uniform structure.

For a projection $p : X \to B$ and $W \in \tau$, let μ_W be a non-empty family of coverings of X_W. We say that $\{\mu_W\}_{W \in \tau}$ is a *system of coverings* of $\{X_W\}_{W \in \tau}$. (For this, we briefly use the notations $\{\mu_W\}$ and $\{X_W\}$). Let $\mathcal{U}$ and $\mathcal{V}$ be families of subsets of a set X. If $\mathcal{V}$ refines $\mathcal{U}$ in the usual sense, we denote $\mathcal{V} < \mathcal{U}$. Let us define the notion of fibrewise covering uniformity.

Definition 2.2. Let $p : X \to B$ be a projection, and $\mu = \{\mu_W\}$ be a system of coverings of $\{X_W\}$. We say that the system $\{\mu_W\}$ is a *fibrewise covering uniformity* (and a pair (X, μ) or $(X, \{\mu_W\})$) is a *fibrewise covering uniform space*) if the following conditions are satisfied:

(C1) Let $\mathcal{U}$ be a covering of X_W and for each $b \in W$ there exist $W' \in N(b)$ and $\mathcal{V} \in \mu_{W'}$ such that $W' \subset W$ and $\mathcal{V} < \mathcal{U}$. Then $\mathcal{U} \in \mu_W$.

(C2) For each $\mathcal{U}_i \in \mu_W, i = 1, 2$, there exists $\mathcal{U}_3 \in \mu_W$ such that $\mathcal{U}_3 < \mathcal{U}_i, i = 1, 2$.

(C3) For each $\mathcal{U} \in \mu_W$ and $b \in W$, there exist $W' \in N(b)$ and $\mathcal{V} \in \mu_{W'}$ such that $W' \subset W$ and $\mathcal{V}$ is a star refinement of $\mathcal{U}$.

(C4) For $W' \subset W$, $\mu_{W'} \supset \mu_W|_{X_{W'}}$, where

$$\mu_W|_{X_{W'}} = \{\mathcal{U}|_{X_{W'}} | \mathcal{U} \in \mu_W\} \text{ and } \mathcal{U}|_{X_{W'}} = \{U \cap X_{W'} | U \in \mathcal{U}\}.$$

For a fibrewise entourage uniformity Ω on X, $D \in \Omega$ and $W \in \tau$, let $\mathcal{U}(D, W) = \{D(x) \cap X_W | x \in X_W\}$. Further let $\mu_W(\Omega)$ be the family of coverings $\mathcal{U}$ of X_W satisfying that for each $b \in W$ there exist $W' \in N(b)$ and $D \in \Omega$ such that $W' \subset W$ and $\mathcal{U}(D, W') < \mathcal{U}$. Then the system $\mu(\Omega) = \{\mu_W(\Omega)\}$ is a fibrewise covering uniformity ([7] Proposition 3.7).

Conversely, for a fibrewise covering uniformity $\mu = \{\mu_W\}$, we can constructed a fibrewise entourage uniformity $\Omega(\mu)$ as follows ([7] Construction 3.8): For $\mathcal{U} \in \mu_W$, $D(\mathcal{U}) = \bigcup\{U_\alpha \times U_\alpha | U_\alpha \in \mathcal{U}\}$. Let $\Omega(\mu)$ be the family of all subsets $D \subset X \times X$ satisfying the following condition:

$\Delta \subset D$, and for every $b \in B$ there exist $W \in N(b)$ and $\mathcal{U} \in \mu_W$ such that $D(\mathcal{U}) \subset D$.

Then $\Omega(\mu)$ is a fibrewise entourage uniformity ([7] Proposition 3.10). Further, we proved the following:

Theorem 2.3. ([7] Theorem 3.11) For a projection $p : X \to B$ and a fibrewise entourage uniformity Ω on X, we have $\Omega = \Omega(\mu(\Omega))$.

For a fibrewise entourage uniformity Ω on X and a fibrewise covering uniformity μ on X, let $\tau(\Omega)$ be the fibrewise topology induced by Ω ([5] Section 13) and $\tau(\mu)$ be the fibrewise topology induced by μ ([7] Proposition 3.8). Then $\tau(\Omega) = \tau(\mu(\Omega))$ and $\tau(\mu) = \tau(\Omega(\mu))$ ([7] Proposition 3.12).

