SOME COMPLETE-TYPE MAPS (General and geometric topology today and their problems)

Bai, Yun-Feng; Miwa, Takuo

数理解析研究所講究録 (2008), 1578: 22-27

Kyoto University
SOME COMPLETE-TYPE MAPS

Yun-Feng Bai
Department of Mathematics, Shimane Univ. (Capital Normal Univ.)

1. Introduction

As well known, in the topological category $TOP$ uniform spaces are studied as
the generalization of metric spaces, compact spaces and topological groups. In the
fibrewise category $TOP_B$ with the base space $B$, the study of fibrewise uniform space
in $TOP_B$ is found in James [5] Ch.3 and Konami-Miwa [6], [7]. Especially in [6] and
[7], they studied the fibrewise uniform spaces by using coverings, and proved in [7]
the equivalence of fibrewise uniform spaces by using entourages (in [5]) and their
one (in [7]). The study of metrizable maps in $TOP_B$ is found in [11], [9], [2], [8] and
[3]. But for a metrizable map $p : X \rightarrow B$, the study of fibrewise uniformity on $X$
has not been done.

In this paper, we announce the existence of fibrewise uniformities on some metriz-
able maps, and study the relations between the completeness induced by a trivial
metric and the one defined by fibrewise uniformities. Further, we discuss the rela-
tions between completely metrizable maps and Čech-complete maps.

2. Preliminaries

In this section, we refer to the notions and notations in Fibrewise Topology. For
the definitions of undefined terms and notations, see [4], [3], [7] and [5].

Throughout this paper, we will use the abbreviation $nbd(s)$ for neighborhood(s).
Let $B$ be a topological space with a fixed topology $\tau$. For each $b \in B$, $N(b)$ is the
family of all open nbds of $b$, and $N, Q, R$ and $I$ are the sets of all natural numbers,
all rational numbers, all real numbers and the unit interval, respectively. In this
paper, we assume that $(B, \tau)$ is a regular space, all spaces are topological spaces
and all maps are continuous.

For a map $p : X \rightarrow B$ and each $b \in B$, the fibre over $b$ is the subset $X_b = p^{-1}(b)$
of $X$. Also for each subset $B'$ of $B$, we denote $X_{B'} = p^{-1}B'$. For a filter $\mathcal{F}$ on $X$,
by a $b$-filter on $X$ we mean a pair $(b, \mathcal{F})$ such that $b$ is a limit point of the filter
$p_*(\mathcal{F})$ on $B$, where $p_*(\mathcal{F})$ is the filter generated by the family $\{p(F) | F \in \mathcal{F}\}$. By
an adherence point of a $b$-filter $\mathcal{F}$ ($b \in B$) on $X$, we mean a point of the fibre $X_b$.
which is an adherence point of \( \mathcal{F} \) as a filter on \( X \). For a projection \( p : X \to B \) and \( W \subset B \), we use the notation \( X_W \times X_W = X_W^2 \) and \( X \times X = X^2 \). For \( D, E \subset X^2 \), \( D \circ E = \{(x, z) \mid \exists y \in X \text{ such that } (x, y) \in D, (y, z) \in E \} \) and \( D(x) = \{y | (x, y) \in D\} \). For a family \( \mathcal{U} \) of subsets of a set \( X \) and a subset \( A \) of \( X \), \( \mathcal{U}|_A = \{U \cap A | U \in \mathcal{U}\} \).

Next, according to [11] let us refer to (completely) trivially metrizable maps. For a map \( p : X \to B \) with a pseudometric \( \rho \) on \( X \) is called a trivial metric (T-metric, for short) on \( p \) if the restriction of \( \rho \) to every fibre \( p^{-1}(b), \, b \in B \), is a metric and \( p^{-1}\tau \cup \tau_{\rho} \), where \( \tau_{\rho} \) is the topology on \( X \) generated by \( \rho \), is a subbase of the topology of \( X \). A map \( p : X \to B \) is called trivially metrizable (a TM-map, for short) if there exists a T-metric on \( p \). A T-metric on a map \( p : X \to B \) is called complete (a CT-metric, or short) if

(*) For any \( b \)-filter \( \mathcal{F} \), \( b \in B \), on \( X \) containing elements of arbitrary small diameter, \( \mathcal{F} \) has adherence points.