3. Fibrewise covering uniformities on TM-maps

For a TM-map $p : X \to B$ parallel to a metric space (M, ρ), let $e : X \to B \times M$ be the embedding. For each $n \in \mathrm{N}$, let $\mathcal{U}_n$ be the family $\{U(x, \frac{1}{n}) | x \in M\}$, where $U(x, \frac{1}{n}) = \{y \in M | \rho(x, y) < \frac{1}{n}\}$ and $\mathcal{W}_n = \{e^{-1}(B \times U) | U \in \mathcal{U}_n\}$. Then for each $W \in \tau$, let $\mu_W = \{\mathcal{U} | \bigcup \mathcal{U} = X_W$ and for each $b \in W$ there exists $n \in \mathbf{N}$ and $W' \in N(b)$ with $W' \subset W$ such that $\mathcal{W}_n | X_{W'} < \mathcal{U}\}$.

Since μ_W and μ constructed above are induced by the metric ρ on M (on X), we call this $\mu = \{\mu_W\}$ a *fibrewise covering uniformity on X induced* by the metric ρ, and denoted by $\mu_\rho = \{\mu_W\}_\rho$. Further, by the construction of $\{\mathcal{W}_n | n \in \mathrm{N}\}$ in the above, we say that the family $\{\mathcal{W}_n | n \in \mathrm{N}\}$ is the *standard developable covering* (*sd*-covering, for short) on X induced by ρ. (Note that we exclusively use the notation $\{\mathcal{W}_n | n \in \mathrm{N}\}$ as *sd*-covering induced by ρ in this paper.)

Theorem 3.1. For a TM-map $p : X \to B$ with a T-metric ρ, the system $\mu_\rho = \{\mu_W\}_\rho$ is a fibrewise covering uniformity on X induced by ρ.

4. Equivalence of some completeness on TM-maps

Definition 4.1. ([5] Definition 14.1) For a map $p : X \to B$, let Ω be a fibrewise entourage uniformity on X.
(1) A subset M of X is said to be *D-small*, where $D \subset X^2$, if M^2 is contained in D.
(2) A *b*-filer $\mathcal{F}$, where $b \in B$, is *Cauchy* if $\mathcal{F}$ contains a D-small members for each $D \in \Omega$. (We call $\mathcal{F}$ J-Cauchy with respect to Ω (w.r.t. Ω, for short), for convenience' sake.)

We shall define a new notion of Cauchy *b*-filter in fibrewise covering uniformity $\mu = \{\mu_W\}$ on X.

Definition 4.2. For a map $p : X \to B$, let $\mu = \{\mu_W\}$ be a fibrewise covering uniformity on X. A *b*-filer $\mathcal{F}$, where $b \in B$, is *Cauchy* if for each $W \in N(b)$ and $\mathcal{U} \in \mu_W$ there exist $F \in \mathcal{F}$ and $U \in \mathcal{U}$ such that $F \subset U$. (We call $\mathcal{F}$ CU-Cauchy with respect to μ (w.r.t. μ, for short), for convenience' sake.)

Theorem 4.3. For a map $p : X \to B$, let Ω be a fibrewise entourage uniformity on X. Then for each $b \in B$, a *b*-filer $\mathcal{F}$ is J-Cauchy w.r.t. Ω if and only if it is CU-Cauchy w.r.t. $\mu(\Omega)$.

For a space X, let $\Upsilon = \{\Phi_\alpha | \alpha \in \Lambda\}$ be a family of families of subsets of X. We say that a family Ψ of subsets of X is *subordinated* to the family Υ if for each $\alpha \in \Lambda$ there exists $U_\alpha \in \Phi_\alpha$ and $V \in \Psi$ such that $V \subset U_\alpha$.

Definition 4.4. Let $p : X \to B$ be a TM-map with a T-metric ρ.
(1)([11]) The map p is *complete* if for any b-filter $\mathcal{F}$, $b \in B$, on X subordinated to the *sd*-covering $\{\mathcal{W}_n | n \in \mathrm{N}\}$ induced by ρ, it has adherence points. (We call this "complete" *P-complete*, and also call this b-filter satisfying this condition *P-Cauchy* w.r.t ρ.)
(2)([5] Definition 14.10) The map p is *complete* if for each $b \in B$ any J-Cauchy b-filter $\mathcal{F}$ w.r.t. $\Omega(\mu_\rho)$ converges. (We call this "complete" *J-complete.*)

Theorem 4.5. For a TM-map $p : X \to B$ with a T-metric ρ and each $b \in B$, a b-filer $\mathcal{F}$ is a P-Cauchy w.r.t. ρ if and only if it is a J-Cauchy w.r.t. Ω_ρ.

5. Complete TM-maps and Čech-complete maps

Definition 5.1. A T_2-compactifiable map $p : X \to B$ is *Čech-complete* if for each $b \in B$, there exists a countable family $\{\mathcal{A}_n\}_{n \in \mathrm{N}}$ of open (in X) covers of X_b with the property that every b-filter $\mathcal{F}$ which is subordinated to the family $\{\mathcal{A}_n\}_{n \in \mathrm{N}}$ has an adherence point.

Proposition 5.2. (1) ([1] Theorem 6.1) Every locally compact map is Čech-complete .
(2) ([1] Theorem 4.1) For T_2-compactifiable maps $p : X \to B$, $q : Y \to B$ and a pefect morphism $f : p \to q$, p is Čech-complete if and only if q is Čech-complete .

Lemma 5.3. Every TM-map $p : X \to B$ is a $T_{3\frac{1}{2}}$-map.

By this lemmm, every TM-map is $T_{3\frac{1}{2}}$-compactifiable. For complete TM-maps, we can prove the following.

Theorem 5.4. If $p : X \to B$ is a complete TM-map, then p is Čech-complete.

6. MT-maps and some problems

About the relations of TM-maps and MT-maps, we have the following.
(a) A closed TM-map is an MT-map.
(b) There exists a compact MT-map which is not a TM-map.
(c) There exists (complete) TM-maps which are not closed, so not MT-maps.

Theorem 6.1. If $p : X \to B$ is a closed TM-map, then p is an MT-map.

As discussed in section 5, there seems to exist many problems about relations between metrizable maps and completeness. As an attempt to the problems, we define a new notion of D-complete MT-maps. For an MT-map $p : X \to B$, we use the following notation: $\{\{\mathcal{U}_n(b)\}_{n\in\mathbf{N}} | b \in B\}$ is a p-development, where $\{\mathcal{U}_n(b)\}_{n\in\mathbf{N}}$ is a b-development. First, we recall some definitions and theorems of MT-maps according to [3].

Definition 6.2. (1)([3] Def. 2.8) For a map $p : X \to B$, a sequence $\{\mathcal{U}_n\}_{n\in\mathbf{N}}$ of open (in X) covers of X_b, $b \in B$, is said to be a *b-development* if for every $x \in X_b$ and every $U \in N(x)$, there exists $n \in \mathbf{N}$ and $W \in N(b)$ such that $x \in st(x, \mathcal{U}_n) \cap X_W \subset U$. The map p is said to have a *p-development* if it has a b-development for every $b \in B$.
(2)([3] Def. 2.9) A closed map $p : X \to B$ is said to be an *MT-map* if it is collectionwise normal and has a p-development.

Definition 6.3. For an MT-map $p : X \to B$ equipped with p-development $\{\{\mathcal{U}_n(b)\}_{n\in\mathbf{N}} | b \in B\}$, we call p *D-complete with respect to the p-development* if for each $b \in B$ every b-filter $\mathcal{F}$ subordinated to $\{\mathcal{U}_n(b)\}_{n\in\mathbf{N}}$ has adherence points.

Problem 6.4. For an MT-map $p : X \to B$, let $\{\{\mathcal{U}_n(b)\}_{n\in\mathbf{N}} | b \in B\}$ be a p-development.
(1) Is there a fibrewise (covering) uniformity on X related to the p-development?
(2) If Problem (1) had an affirmative answer, then is the J-completion of p w.r.t. the fibrewise (covering) uniformity on X equivalent to D-completion?

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