A map \( p : X \to B \) is called completely trivially metrizable (a complete TM-map, for short) if there exists a CT-metric on it.

A map \( p : X \to B \) is called (resp. closely) parallel to a space \( Z \) if there exists an embedding \( e : X \to B \times Z \) such that \( p = \pi \circ e \), where \( \pi : B \times Z \to B \) is the projection (see [10]).

The following are proved in [11]: A map \( p : X \to B \) is a TM-map if and only if \( p \) is parallel to a metrizable map, and \( p \) is a complete TM-map if and only if it is closely parallel to a completely metrizable (i.e., metrizable by complete metric) space.

**Remark:** By these, for a TM-map \( p : X \to B \) there exists a metric space \( (M, \rho) \) and an embedding \( e : X \to B \times M \) such that \( p = \pi \circ e \). Then it is easy to see that we can define a T-metric (pseudometric) \( \rho' \) on \( X \) by \( \rho'(x, y) = \rho(\pi \circ e(x), \pi \circ e(y)) \), and vice versa. So, we can identify \( \rho \) on \( M \) and \( \rho' \) on \( X \) in the above meaning. In latter sections, we use the same notation \( \rho \) on \( M \) and on \( X \).

We shall conclude this section by referring to fibrewise uniformities according to [7]. First, we recall the following definition.

**Definition 2.1.** Let \( p : X \to B \) be a projection, and \( \Delta \) be the diagonal of \( X \times X \). A fibrewise entourage uniformity on \( X \) is a filter \( \Omega \) on \( X \times X \) satisfying the following four conditions:

1. \( \Delta \subset D \) for every \( D \in \Omega \).
2. Let \( D \in \Omega \). Then for each \( b \in B \) there exist \( W \in N(b) \) and \( E \in \Omega \) such that \( E \cap X_W^2 \subset D^{-1} \).
3. Let \( D \in \Omega \). Then for each \( b \in B \) there exist \( W \in N(b) \) and \( E \in \Omega \) such that

\[
(E \cap X_W^2) \circ (E \cap X_W^2) \subset D
\]

4. If \( E \subset X \times X \) satisfies that for each \( b \in B \) there exist \( W \in N(b) \) and \( D \in \Omega \) such that \( D \cap X_W^2 \subset E \), then \( E \in \Omega \).
Note that in [5] Section 12, a filter $\Omega$ on $X \times X$ satisfying (J1), (J2) and (J3) is called a fibrewise uniform structure on $X$. So, the notion of a fibrewise entourage uniformity is slightly stronger than one of a fibrewise uniform structure.

For a projection $p : X \to B$ and $W \in \tau$, let $\mu_W$ be a non-empty family of coverings of $X_W$. We say that $\{\mu_W\}_{W \in \tau}$ is a system of coverings of $\{X_W\}_{W \in \tau}$. (For this, we briefly use the notations $\{\mu_W\}$ and $\{X_W\}$). Let $\mathcal{U}$ and $\mathcal{V}$ be families of subsets of a set $X$. If $\mathcal{V}$ refines $\mathcal{U}$ in the usual sense, we denote $\mathcal{V} \prec \mathcal{U}$. Let us define the notion of fibrewise covering uniformity.

**Definition 2.2.** Let $p : X \to B$ be a projection, and $\mu = \{\mu_W\}$ be a system of coverings of $\{X_W\}$. We say that the system $\{\mu_W\}$ is a fibrewise covering uniformity (and a pair $(X, \mu)$ or $(X, \{\mu_W\})$ is a fibrewise covering uniform space) if the following conditions are satisfied:

(C1) Let $\mathcal{U}$ be a covering of $X_W$ and for each $b \in W$ there exist $W' \in N(b)$ and $\mathcal{V} \in \mu_W$ such that $W' \subset W$ and $\mathcal{V} \prec \mathcal{U}$. Then $\mathcal{U} \in \mu_W$.

(C2) For each $\mathcal{U}_i \in \mu_W$, $i = 1, 2$, there exists $\mathcal{U}_3 \in \mu_W$ such that $\mathcal{U}_3 \prec \mathcal{U}_i$, $i = 1, 2$.

(C3) For each $\mathcal{U} \in \mu_W$ and $b \in W$, there exist $W' \in N(b)$ and $\mathcal{V} \in \mu_W$ such that $W' \subset W$ and $\mathcal{V}$ is a star refinement of $\mathcal{U}$.

(C4) For $W' \subset W$, $\mu_W \supseteq \mu_W|_{X_{W'}}$, where

$$\mu_W|_{X_{W'}} = \{U|_{X_{W'}}, U \in \mu_W\} \quad \text{and} \quad U|_{X_{W'}} = \{U \cap X_{W'}, U \in \mathcal{U}\}.$$ 

For a fibrewise entourage uniformity $\Omega$ on $X$, $D \in \Omega$ and $W \in \tau$, let $\mathcal{U}(D, W) = \{D(x) \cap X_W|x \in X_W\}$. Further let $\mu_W(\Omega)$ be the family of coverings $\mathcal{U}$ of $X_W$ satisfying that for each $b \in W$ there exist $W' \in N(b)$ and $D \in \Omega$ such that $W' \subset W$ and $\mathcal{U}(D, W') \prec \mathcal{U}$. Then the system $\mu(\Omega) = \{\mu_W(\Omega)\}$ is a fibrewise covering uniformity ([7] Proposition 3.7).

Conversely, for a fibrewise covering uniformity $\mu = \{\mu_W\}$, we can constructed a fibrewise entourage uniformity $\Omega(\mu)$ as follows ([7] Construction 3.8): For $\mathcal{U} \in \mu_W$, $D(\mathcal{U}) = \cup\{U_\alpha \times U_\alpha|U_\alpha \in \mathcal{U}\}$. Let $\Omega(\mu)$ be the family of all subsets $D \subset X \times X$ satisfying the following condition:

$$\Delta \subset D, \text{ and for every } b \in B \text{ there exist } W \in N(b) \text{ and } \mathcal{U} \in \mu_W \text{ such that } D(\mathcal{U}) \subset D.$$ 

Then $\Omega(\mu)$ is a fibrewise entourage uniformity ([7] Proposition 3.10). Further, we proved the following:

**Theorem 2.3.** ([7] Theorem 3.11) For a projection $p : X \to B$ and a fibrewise entourage uniformity $\Omega$ on $X$, we have $\Omega = \Omega(\mu(\Omega))$.

For a fibrewise entourage uniformity $\Omega$ on $X$ and a fibrewise covering uniformity $\mu$ on $X$, let $\tau(\Omega)$ be the fibrewise topology induced by $\Omega$ ([5] Section 13) and $\tau(\mu)$ be the fibrewise topology induced by $\mu$ ([7] Proposition 3.8). Then $\tau(\Omega) = \tau(\mu(\Omega))$ and $\tau(\mu) = \tau(\Omega(\mu))$ ([7] Proposition 3.12).
3. Fibrewise covering uniformities on TM-maps

For a TM-map $p : X \to B$ parallel to a metric space $(M, \rho)$, let $e : X \to B \times M$ be the embedding. For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be the family $\{U(x, \frac{1}{n})|x \in M\}$, where $U(x, \frac{1}{n}) = \{y \in M|\rho(x, y) < \frac{1}{n}\}$ and $W_n = \{e^{-1}(B \times U)|U \in \mathcal{U}_n\}$. Then for each $W \in \tau$, let $\mu_W = \{U|\bigcup U = X_W\}$ and for each $b \in W$ there exists $n \in \mathbb{N}$ and $W' \in N(b)$ with $W' \subseteq W$ such that $W_n|_{X_{W'}} < U$.

Since $\mu_W$ and $\mu$ constructed above are induced by the metric $\rho$ on $M$ (on $X$), we call this $\mu = \{\mu_W\}$ a fibrewise covering uniformity on $X$ induced by the metric $\rho$, and denoted by $\mu_\rho = \{\mu_W\}_\rho$. Further, by the construction of $\{W_n|n \in \mathbb{N}\}$ in the above, we say that the family $\{W_n|n \in \mathbb{N}\}$ is the standard developable covering (sd-covering, for short) on $X$ induced by $\rho$. (Note that we exclusively use the notation $\{W_n|n \in \mathbb{N}\}$ as sd-covering induced by $\rho$ in this paper.)

**Theorem 3.1.** For a TM-map $p : X \to B$ with a $T$-metric $\rho$, the system $\mu_\rho = \{\mu_W\}_\rho$ is a fibrewise covering uniformity on $X$ induced by $\rho$.

4. Equivalence of some completeness on TM-maps

**Definition 4.1.** ([5] Definition 14.1) For a map $p : X \to B$, let $\Omega$ be a fibrewise entourage uniformity on $X$.

1. A subset $M$ of $X$ is said to be $D$-small, where $D \subseteq X^2$, if $M^2$ is contained in $D$.
2. A $b$-filter $\mathcal{F}$, where $b \in B$, is Cauchy if $\mathcal{F}$ contains a $D$-small members for each $D \in \Omega$. (We call $\mathcal{F}$ $J$-Cauchy with respect to $\Omega$ (w.r.t. $\Omega$, for short), for convenience' sake.)

We shall define a new notion of Cauchy $b$-filter in fibrewise covering uniformity $\mu = \{\mu_W\}$ on $X$.

**Definition 4.2.** For a map $p : X \to B$, let $\mu = \{\mu_W\}$ be a fibrewise covering uniformity on $X$. A $b$-filter $\mathcal{F}$, where $b \in B$, is Cauchy if for each $W \in N(b)$ and $\mathcal{U} \in \mu_W$ there exist $F \in \mathcal{F}$ and $U \in \mathcal{U}$ such that $F \subseteq U$. (We call $\mathcal{F}$ $CU$-Cauchy with respect to $\mu$ (w.r.t. $\mu$, for short), for convenience' sake.)

**Theorem 4.3.** For a map $p : X \to B$, let $\Omega$ be a fibrewise entourage uniformity on $X$. Then for each $b \in B$, a $b$-filter $\mathcal{F}$ is $J$-Cauchy w.r.t. $\Omega$ if and only if it is $CU$-Cauchy w.r.t. $\mu(\Omega)$.

For a space $X$, let $\Psi = \{\Phi_\alpha|\alpha \in \Lambda\}$ be a family of families of subsets of $X$. We say that a family $\Psi$ of subsets of $X$ is subordinated to the family $\Psi$ if for each $\alpha \in \Lambda$ there exists $U_\alpha \in \Phi_\alpha$ and $V \in \Psi$ such that $V \subseteq U_\alpha$. 


Definition 4.4. Let $p : X \to B$ be a $TM$-map with a $T$-metric $\rho$.

1. ([11]) The map $p$ is complete if for any $b$-filter $F$, $b \in B$, on $X$ subordinated to the $sd$-covering $\{W_n | n \in N\}$ induced by $\rho$, it has adherence points. (We call this "complete" $P$-complete, and also call this $b$-filter satisfying this condition $P$-Cauchy w.r.t. $\rho$.)

2. ([5] Definition 14.10) The map $p$ is complete if for each $b \in B$ any $J$-Cauchy $b$-filter $F$ w.r.t. $\Omega(\mu_\rho)$ converges. (We call this "complete" $J$-complete.)

Theorem 4.5. For a $TM$-map $p : X \to B$ with a $T$-metric $\rho$ and each $b \in B$, a $b$-filter $F$ is a $P$-Cauchy w.r.t. $\rho$ if and only if it is a $J$-Cauchy w.r.t. $\Omega_\rho$.

5. COMPLETE $TM$-MAPS AND ČECH-COMPLETE MAPS

Definition 5.1. A $T_2$-compactifiable map $p : X \to B$ is Čech-complete if for each $b \in B$, there exists a countable family $\{A_n\}_{n \in N}$ of open (in $X$) covers of $X_b$ with the property that every $b$-filter $F$ which is subordinated to the family $\{A_n\}_{n \in N}$ has an adherence point.

Proposition 5.2. (1) ([1] Theorem 6.1) Every locally compact map is Čech-complete.

(2) ([1] Theorem 4.1) For $T_2$-compactifiable maps $p : X \to B$, $q : Y \to B$ and a perfect morphism $f : p \to q$, $p$ is Čech-complete if and only if $q$ is Čech-complete.

Lemma 5.3. Every $TM$-map $p : X \to B$ is a $T_{3\frac{1}{2}}$-map.

By this lemma, every $TM$-map is $T_{3\frac{1}{2}}$-compactifiable. For complete $TM$-maps, we can prove the following.

Theorem 5.4. If $p : X \to B$ is a complete $TM$-map, then $p$ is Čech-complete.

6. MT-MAPS AND SOME PROBLEMS

About the relations of $TM$-maps and $MT$-maps, we have the following.

(a) A closed $TM$-map is an $MT$-map.

(b) There exists a compact $MT$-map which is not a $TM$-map.

(c) There exists (complete) $TM$-maps which are not closed, so not $MT$-maps.

Theorem 6.1. If $p : X \to B$ is a closed $TM$-map, then $p$ is an $MT$-map.
As discussed in section 5, there seems to exist many problems about relations between metrizable maps and completeness. As an attempt to the problems, we define a new notion of $D$-complete $MT$-maps. For an $MT$-map $p : X \to B$, we use the following notation: $\{{\mathcal{U}_n(b)}_{n\in N}|b \in B\}$ is a $p$-development, where $\{{\mathcal{U}_n(b)}_{n\in N}|b \in B\}$ is a $b$-development. First, we recall some definitions and theorems of $MT$-maps according to [3].

**Definition 6.2.** (1) ([3] Def. 2.8) For a map $p : X \to B$, a sequence $\{{\mathcal{U}_n}\}_{n \in N}$ of open (in $X$) covers of $X_b, b \in B$, is said to be a $b$-development if for every $x \in X_b$ and every $U \in N(x)$, there exists $n \in N$ and $W \in N(b)$ such that $x \in st(x, \mathcal{U}_n) \cap X_W \subset U$. The map $p$ is said to have a $p$-development if it has a $b$-development for every $b \in B$.
(2) ([3] Def. 2.9) A closed map $p : X \to B$ is said to be an $MT$-map if it is collectionwise normal and has a $p$-development.

**Definition 6.3.** For an $MT$-map $p : X \to B$ equipped with $p$-development $\{{\mathcal{U}_n(b)}_{n \in N}|b \in B\}$, we call $p$ $D$-complete with respect to the $p$-development if for each $b \in B$ every $b$-filter $\mathcal{F}$ subordinated to $\{{\mathcal{U}_n(b)}_{n \in N}|b \in B\}$ has adherence points.

**Problem 6.4.** For an $MT$-map $p : X \to B$, let $\{{\mathcal{U}_n(b)}_{n \in N}|b \in B\}$ be a $p$-development.
(1) Is there a fibrewise (covering) uniformity on $X$ related to the $p$-development?
(2) If Problem (1) had an affirmative answer, then is the $J$-completion of $p$ w.r.t. the fibrewise (covering) uniformity on $X$ equivalent to $D$-completion?

**REFERENCES**